Some geometrical aspects of entanglement in Holography & CFT (from the lattice)

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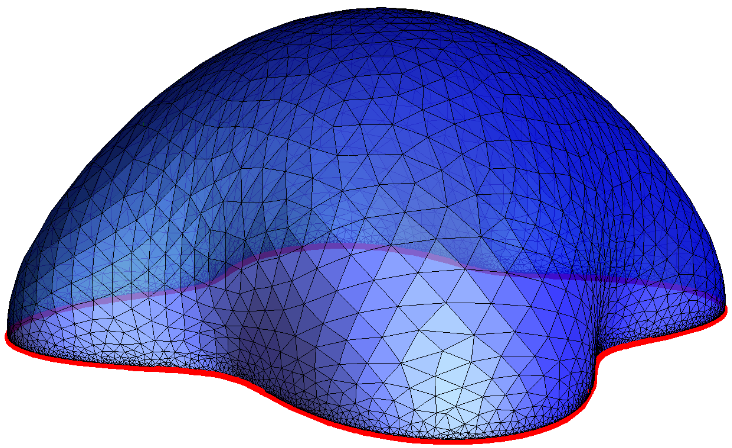
Holography and Quantum Information

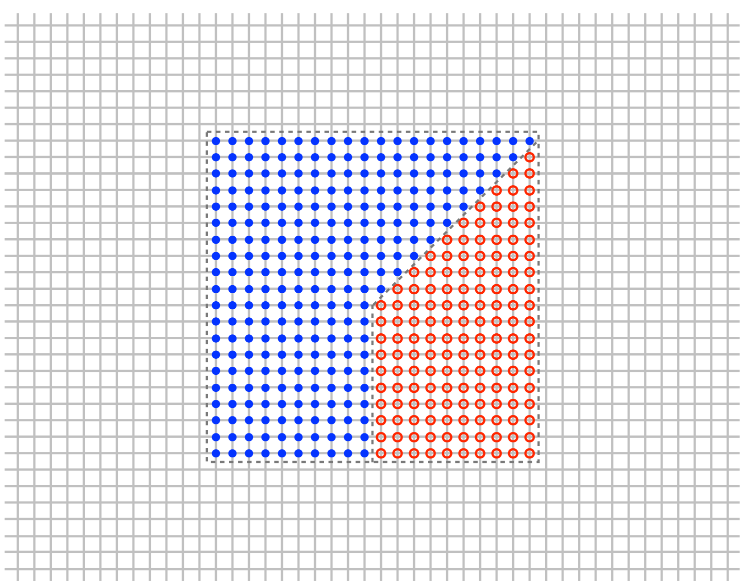
Outline



$\boxed{\text{Holographic entanglement entropy}}$

(HEE) in AdS_4/CFT_3

- Domains with arbitrary shapes



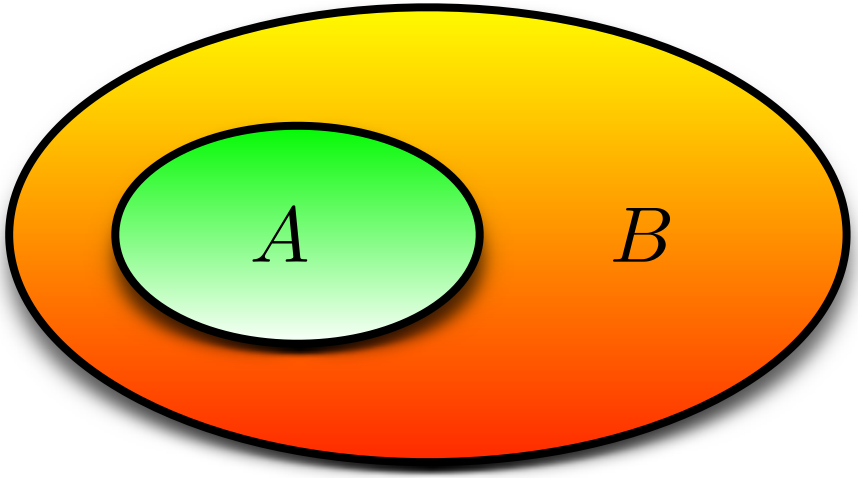
$\boxed{\text{Entanglement negativity}}$

of adjacent regions

in a 2D harmonic lattice (massless)

- Area law
- Corner contributions

Entanglement entropy



- Quantum system in a state ρ
- Bipartite Hilbert space

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

- Reduced density matrix

$$\boxed{\rho_A \equiv \mathrm{Tr}_B\, \rho}$$

- Entanglement entropy

$$\boxed{S_A = -\,\mathrm{Tr}_A(\rho_A \log \rho_A)}$$

- Pure states: $S_A = S_B$

Entanglement entropy is a measure of the bipartite entanglement

- Area law in QFT_d

Important exceptions exist
(e.g. $1+1$ CFTs)

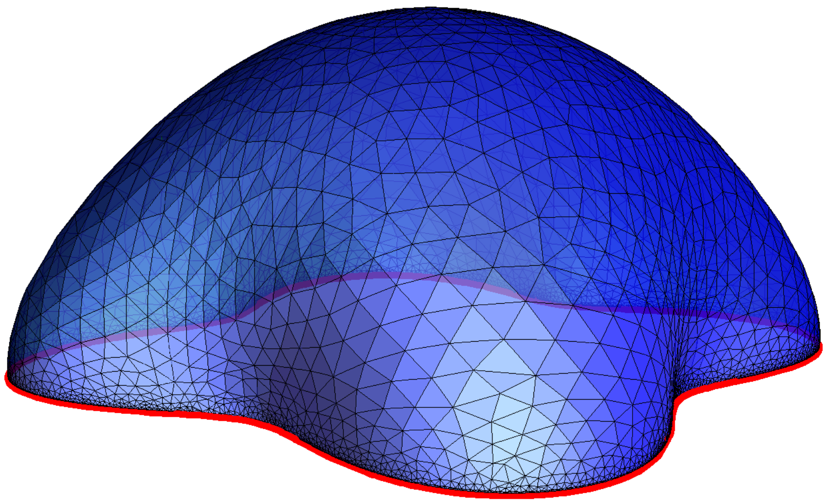
$$S_A \propto \frac{\mathrm{Area}(\partial A)}{\varepsilon^{d-2}} + \dots$$

Holographic Entanglement Entropy in AdS(4)/CFT(3)

- Constant time slice in AdS_{d+1}
  Hypersurfaces γ_A s.t. $\partial\gamma_A = \partial A$
  Find the *minimal area* surface $\hat{\gamma}_A$
  [Ryu, Takayanagi, (2006)]

$$\boxed{S_A = \frac{\text{Area}(\hat{\gamma}_\varepsilon)}{4G_N^{(d+1)}}}$$

- Holographic dual of Wilson loops [Maldacena, (1998)]
- Expansion of the area as $\varepsilon \to 0$
  [Graham, Witten, (1999)]
- E.g.: AdS_4 $\quad ds^2 = \frac{1}{z^2}\big(-dt^2 + dz^2 + d\boldsymbol{x}^2\big)$
- Asymptotically AdS_4 geometries



$$\boxed{\mathcal{A}[\hat{\gamma}_\varepsilon] = \frac{P_A}{\varepsilon} - F_A + o(1)}$$

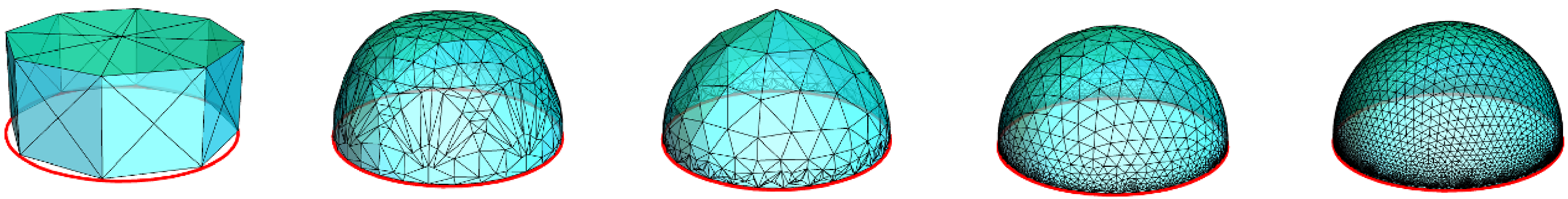
- Various non trivial checks. E.g. strong subadditivity [Headrick, Takayanagi, (2007)]
- Simply connected domains analytically solved: spheres and infinite strips
- Domains A obtained as small perturbations of the sphere
  [Hubeny, (2012)] [Klebanov, Nishioka, Pufu, Safdi, (2012)] [Allais, Mezei, (2014)]

HEE in AdS(4) with Surface Evolver

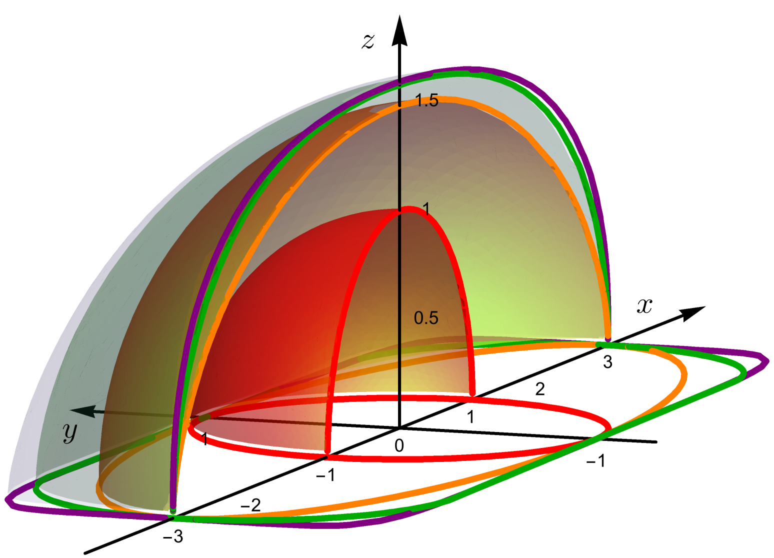
- Generic shape for ∂A [Fonda, Giomi, Salvio, E.T., (2014)] [Fonda, Seminara, E.T., (2015)]

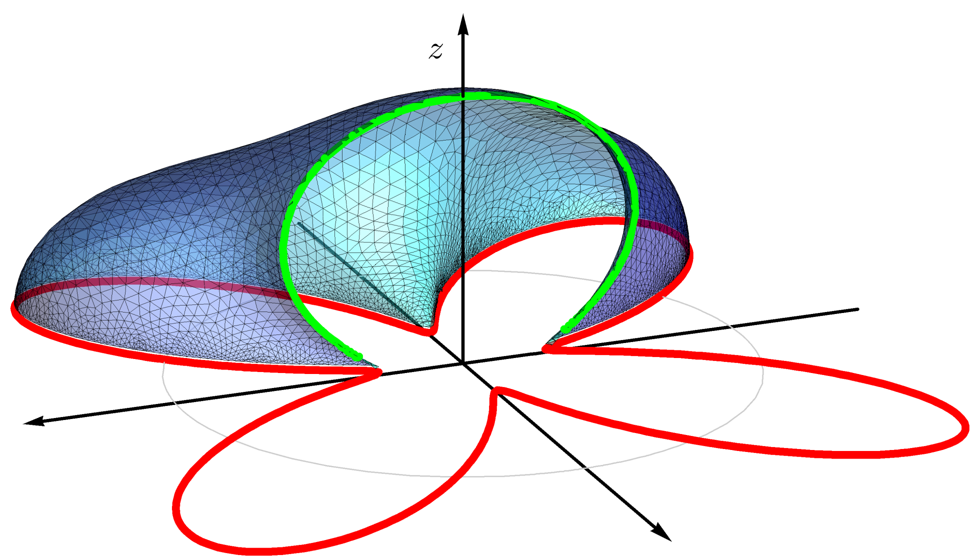
  Numerical analysis based on *Surface Evolver* (developed by Ken Brakke)

  E.g.: when A is a disk



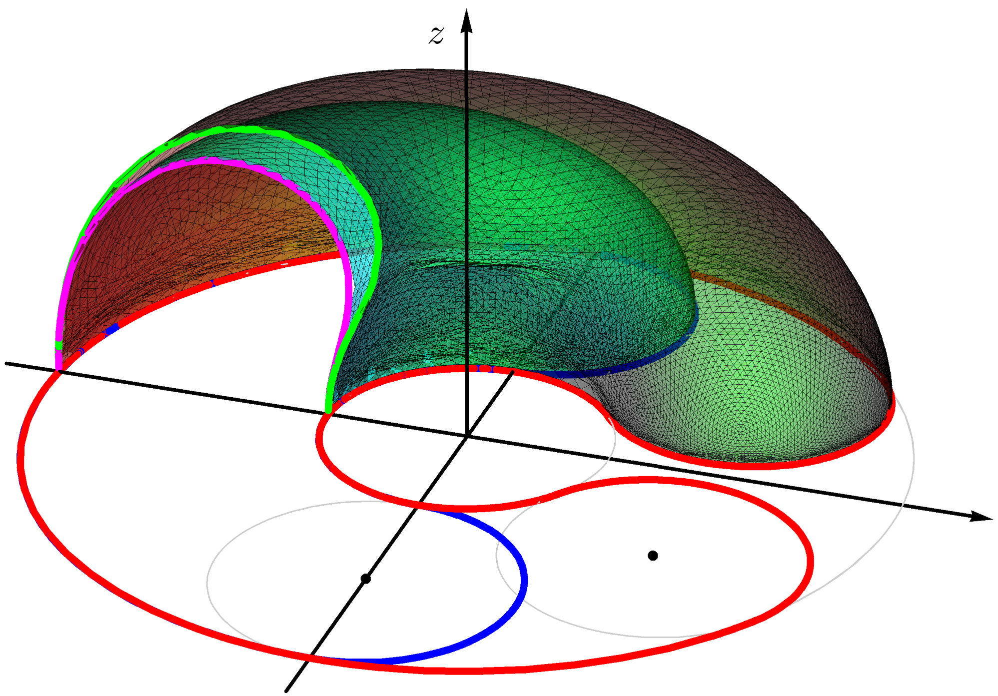
- Domains with generic boundaries can be studied

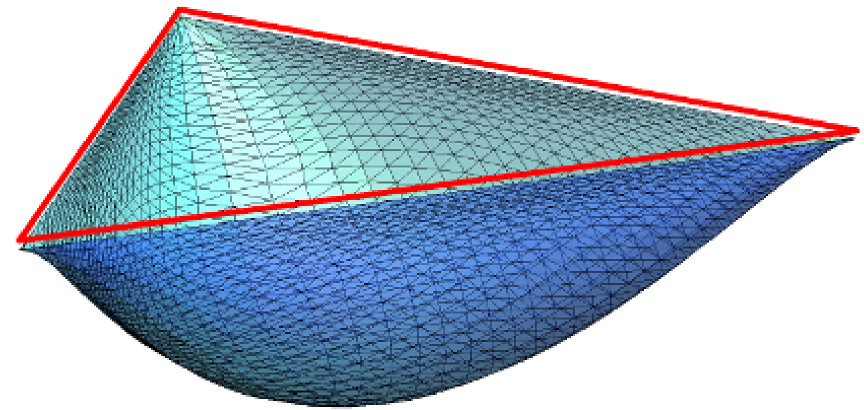




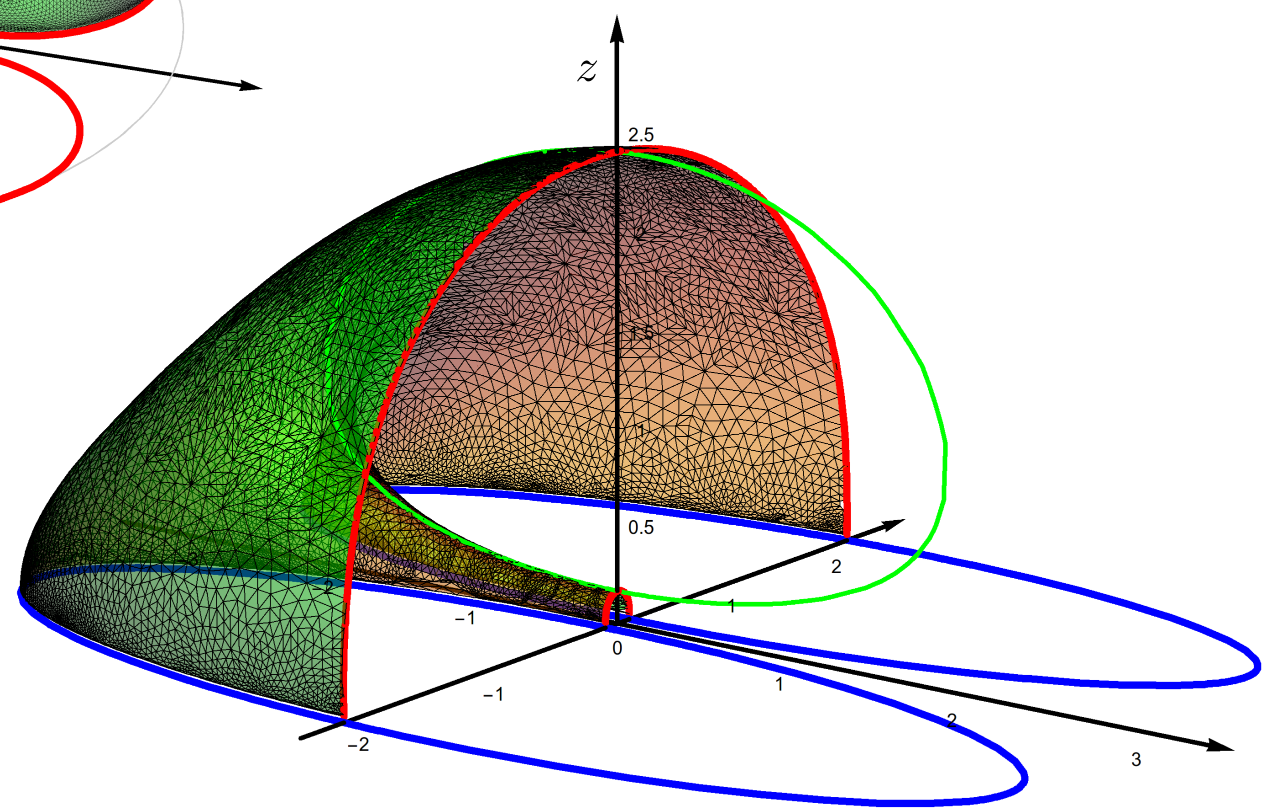
Minimal area surfaces in AdS(4)

- Pathwise connected domains A
  (also with non smooth ∂A)



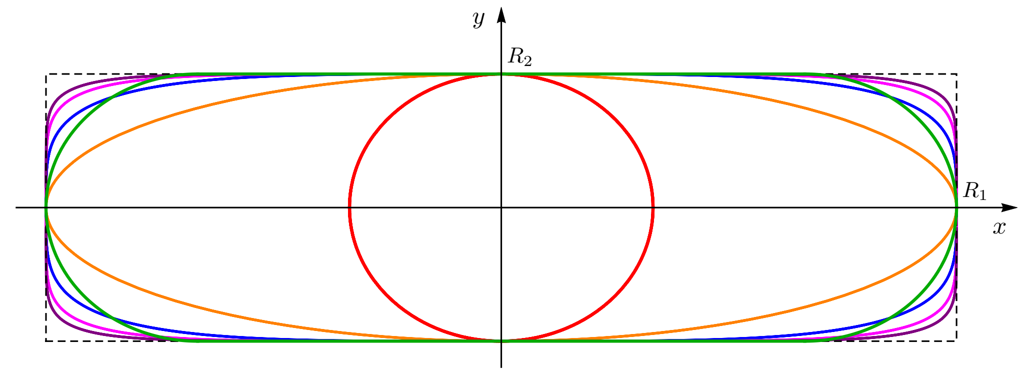


- Disjoint regions
  $(A = A_1 \cup A_2)$

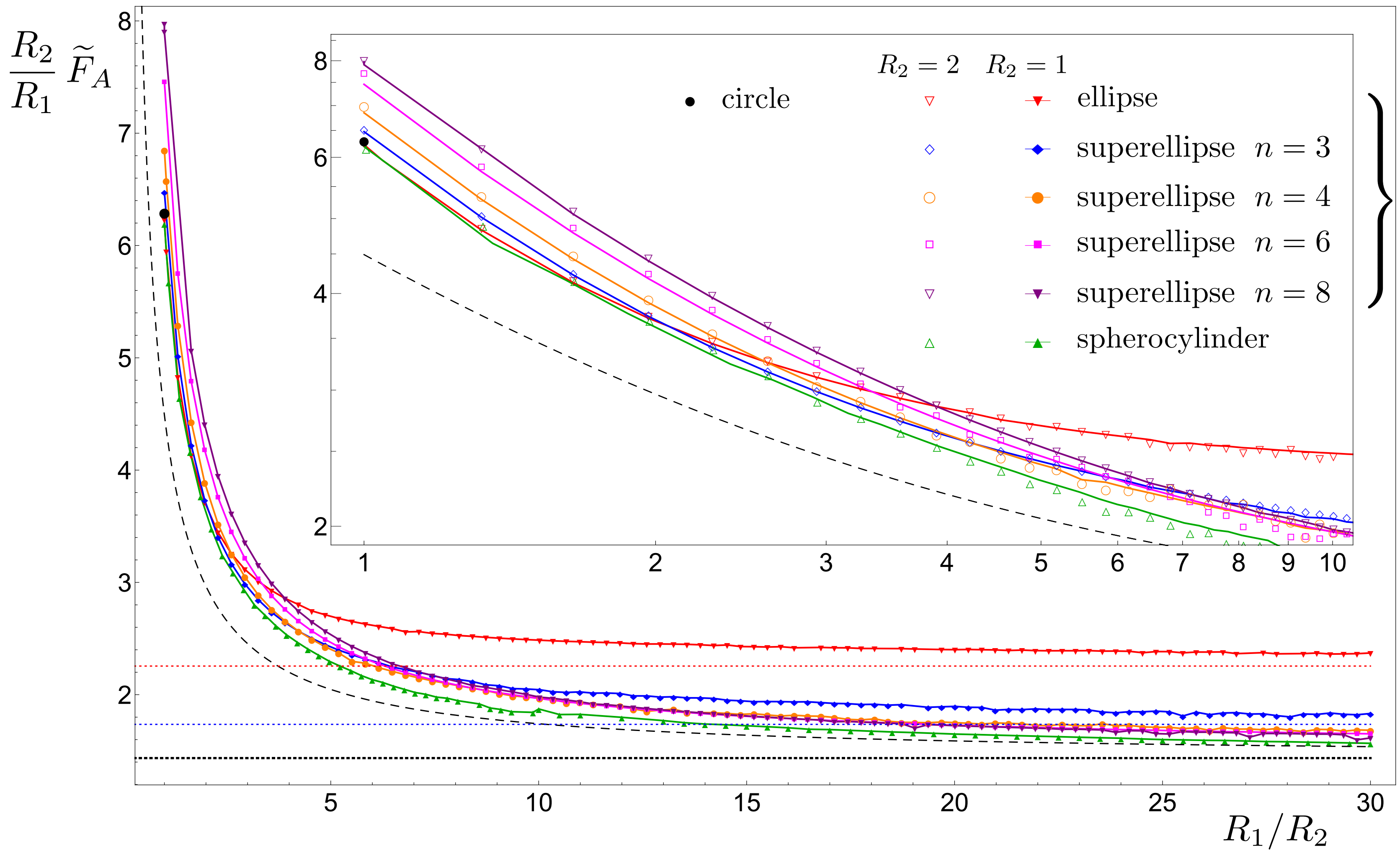


HEE in AdS(4). From the disk to the infinite strip

$$\boxed{\mathcal{A}_A = \frac{P_A}{\varepsilon} - F_A + o(1) \equiv \frac{P_A}{\varepsilon} - \widetilde{F}_A}$$



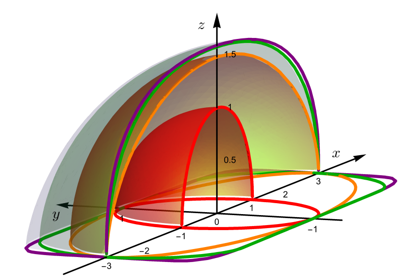
[Fonda, Giomi, Salvio, E.T., (2014)]



Superellipses:

$$\frac{|x|^n}{R_1^n} + \frac{|y|^n}{R_2^n} = 1$$

squircles: $R_1 = R_2$



HEE in AdS(4) & Willmore energy

- Willmore energy of a closed smooth surface $\Sigma_g \subset \mathbb{R}^3$

$$\boxed{\mathcal{W}[\Sigma_g] \equiv \frac{1}{4}\int_{\Sigma_g} \left(\mathrm{Tr}\widetilde{K}\right)^2 d\tilde{A}}$$

[Willmore, (1965)]

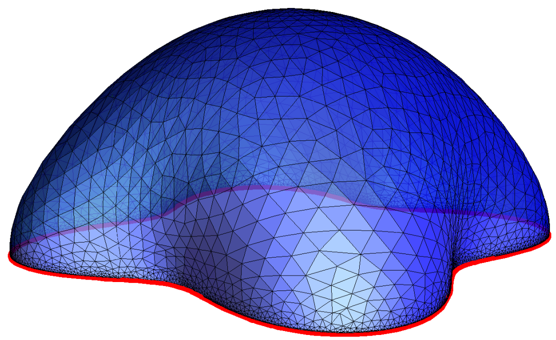
- Minimal area surface $\hat{\gamma}_A \subset \mathbb{H}^3$ has $\mathrm{Tr}K = 0$

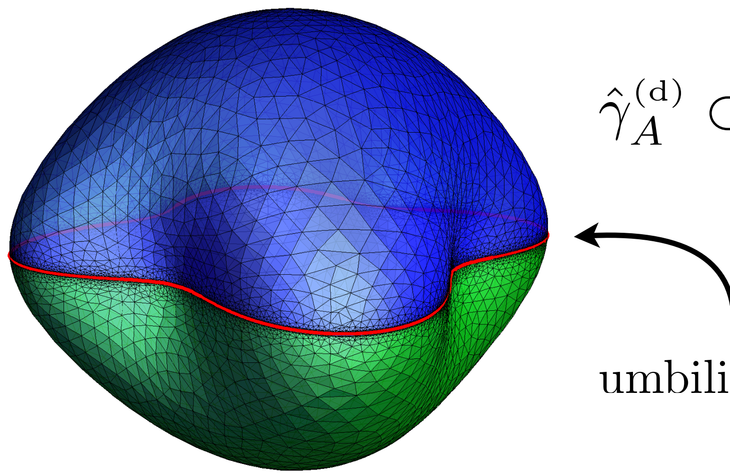
Consider $\hat{\gamma}_A \subset \mathbb{R}^3$

[Babich, Bobenko, (1993)]
[Alexakis, Mazzeo, (2010)]

$$\boxed{F_A = \mathcal{W}[\hat{\gamma}_A] = \int_{\hat{\gamma}_A} \frac{(\tilde{n}^z)^2}{z^2}\, d\tilde{A} = \frac{1}{2}\mathcal{W}[\hat{\gamma}_A^{(\mathrm{d})}]}$$

$\hat{\gamma}_A \subset \mathbb{R}^3$





$\hat{\gamma}_A^{(\mathrm{d})} \subset \mathbb{R}^3$

umbilic line

- Since $\mathcal{W}[\Sigma_g] \geqslant 4\pi$ (saturated only by round spheres) [Willmore, (1965)]

HEE is maximised by the disk for a given perimeter P_A, i.e. $F_A \geqslant 2\pi$

HEE in asymptotically AdS(4) static spacetimes

[Fonda, Seminara, E.T., (2015)]

- Take $ds^2|_{t=\text{const}} = g_{\mu\nu}\, dx^\mu dx^\nu$ with $g_{\mu\nu} = e^{2\varphi}\, \tilde{g}_{\mu\nu}$ and $\varphi = -\log(z) + \dots$

  The metric $\tilde{g}_{\mu\nu}$ is asymptotically flat as $z \to 0$

  - $\hat{\gamma}_A$ extremal area surface

$$\mathrm{Tr} K = 0 \qquad \Longleftrightarrow \qquad \boxed{\big(\mathrm{Tr}\widetilde{K}\big)^2 = 4(\tilde{n}^\lambda \partial_\lambda \varphi)^2}$$

The unit vector $\tilde{n}^\mu$ is normal to $\hat{\gamma}_A \subset \tilde{\mathcal{M}}_3$ (defined by $\tilde{g}_{\mu\nu}$)

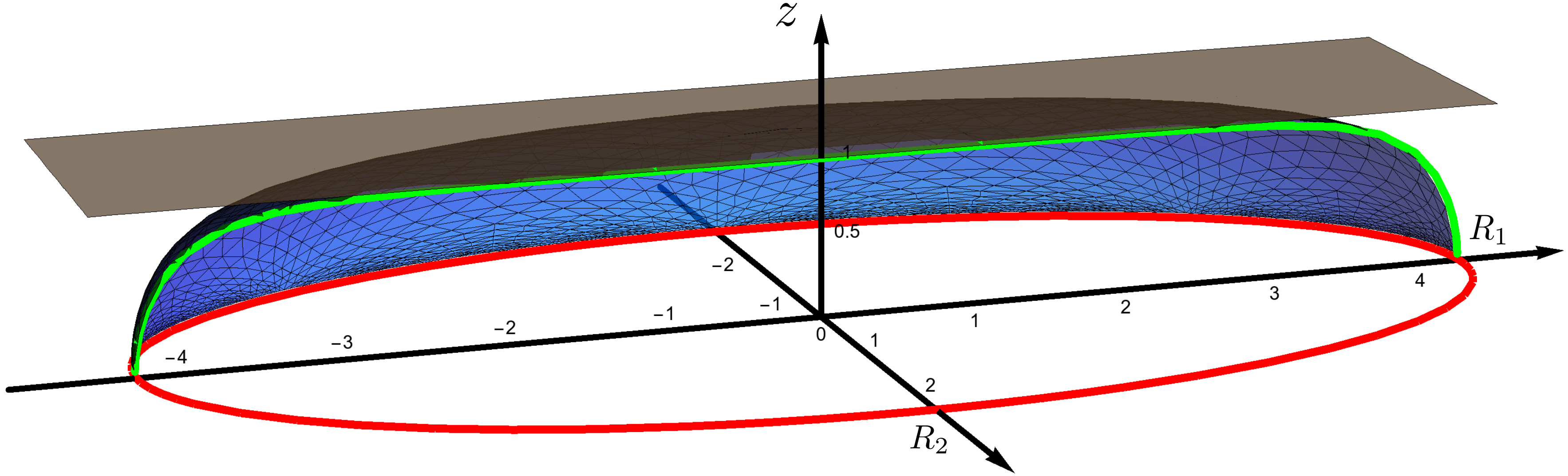
- Generalising the result for AdS$_4$, one finds

$$\boxed{F_A = \int_{\hat{\gamma}_A} \left[\frac{1}{2}\big(\mathrm{Tr}\widetilde{K}\big)^2 + \widetilde{\nabla}^2\varphi - e^{2\varphi} - \tilde{n}^\mu \tilde{n}^\nu\, \widetilde{\nabla}_\mu \widetilde{\nabla}_\nu \varphi \right] d\tilde{\mathcal{A}}}$$

- AdS$_4$: the formula involving the Willmore energy is recovered

HEE in asymptotically AdS(4) black holes

- $$ds^2 = \frac{1}{z^2}\left(-f(z)\,dt^2 + \frac{dz^2}{f(z)} + d\boldsymbol{x}^2\right) \qquad\qquad f(z) = 1 - Mz^3 + Q^2 z^4$$



$$F_A = \int_{\hat{\gamma}_A} \frac{1}{z^2}\left[\left(1 + \frac{z f'(z)}{2f(z)}\right)(\tilde{n}^z)^2 + f(z) - \frac{z f'(z)}{2} - 1\right] d\tilde{\mathcal{A}}$$

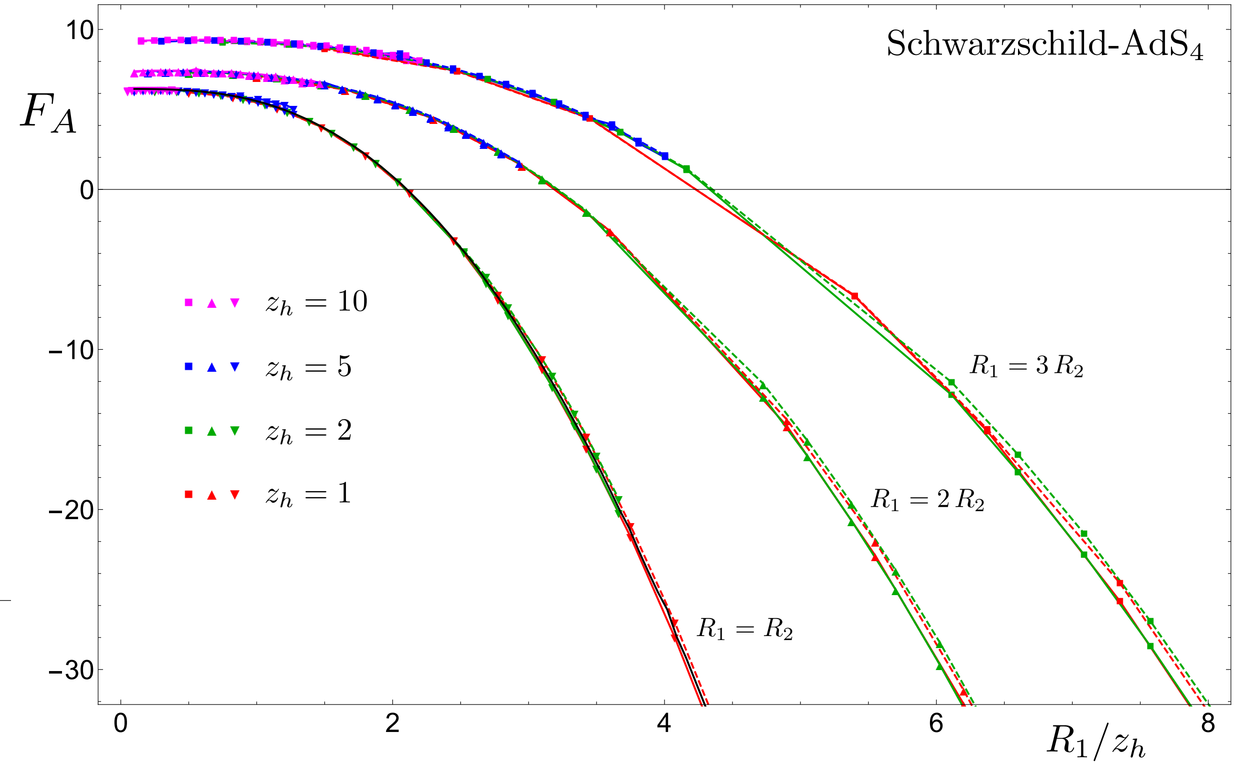
- Large domains A: the highest value of z on $\hat{\gamma}_A$ is $z_* \lesssim z_h$

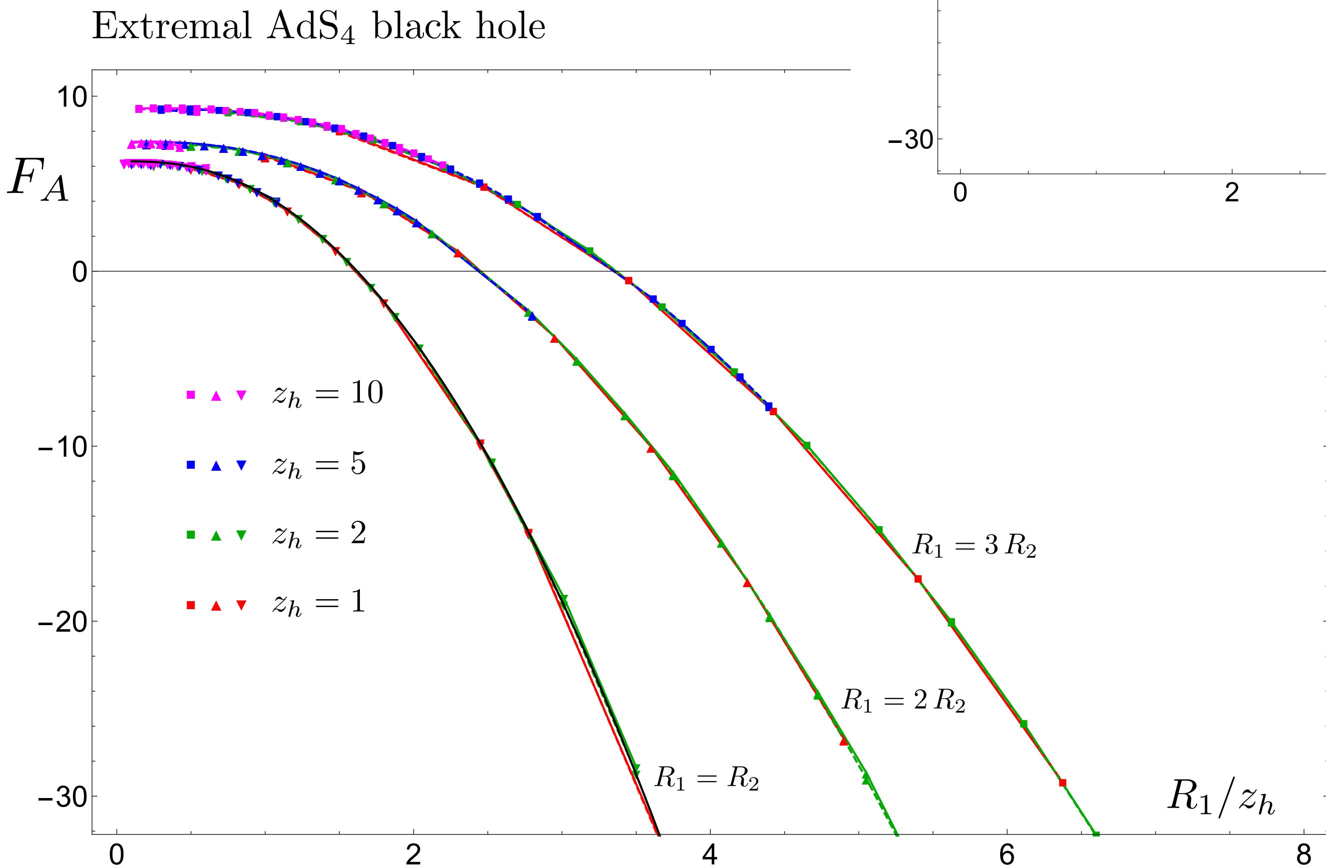
  $F_A \simeq F_A^{\text{cyl}}$, i.e. F_A evaluated on the cylinder with $0 \leqslant z \leqslant z_*$ built on ∂A

  $\Longrightarrow \quad F_A^{\text{cyl}} = -\,\text{Area}(A)/z_h^2 + \dots$

HEE in asymptotically AdS(4) black holes. Ellipses

- Domains A
  delimited by ellipses





HEE in asymptotically AdS(4) domain wall geometries

- $$ds^2 = \frac{1}{z^2}\left(\frac{-\,dt^2 + d\boldsymbol{x}^2}{p(z)} + dz^2\right) \qquad p(z) = \big[1 + (z/z_{\mathrm{RG}})^{\alpha}\big]^{2\gamma} \qquad \begin{array}{l} \alpha > 0 \\ \gamma > 0 \end{array}$$

Holographic RG flow

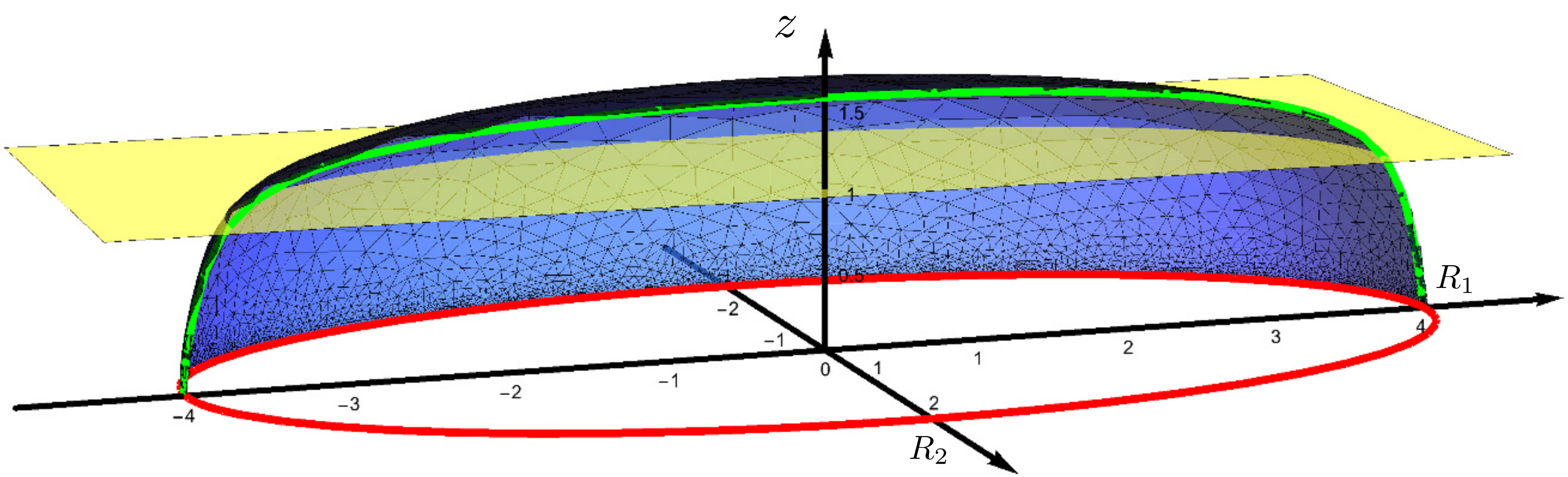
$z/z_{\mathrm{RG}} \ll 1$ UV regime: AdS_4 with $L_{\mathrm{UV}} = 1$

$z/z_{\mathrm{RG}} \gg 1$ IR regime: AdS_4 with $L_{\mathrm{IR}} = 1/(1+\gamma\alpha) < L_{\mathrm{UV}}$

[Freedman, Gubser, Pilch, Warner, (1999)] [Girardello, Petrini, Porrati, Zaffaroni, (1998); (1999)]

HEE: see [Myers, Sinha, (2010)] [Albash, Johnson, (2010)] [Myers, Singh, (2012)] [Liu, Mezei, (2012)]

- Generic shapes [Fonda, Seminara, E.T., (2015)]



$$F_A = \int_{\hat{\gamma}_A} \frac{1}{z^2}\left[\left(1 + \frac{z\,p'(z)}{2\,p(z)}\right)(\tilde{n}^z)^2 + \frac{z\,p'(z)}{2\,p(z)}\right] d\tilde{\mathcal{A}}$$

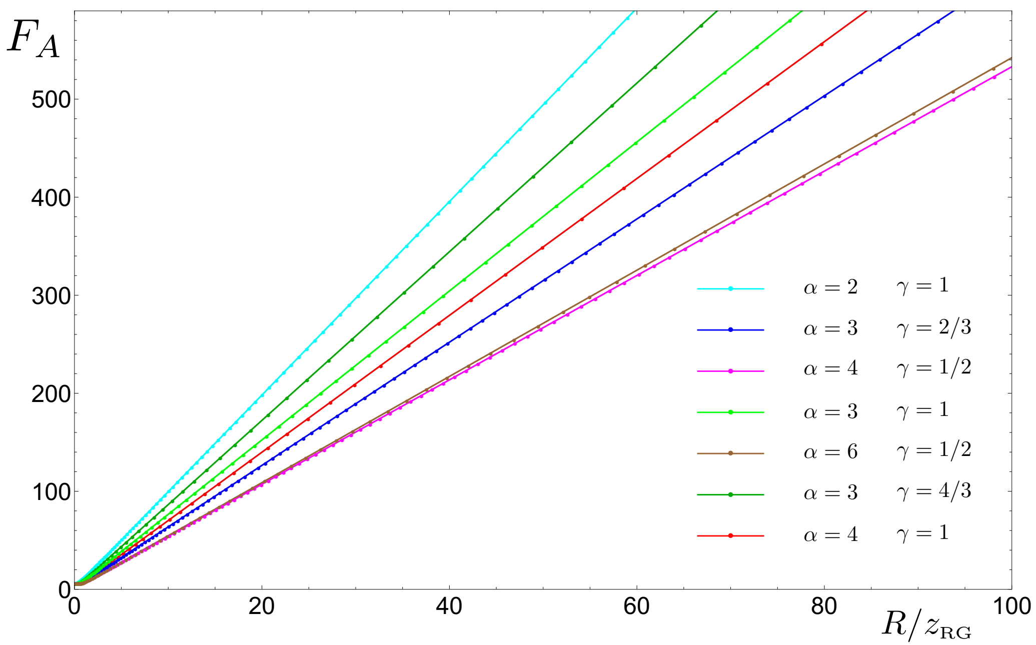
Domain wall geometries: disk & F-theorem

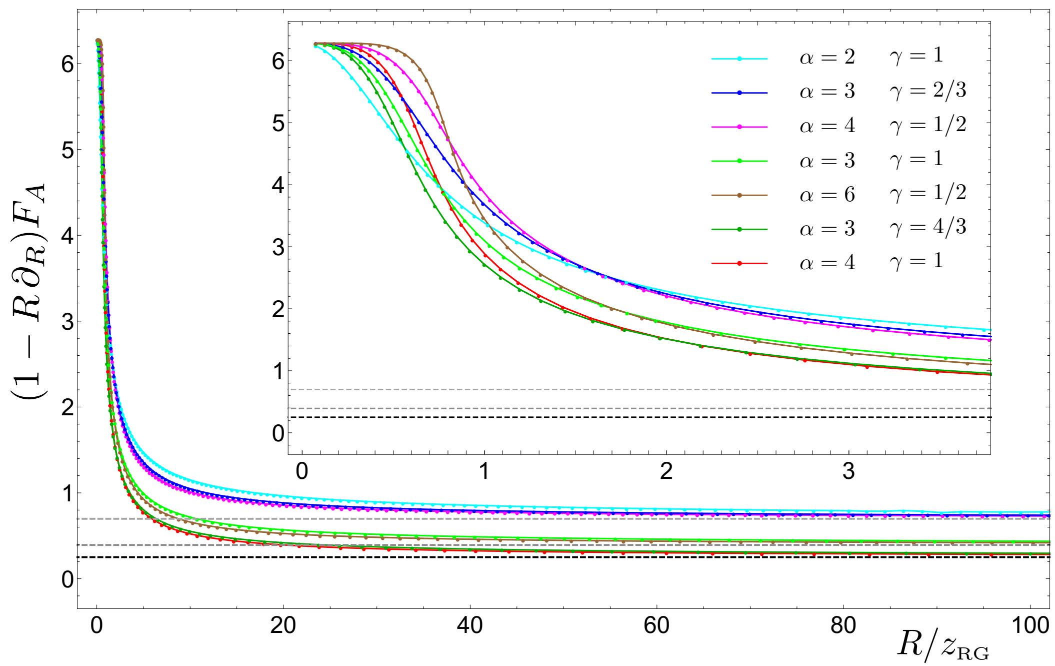
- In $2+1$ dimensions, when A is a disk

[Jafferis, Klebanov, Pufu, Safdi, (2011)]
[Casini, Huerta, (2012)] [Liu, Mezei, (2012)]
[Myers, Sinha, (2010)]

$$\boxed{C \equiv (R\,\partial_R - 1) S_A} \qquad C_{\text{UV}} \geqslant C_{\text{IR}}$$

- Domain wall geometries: [Myers, Singh, (2012)] [Liu, Mezei, (2012)] [Fonda, