YITP

arXiv:1603.09543

Sine-square deformation(SSD) and Mobius quantization of two-dimensional conformal field theory

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thanks Hosho Katsura(Univ. Tokyo)
Tsukasa Tada(RIKEN)

Sine-square deformation (SSD)

- ✓ SSD : smooth cutoff to suppress the boundary scattering

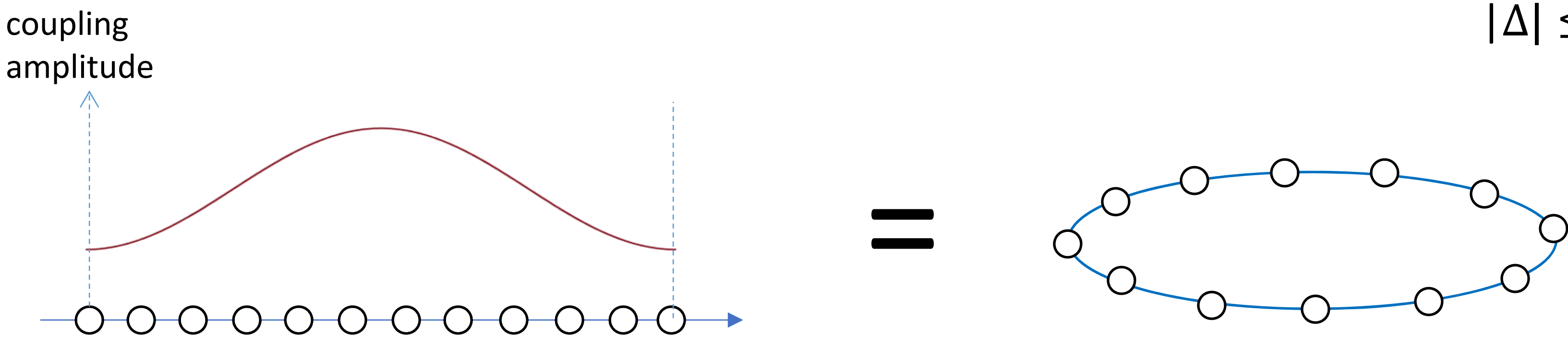
Gendiar, Krcmar and Nishino, Prog. Theor. Phys. 122, 953(2009)

Hikihara and Nishino, Phys. Rev. B 83, 060414(2011)

e.g. spin chains

$$H = \sum \left(\sin \frac{\pi}{L} n\right)^2 (S_n^x S_{n+1}^x + S_n^y S_{n+1}^y + \Delta S_n^z S_{n+1}^z) - h \sum \left(\sin \frac{\pi}{L} n\right)^2 S_n^z$$

