Equilibration in Matrix Quantum Mechanics

Yasuhiro Sekino（Takushoku U.）

w/ Robert Hübener and Jens Eisert (Free U. of Berlin)

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The model studied in this work

- Quantum mechanics of
  two 2 x 2 Hermitian matrices X_1, X_2 (momenta P_1, P_2)
- Hamiltonian:

$$H = \frac{1}{2}\mathrm{Tr}\left\{(P_1)^2 + (P_2)^2 - \frac{1}{2}[X_1, X_2]^2\right\}$$

- SU(2) gauge symmetry:
  Physical states are required to be invariant under

$$X_i \to U^\dagger X_i U \qquad U \in SU(2)$$

- Simple, but not (known to be) integrable.

Matrix quantum mechanics: theory of D0-branes

- Collection of *N* D-branes are described by *N* x *N* hermitian matrices.
  - diagonal elements: position of D-branes
  - off-diag. elements: strings between D-branes

- Novel features of D-branes:
  - Non-commutativity of coordinates
  - SU(*N*) gauge symmetry

- Important issues:
  - Fast scrambling [Sekino-Susskind, '08]: $t_* \sim \log N$
    Due to non-locality in the space of matrix elements?
  - Change of reference frames: gauge symmetry?

The full theory for D0-branes (BFSS Matrix theory)

Dimensional reduction of supersymmetric SU(*N*)
Yang-Mills theory from (9+1)D

$$H = \mathrm{Tr}\left[\frac{1}{2}P^iP^i - \frac{1}{4}[X^i, X^j]^2 + \frac{1}{2}\theta^T\gamma_i[X^i, \theta]\right]$$

Proposed to be an exact description of M-theory.

[Banks-Fischler-Shenker-Susskind, '96]

Difficult to solve:

- No mass term; Interaction connects every elements.
- Perturbation is useful only in limited cases.

Known facts:

- Flat direction for $[X_i, X_j]=0$, due to SUSY cancellation; continuous spectrum [de Wit-Luscher-Nicolai '89, …]

Known facts

- Evidence for correspondence w/ gravity from Monte-Carlo
  - Reproduces black hole thermodynamics
    [Hanada, Nishimura, Ishiki, Hyakutake, Takeuchi, Miwa, '08-]
  - Gauge/gravity correspondence: correlation functions
    [Sekino, Yoneya, '99-, Hanada, Nishimura, Sekino, Yoneya, '09-]
- Analysis of classical dynamics:
  [Asplund, Berenstein, Dzienkowski, '12-, Gur-Ari, Hanada, Shenker, '16]

Direct quantum mechanical study is desirable
(find energy eigenstates, and study unitary evolution).

As a toy model, we study the d=2, N=2 bosonic model.

Degrees of freedom

- We parametrize the 2x2 matrices as: $X_i = \sum_{a=1}^{3} x_a^i \frac{\sigma^a}{2}$
- Each matrix is a vector in the "internal space" R^3: $\vec{x}_1, \ \vec{x}_2$
- Gauge symmetry SU(2)=SO(3) acts as rotation in this R^3.
(Gauge transformation: simultaneous rotation of $\vec{x}_1, \ \vec{x}_2$)
- Hamiltonian:

$$H = \frac{1}{4}\left\{(\vec{p}_1)^2 + (\vec{p}_2)^2 + (\vec{x}_1 \times \vec{x}_2)^2\right\}$$

(Have been considered as a prototypical model of chaos)

Gauge invariant wave function

- Total angular momentum (in internal space) =0.

- Gauge invariant variables:
  r_1, r_2, θ_{12} (angle between $\vec{x}_1, \ \vec{x}_2$)

$$(r_1)^2 = 2\mathrm{Tr}(X_1 X_1), \quad (r_2)^2 = 2\mathrm{Tr}(X_2 X_2),$$
$$r_1 r_2 \cos\theta_{1,2} = \vec{x}_1 \cdot \vec{x}_2 = 2\mathrm{Tr}(X_1 X_2)$$

- Gauge invariant wave fn.: parametrized by spherical harmonics

$$\Psi(r_1, \theta_1, \phi_1; r_2, \theta_2, \phi_2)$$
$$= \frac{1}{r_1 r_2} \sum_{\ell} \frac{\psi_\ell(r_1, r_2)}{\sqrt{2\ell+1}} \sum_{m=-\ell}^{\ell} (-1)^{\ell-m} Y_{\ell,m}(\theta_1, \phi_1) Y_{\ell,-m}(\theta_2, \phi_2)$$

One function $\psi_\ell(r_1, r_2)$ for each $\ell \ (= 0, 1, 2, \ldots)$

Schrödinger equation

- $H\Psi = E\Psi$

$$\sum_{\ell'} \left[\left\{ -\partial_{r_1}^2 - \partial_{r_2}^2 + \left(\frac{1}{r_1^2} + \frac{1}{r_2^2} \right) \ell(\ell+1) \right\} \delta_{\ell,\ell'} + r_1^2 r_2^2 A_{\ell,\ell'} \right] \psi_{\ell'}(r_1, r_2) = E \psi_\ell(r_1, r_2)$$

  - 1^{st} term (from kinetic term): diagonal in ℓ .
  - 2^{nd} term (from commutator) : shifts ℓ by ±2 (Interaction is local in the ℓ space.)

- $A_{\ell\ell'}$ is roughly the 2^{nd} order difference operator

$$\Delta_{\ell,\ell'} = \delta_{\ell',\ell+2} + \delta_{\ell',\ell-2} - 2\delta_{\ell',\ell}$$

  - The ℓ system is roughly the discretized harmonic oscillator (with r_1, r_2 dependent mass, spring constant)

Conserved charge

- SO(2) angular momentum (<u>in target space</u>):

$$Q = \vec{x}_1 \cdot \vec{p}_2 - \vec{x}_2 \cdot \vec{p}_1$$

- Representation of *Q* as an operator acting on $\psi_\ell(r_1, r_2)$

$$Q = \sum_\ell \Big(q_{\ell+1,\ell} |\ell+1\rangle\langle\ell| + q_{\ell-1,\ell} |\ell-1\rangle\langle\ell| \Big)$$

$$q_{\ell+1,\ell} = q^\dagger_{\ell,\ell+1} = i \left\{ \frac{\ell}{\sqrt{4\ell^2-1}} (r_1 \partial_{r_2} - r_2 \partial_{r_1}) - \frac{\ell^2}{\sqrt{4\ell^2-1}} \left(\frac{r_1}{r_2} - \frac{r_2}{r_1} \right) \right\}$$

- [*H*, *Q*]=0: *H* and *Q* are simultaneously diagonalizable.
- Eigenvalues of *Q* are even integers.

Analysis of the spectrum

We numerically diagonalize the Hamiltonian.

- Hilbert space: $\mathcal{H} = L^2(R^+) \otimes L^2(R^+) \otimes l^2$

- Truncate the Hilbert space:
  Take the first h_0=107 levels of the oscillators for r_1, r_2, and the first l_0=156 of l.
  Checked the cutoff independence by varying h_0 and l_0

- The locality of interaction has been essential for the precision of the analysis. (Sparse matrices are easier to diagonalize than general ones.)

We found about 800 energy eigenstates.

For each energy eigenstate,

- Identify conserved charge : $\langle Q^2 \rangle$
- Compute the spatial extent of the state:

$$L^2 = \langle (r_1)^2 + (r_2)^2 \rangle = 2\langle \mathrm{Tr}((X_1)^2 + (X_2)^2) \rangle$$

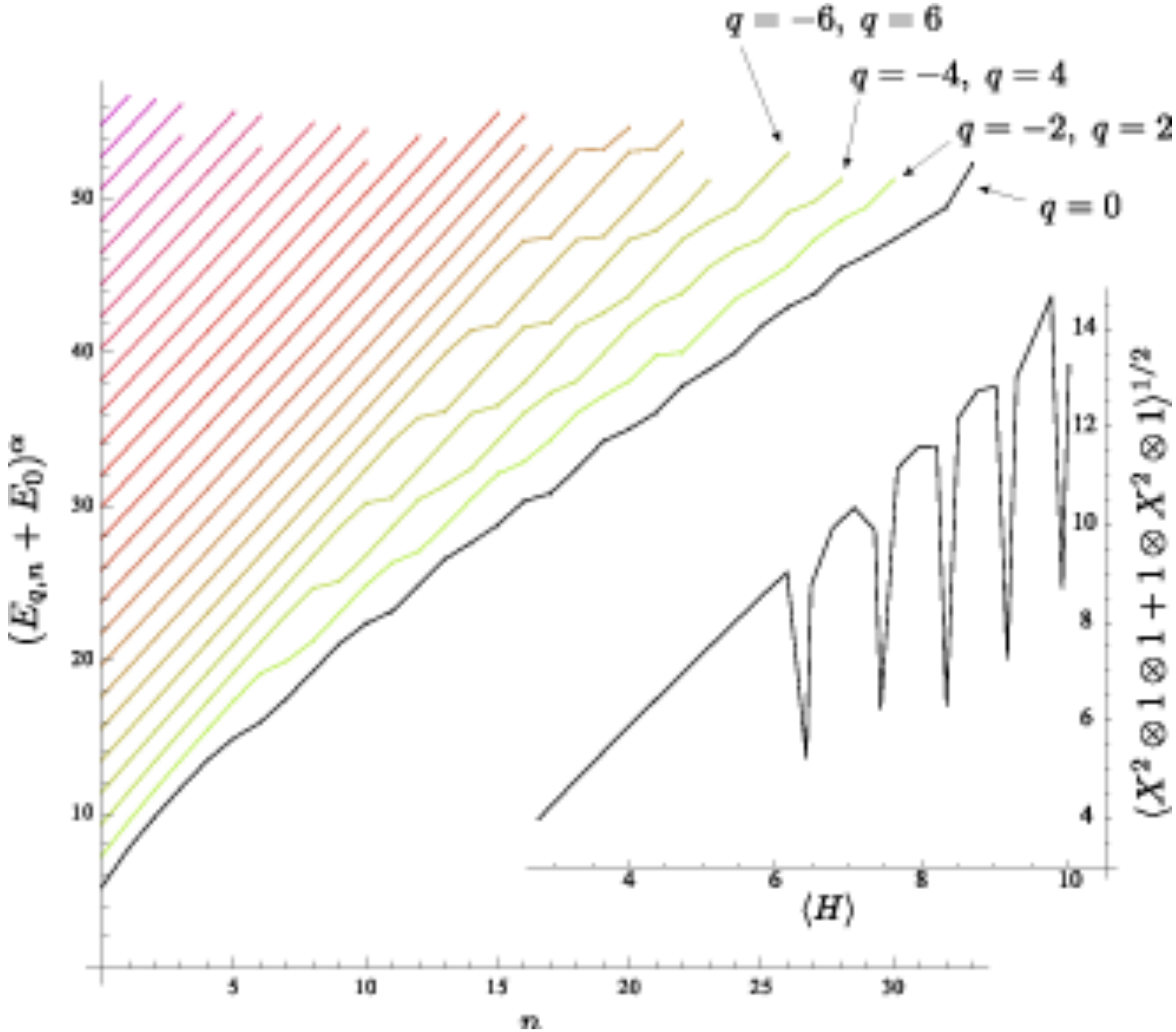
Label the states by $E_{q,n}$

- q : Eigenvalue of Q (even integers)
- n : level number in a given q sector

All the states are bound states

- No flat direction in the bosonic model

Spectrum: power law $(E+E_0)^\alpha = a + b|q| + cn$



Our result is consistent with $1.5 \leq \alpha \leq 2.3$.
Best fit: α=1.62

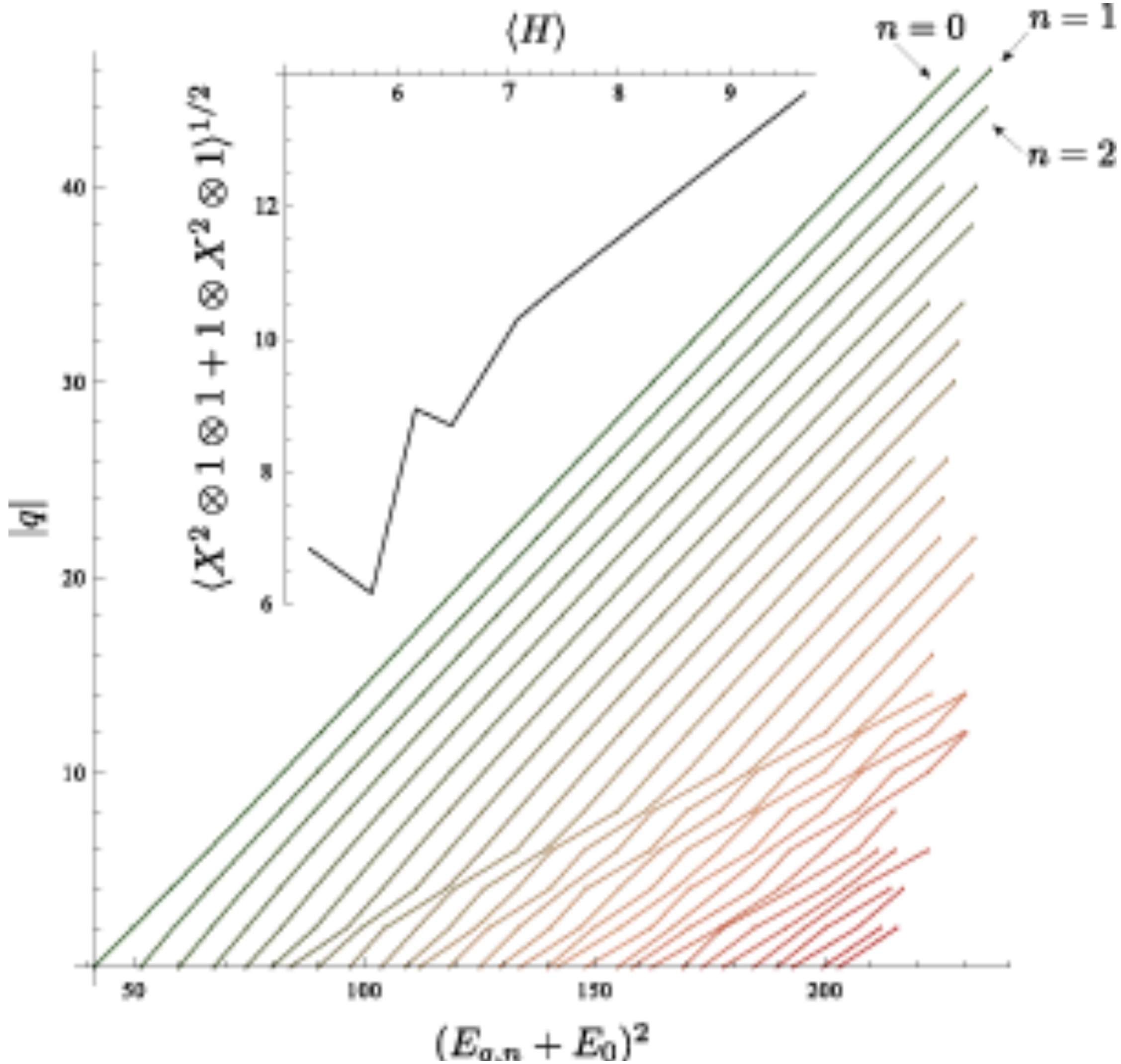
Size is mostly proportional to *E*

There are states with exceptionally small size.

q versus $(E+E_0)^2$ (Chew-Frautschi plot): Same data

- Behavior very similar to the "Regge trajectory"

Small figure:
size vs E
for n=11



Equilibration

- With the knowledge of the spectrum, we can make some exact statements about equilibration:
  1. Bound on the "effective dimension"
  2. There are states with long equilibration time
- We consider sector with a fixed conserved charge, q.

- "Equilibration": For most times, expectation values take values as if the system was in the time-averaged state ω:

$$\omega := \lim_{T\to\infty} \frac{1}{T}\int_0^T e^{-itH}\rho_0\, e^{itH} dt, \quad \rho_0 := |\psi(0)\rangle\langle\psi(0)|$$

$$= \sum_n |\langle\psi(0)|q,n\rangle|^2\, |q,n\rangle\langle q,n|$$

Effective dimension

- "Effective dimension":

$$d_{\text{eff}} := \frac{1}{\text{Tr}(\omega^2)} = \sum_n |\langle\psi(0)|n\rangle|^{-4}$$

(measure of how many pure states contribute to *ω*)

- Large d_{eff} : expectation values tend to stay close to those of the time average. [Linden, Popescu, Short, Winter, 2009]
  – For any d_S dimensional subsystem S, the expected deviation from time averaged state ω_s is bounded as:

$$\mathbb{E}\|\rho_S(t) - \omega_S\|_1 \leq \frac{d_S}{d_{\text{eff}}^{1/2}}$$

($\mathbb{E}$: expectated value; $||\ \cdot\ ||_1$: "trace norm")

Effective dimension in our model

- Fact: For random states in the energy window [E, E+Δ] (whose # of states: d_Δ), effective dimension is bounded by

$$\mathbb{E}\left(d_{\text{eff}}\right) \geq \frac{d_\Delta + 1}{2}$$

- Number of states d_Δ in our model ($1.5 \leq \alpha \leq 2.3$):

$$(E+E_0)^\alpha \leq \tilde{a} + cn \quad \Rightarrow \quad d_\Delta \geq \frac{2\Delta(E+E_0)^{\alpha-1}}{c}$$

- Thus, d_{eff} grows with E:

$$\mathbb{E}(d_{\text{eff}}) \geq \frac{\alpha\Delta(E+E_0)^{\alpha-1}}{2c} + \frac{1}{2}$$

For large initial energy, one expects strong equilibration.

Equilibration time

- There are special states for which equilibration takes arbitrarily long time.
- Consider initial state and observable (*I*=states in [*E*, *E*+Δ]):

$$|\psi(0)\rangle = \frac{1}{\sqrt{d_\Delta}} \sum_{n \in I} |n\rangle, \quad O = \sum_{i,j \in I} (\delta_{i,j-1} + \delta_{i,j+1}) |i\rangle\langle j|$$

- The difference from time-average is bounded from below:

$$\begin{aligned} \frac{|\mathrm{Tr}[O(\rho(t) - \omega)]|}{\|O\|} &\geq 1 - \frac{1}{2d_\Delta} \sum_{(i,i+1) \in I \times I} (E_{i+1} - E_i)^2 t^2 \\ &\geq 1 - \frac{c^2(d_\Delta - 1)}{2\alpha^2 d_\Delta} \frac{t^2}{(E + E_0)^{2(\alpha-1)}} \end{aligned}$$

(Equilibration time can be arbitrarily long, if *E* is large.)

Conclusions

Found spectrum of the d=2, N=2 bosonic matrix model:

- Regge-like trajectories (It is remarkable that this can be found by straightforward diagonalization.)
- Higher trajectories (not only leading ones) are seen, probably because of the low dimensionality.
- States with small size: due to non-trivial dynamics.

Derived properties about equilibration:

- Effective dimension, equilibration time scale

Our method should be applicable to higher d and N:

- Locality in the space of "angular momentum" persists.