$$|\Delta| \leq 1$$

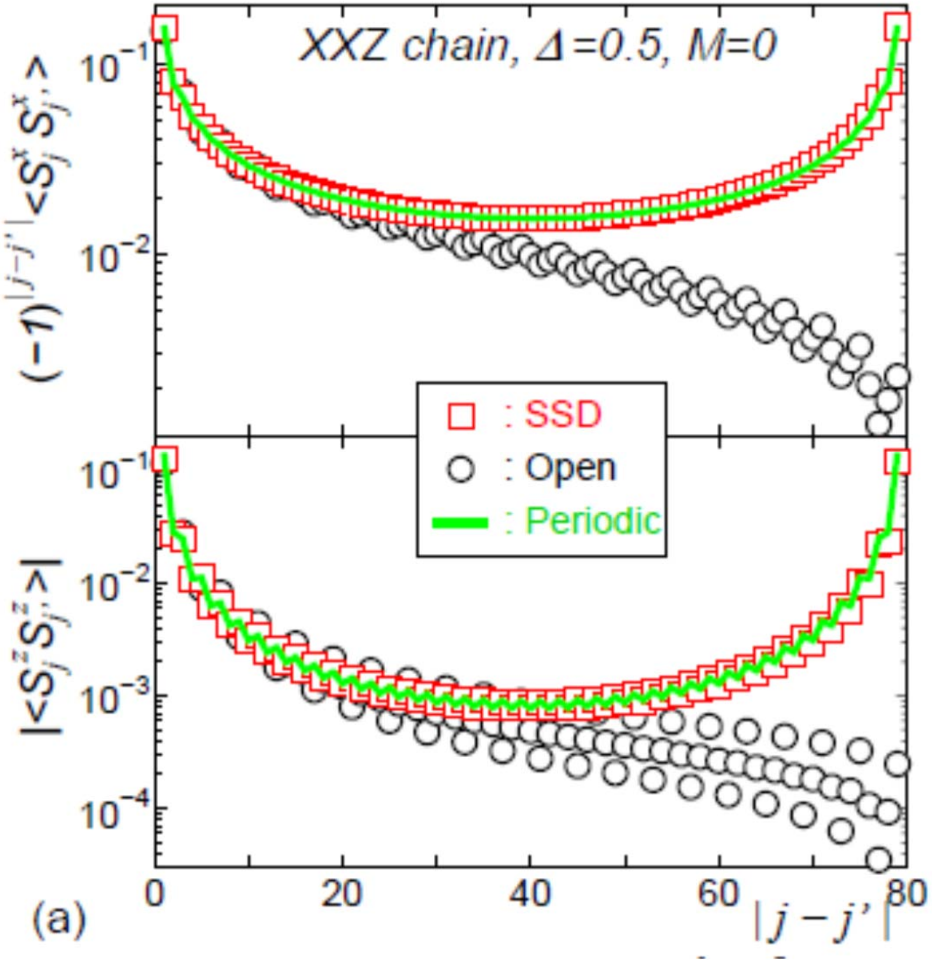


SSD system(open B.C.)

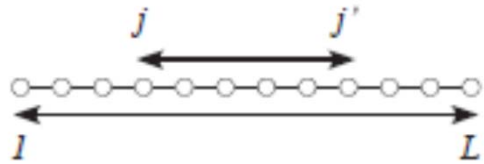
uniform system with periodic B.C.

The groundstate wavefunctions are identical to each other!

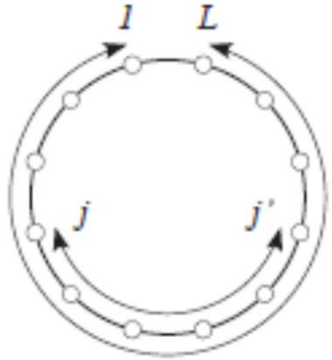
Correlation functions of the XXZ chain(massless regime)



Open / SSD



Periodic



Hikihara and Nishino, Phys. Rev. B 83, 060414(2011)

The overlap of the wavefunctions between SSD and PBC systems is 1, within numerical accuracy!

SSD

- ✓ the exact equivalence between SSD and PBC systems
  for the gapless groundstate *Hikihara and Nishino, Phys. Rev. B **83**, 060414(2011)*

- ✓ exact example:

XY model, *Katsura, J. Phys. A: Math. theror. **44**, 252011(2011)*

free fermion(lattice) *Maruyama, Katsura and Hikihara, Phys. Rev. B **84**, 165132(2011)*

free fermion(non-rela) *Okunishi, Katsura, J.Phys.A:Math.Teore. **48** (2015) 445208*

- ✓ applications grand canonical approach for magnetization curves

*Shibata and Hotta, Phys. Rev.B **84**, 115116(2011)*
*Hotta Nishimoto and Shibata, Phys. Rev. B **87**, 115128 (2013)*

- ✓ CFT *H. Katsura, J. Phys. A:Math. Theore. 45, 115003 (2012)*
- ✓ string theory/CFT *Tada, arXiv:1404.6346[hep-th]*
  Ishibashi and Tada, arXiv:1504.00138[hep-th]
  Ishibashi and Tada, arXive: 1602.01190[hep-th]

Lattice free fermion

Maruyama, Katsura and Hikihara, Phys. Rev. B 84, 165132(2011)

$$H = \sum -\left(\sin\frac{\pi}{L}n\right)^2 (c_n^+ c_{n+1} + c_n\, c_{n+1}^+) - \mu \sum \left(\sin\frac{\pi}{L}(n - {}^1/_2)\right)^2 c_n^+ c_n$$

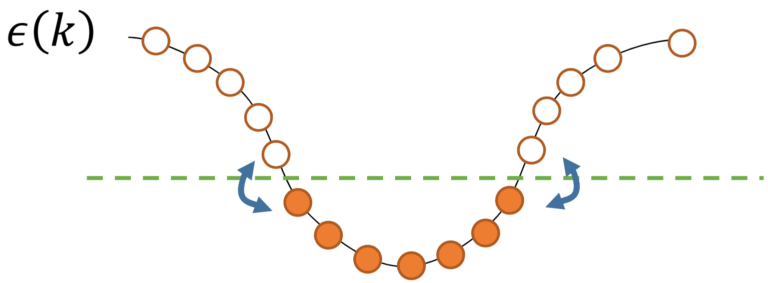
Fourier transform. $$H = \frac{1}{2}H_0 - \frac{1}{4}(H_+ + H_-)$$

$$H_0 = \sum_k \epsilon(k) c_k^+ c_k$$

uniform $\epsilon(k) = -(2\cos k - \mu)$

$$H_\pm = \sum_k e^{\mp i\frac{\pi}{L}} \epsilon(k \mp \frac{\pi}{L}) c_k^+ c_{k\mp 1}$$

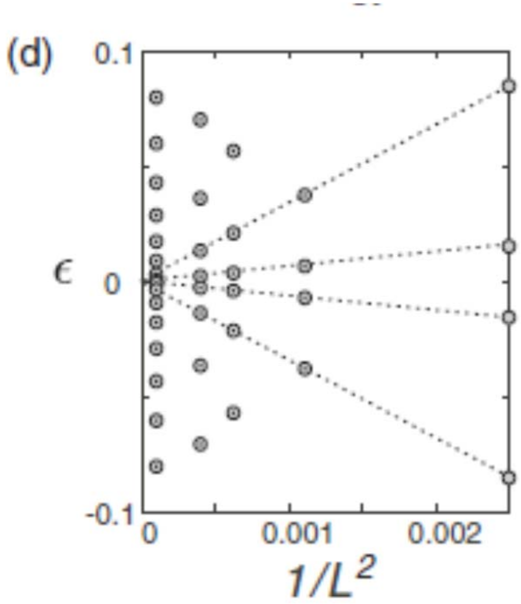
spatial deformation



Nearest-neighbor hopping in the momentum space!

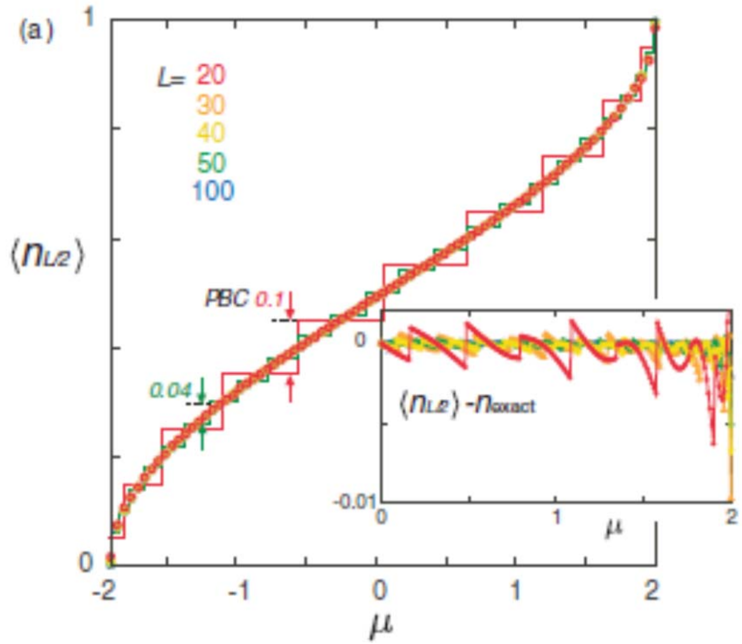
$H_\pm$ can annihilates the Fermi sea, if the chemical potential is appropriately chosen

excitations



1/L^2 dependence of
low-energy excitations

drastically reduce
the finite size effect



(Almost) continuous $\mu - \rho$ curve

*Hotta Nishimoto and Shibata, Phys. Rev. B **87**, 115128 (2013)*
application: grandcanonical approach

<u>Note</u>

Non-relativistic free fermion: SUSY quantum mechanics

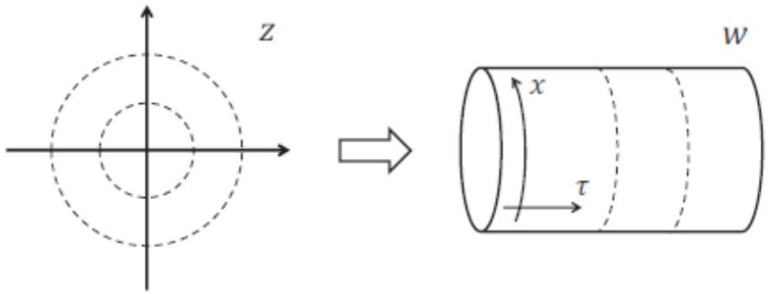
➡ exact 1/L^2 dependence of excitations

*Okunishi, Katsura, J.Phys.A:Math.Teore. **48** (2015) 445208*

CFT Hamiltonian

H. Katsura J. Phys. A: Math. Thoer. 45, 115003 (2012)

Hamiltonian for the cylinder



$$H_0 = \int_0^l \frac{dw}{2\pi}(T(w) + \bar{T}(\bar{w}))$$

$$w = \frac{l}{2\pi}\log(z)$$

$$H_\pm = \int_0^l \frac{dw}{2\pi}(e^{2\pi w/l}T(w) + e^{2\pi \bar{w}/l}\bar{T}(\bar{w}))$$

SSD Hamiltonian $\quad H_{SSD} = \frac{\pi}{l}[L_0 + \bar{L}_0 - \frac{1}{2}(L_1 + L_{-1} + \bar{L}_1 + \bar{L}_{-1})]$

SL(2,C) invariance of CFT ⇨ $L_{\pm 1}|0\rangle = 0$

The SSD vacuum is equivalent to that of the uniform system, H_0

SSD/CFT dipolar quantization

Ishibashi and Tada, arXiv:1504.00138[hep-th]
Ishibashi and Tada, arXive: 1602.01190[hep-th]

Regarding H_{SSD} as a Hamiltonian of a CFT

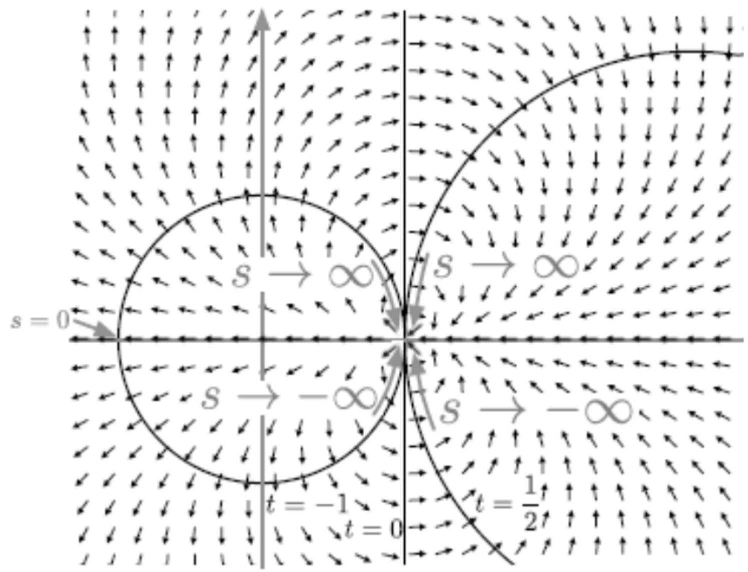
$$\mathcal{L}_0 = L_0 - \frac{1}{2}(L_1 + L_{-1}), \quad \bar{\mathcal{L}}_0 = \bar{L}_0 - \frac{1}{2}(\bar{L}_1 + \bar{L}_{-1})$$

Quantization of the CFT, starting with the classical Virasoro algebra

- Quantization on the dipolar coordinate (singularity at z=1)
- infinite circumference limit

$$-\infty < s < \infty$$

- continuous Virasoro algebra



$$[\mathcal{L}_\kappa, \mathcal{L}_{\kappa'}] = (\kappa - \kappa')\mathcal{L}_{\kappa+\kappa'} + \frac{c}{12}\kappa^3\delta(\kappa + \kappa')$$

Parameterization: uniform and SSD

$$\mathcal{L}_0(\theta) = \cosh(\theta) L_0 - \frac{\sinh(\theta)}{2}(L_1 + L_{-1})$$

θ= 0 : Uniform
∞ : SSD

The CFT vacuum is always the same for any real θ

SL(2,R) invariance of CFT

Lorentz transformation

If we construct a CFT for the Hamiltonian of $\mathcal{L}_0(\theta) + \bar{\mathcal{L}}_0(\theta)$, what happens?

interpolating between radial quantization(uniform system) and dipolar quantization(SSD system).

Classical Virasoro

$$\mathfrak{l}_0 \equiv \cosh(2\theta) l_0 - \sinh(2\theta)\frac{l_{-1} + l_1}{2} = -g(z)\frac{\partial}{\partial z}$$

$$g(z) \equiv -\frac{1}{2}\left[\sinh(2\theta)(z^2 + 1) - 2\cosh(2\theta) z\right]$$

$$\mathfrak{l}_0 f_n(z) = -n f_n(z)$$

$$l_n = -z^{-n+1}\frac{\partial}{\partial z} \quad \Rightarrow \quad \mathfrak{l}_n = -g(z) f_n(z)\frac{\partial}{\partial z}$$

classical Virasoro generators — new classical Virasoro generators

$$f_n(z) = (-)^n\left(\frac{\sinh(\theta) - \cosh(\theta) z}{\cosh(\theta) - \sinh(\theta) z}\right)^n$$

Mobius transformation coordinate

$$\Rightarrow \quad [\mathfrak{l}_n, \mathfrak{l}_{n'}] = (n - n')\mathfrak{l}_{n+n'}$$

classical Virasoro algebra

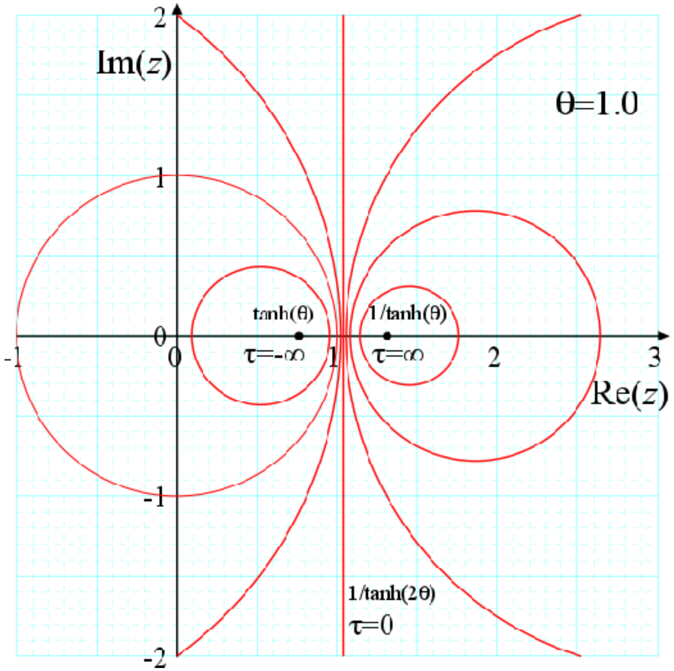
Mobius coordinate

define complex cooridinate $\zeta = \tau + is$

with $-\frac{\partial}{\partial \tau} = \mathrm{L}_0 + \bar{\mathrm{L}}_0 \quad -\frac{\partial}{\partial s} = i(\mathrm{L}_0 - \bar{\mathrm{L}}_0)$

$$\begin{cases} \tau = -\infty: & z = \tanh(\theta) \\ \tau = +\infty: & z = 1/\tanh(\theta) \end{cases}$$

$$-\pi \leq s \leq \pi$$



constant τ contours in z plane

radial quantization ($\theta = 0$)

$\tau = -\infty$: $z = 0$, $\quad \tau = +\infty$: $z = \infty$

dipolar quantization ($\theta = \infty$)

$\tau = \pm\infty$: $z = 1$, essential singularity

time source and sink approach to each other as θ increases

Virasoro charges

$$\mathcal{L}_n = \oint_\tau \frac{dz}{2\pi i} g(z) f_n(z) T(z) = -(-)^n \frac{\sinh(2\theta)}{2\tanh^n(\theta)} \oint_\tau \frac{dz}{2\pi i} \frac{(z-\tanh(\theta))^{n+1}}{(z-1/\tanh(\theta))^{n-1}} T(z)$$

integration path : constant τ contour

$$T(z) = \sum_n z^{-n-2} L_n$$

Relation with the conventional Virasoro generators

$$\mathcal{L}_0 = \cosh(2\theta) L_0 - \sinh(2\theta) \frac{L_1 + L_{-1}}{2}$$

$$\mathcal{L}_1 = \frac{\sinh(2\theta)}{2} \left(\frac{1}{\tanh(\theta)} L_1 - 2L_0 + \tanh(\theta) L_{-1} \right)$$

$$\mathcal{L}_{-1} = \frac{\sinh(2\theta)}{2} \left(\tanh(\theta) L_1 - 2L_0 + \frac{1}{\tanh(\theta)} L_{-1} \right)$$

SL(2) subalgebra

$$[\mathcal{L}_1, \mathcal{L}_{-1}] = 2\mathcal{L}_0, \qquad [\mathcal{L}_0, \mathcal{L}_{\pm 1}] = \mp \mathcal{L}_{\pm 1}$$

For general n(>1), we have series expansion form :

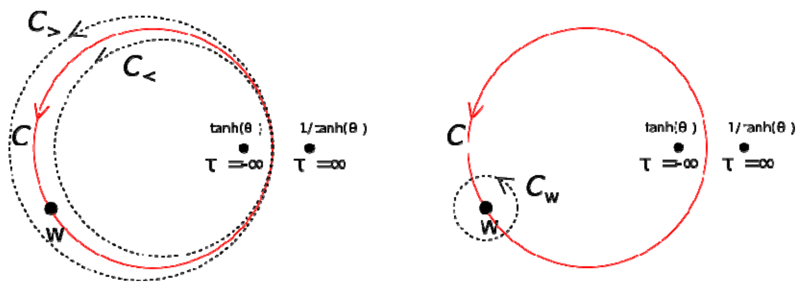
$$\mathcal{L}_n = -(-)^n \frac{\sinh(2\theta)}{2t^n} \sum_l L_l \oint_\tau \frac{dz}{2\pi i} \frac{(z-t)^{n+1}}{(z-1/t)^{n-1}} z^{-l-2} = -(-)^n \frac{\sinh(2\theta)}{2} \sum_l C_l^n(t) L_l$$

$$C_l^n(t) = \begin{cases} (-)^{l+1} \frac{(n+1)!}{(l+1)!(n-l)!} F(n-1, -l-1; n-l+1; t^2)\, t^{n-l-1} & n \geq l \geq -1 \\ (-)^{n+1} \frac{(l-2)!}{(n-2)!(l-n)!} F(-n-1, l-1; l-n+1; t^2)\, t^{l-n-1} & n < l \end{cases}$$

F: Gauss's hypergeometric function , $t \equiv \tanh(\theta)$

$\mathcal{L}_n$ satisfies the Virasoro algebra

commutator ➡ contour integral



$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n} + \frac{c}{12} m(m-1)(m+1)\delta_{n+m,0}$$

“Continuum” limit (connection to the dipolar quantization)

Lorentz transformation contains a diversive factor of scale $N_\theta \equiv \cosh(2\theta)$

overall scale factor

normalized generator $\tilde{\mathcal{L}}_0 \equiv \frac{\mathcal{L}_0}{N_\theta} = L_0 - \frac{\tanh(2\theta)}{2}(L_{-1} + L_1)$

θ=∞ : dipolar quantization

The spectrum of $\mathcal{L}_0$ becomes continuous : $\kappa = n/N_\theta$

The spatial coordinate on the constant τ contours $N_\theta s \longrightarrow \tilde{s}$

$-\pi N_\theta \leq \tilde{s} \leq \pi N_\theta$

continuous Virasoro algebra in the $\theta \to \infty$ limit

$n/N_\theta \longrightarrow \kappa, \quad \mathcal{L}_n/N_\theta \longrightarrow \tilde{\mathcal{L}}_\kappa, \quad N_\theta \delta_{n,0} \longrightarrow \delta(\kappa)$ with $N_\theta \longrightarrow \infty$

$$[\tilde{\mathcal{L}}_\kappa, \tilde{\mathcal{L}}_{\kappa'}] = (\kappa - \kappa')\tilde{\mathcal{L}}_{\kappa+\kappa'} + \frac{c}{12}\kappa^3\delta(\kappa + \kappa')$$

Primary fields

primary field of the scaling dimension h $$\phi'_h(w) = \left(\frac{dz}{dw}\right)^h \phi_h(z)$$

primary state for θ $$|h\rangle_\theta = \phi_h(z = \tanh(\theta))|0\rangle$$

$|h\rangle_\theta$ can be related with $|h\rangle$ through translation operator L_{-1}

$$\phi_h(\tanh(\theta)) = e^{\tanh(\theta)L_{-1}} \phi_h(0) e^{-\tanh(\theta)L_{-1}}$$

$$\mathcal{L}_0|h\rangle_\theta = h|h\rangle_\theta \quad , \quad \mathcal{L}_n|h\rangle_\theta = 0, \quad \text{for} \quad n \geq 1$$

c.f. $$[\mathcal{L}_n, \phi_h(z)] = \; = -(-)^n \frac{\sinh(2\theta)}{2(\tanh(\theta))^n}\left[h\left(\frac{(z-\tanh(\theta))^{n+1}}{(z-\frac{1}{\tanh(\theta)})^{n-1}}\right)' \phi_h(z) - \left(\frac{(z-\tanh(\theta))^{n+1}}{(z-\frac{1}{\tanh(\theta)})^{n-1}}\right)\partial_z\phi_h(z)\right]$$

This primary state is normalizable for $|\theta| < \infty$

However, primary fields at the SSD/dipolar point($\theta = \infty$) is still unknown

conformal mapping approach

We can obtain $\mathcal{L}_n$ without passing through analysis of the Mobius coordinate.

<u>Conformal mapping of SL(2,R)</u>

$$w = -\frac{\sinh(\theta) - \cosh(\theta)z}{\cosh(\theta) - \sinh(\theta)z}$$

$$\mathcal{L}_n = \int \frac{dz}{2\pi i}\frac{dw}{dz}w^{n+1}\left(\frac{dz}{dw}\right)^2 T(z) = -(-)^n \frac{\sinh(2\theta)}{2(\tanh(\theta))^n}\int \frac{dz}{2\pi i}\frac{(z-\tanh(\theta))^{n+1}}{(z-1/\tanh(\theta))^{n-1}}T(z)$$

➡ the same as Mobius quantization

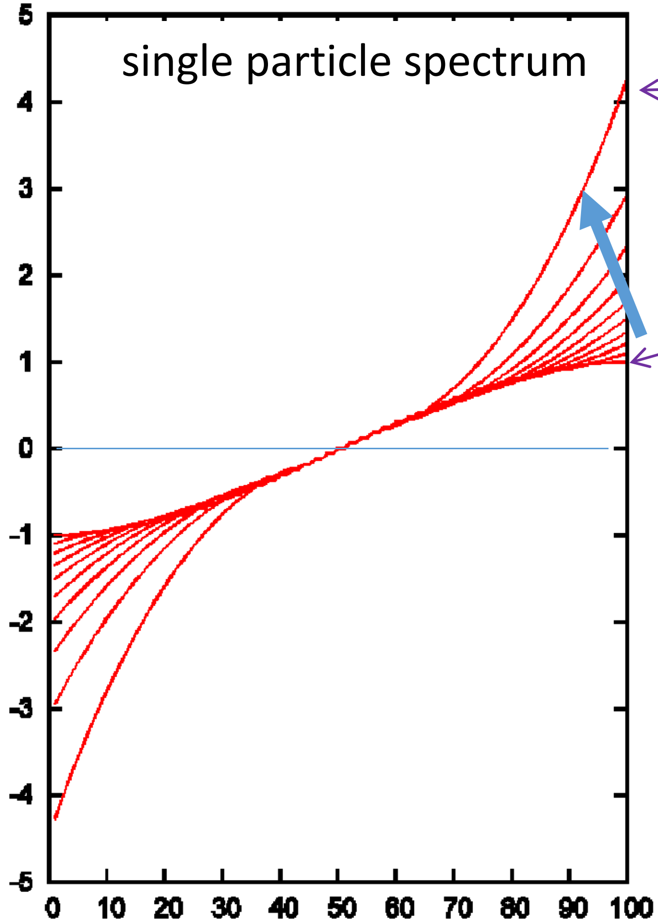
However, the Mobius quantization is essential to reveal the continuum limit of the Virasoro algebra.

simple example: lattice free fermion

$$H = -\sum_n \frac{1}{2}(\cosh(\theta) - \sinh(\theta)\cos\frac{2\pi n}{L})[c_n^\dagger c_{n+1} + h.c.]$$

θ= 0 : Uniform
∞ : SSD

Fermi surface



quantum index

SSD ($\theta \gg 1$)

parabola dispersion

θ

uniform($\theta = 0$)

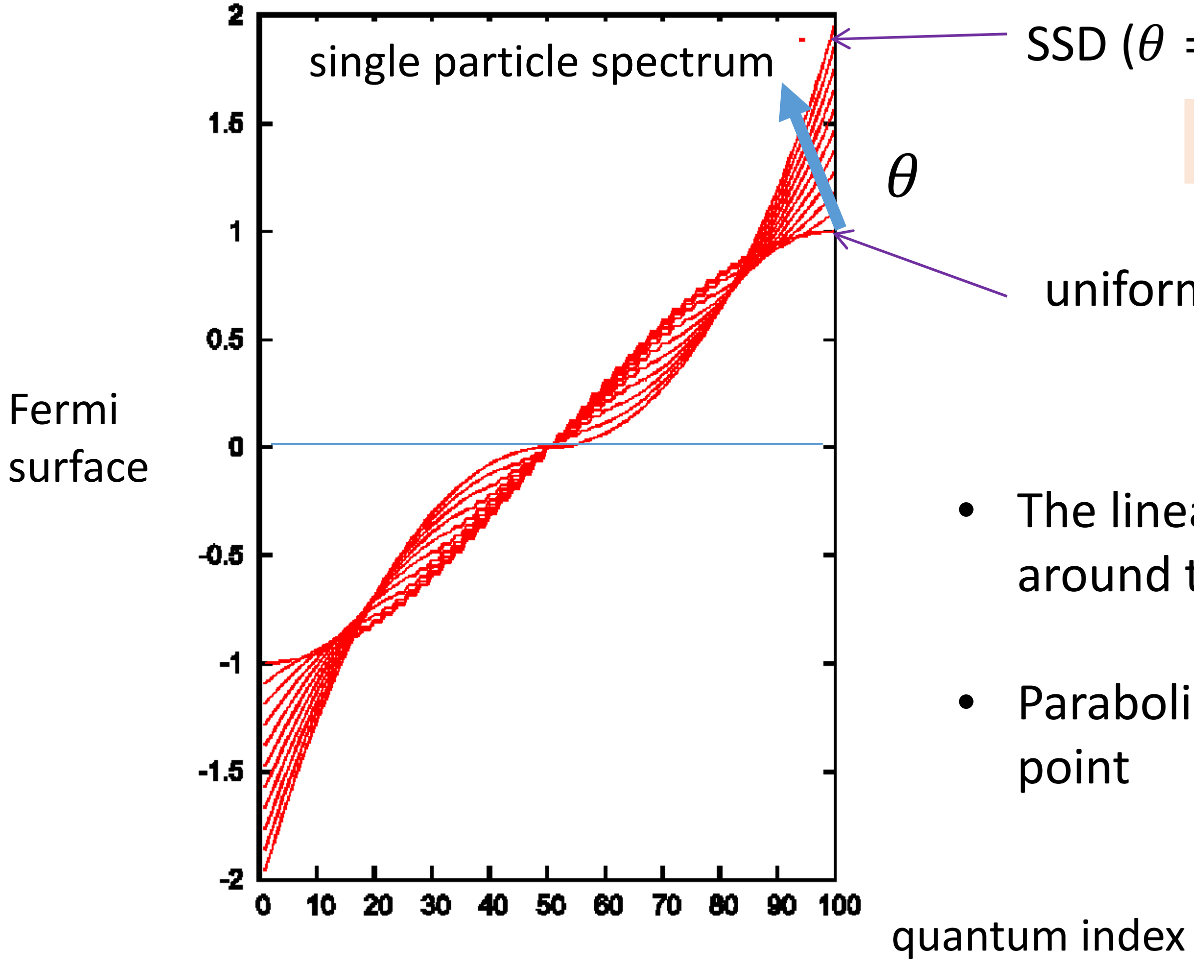
cos dispersion

- The linear dispersion around the Fermi surface is invariant
- High energy states changes into the parabola dispersion due to the SSD effect.

role of N_θ (the same Hamiltonian except for the overall scale)

$$H = -\sum_n \frac{1}{2}(1 - \tanh(\theta)\cos\frac{2\pi n}{L})[c_n^\dagger c_{n+1} + h.c.]$$

θ= 0 : Uniform
∞ : SSD



SSD ($\theta = \infty$)

parabola dispersion

uniform($\theta = 0$)

cos dispersion

- The linear dispersion is compressed around the Ferimi surface.
- Parabolic dispersion at the SSD point

summary

- ✓ We analyzed the SSD problem in terms of 2D CFT
- ✓ The Mobius coordinate of SL(2,R) plays an essential role.
- ✓ For a finite θ, a primary state is well-defined.
- ✓ In the$\theta \to \infty$ limit, we have the continuous Virasoro algebra corresponding to the dipolar quantization .

There are also a couple of remaining mysteries at the SSD point

Problems at the SSD point

- $\mathcal{L}_n$ for finite n collapses to $\mathcal{L}_0$, in the $\theta \to \infty$ limit.

  ⇨ It's difficult to directly see the continuous Virasoro algebra from a finite θ.

- The primary state by the analytic continuation is not normalizable

  ⇨

$$\langle h|e^{\alpha L_1} e^{\alpha L_{-1}}|h\rangle = 1 + \sum_{n=1}^{\infty} B_n$$

This might converges if h<1/2

$$B_n = \frac{\alpha^{2n}}{n!}(n-1+2h)(n-2+2h)\cdots(1+2h)(2h)$$

- $\tau = \pm\infty$ is located at $z = 1$.

  Hemitian conjugate is also nontrivial

  ⇨

$$\begin{aligned}(L_{-1})^\dagger &= L_{-1},\\ (L_0)^\dagger &= 2L_{-1} - L_0,\\ (L_1)^\dagger &= L_1 - 4L_0 + 4L_{-1}.\end{aligned}$$

Ishibashi and Tada, arXive: 1602.01190[hep-th]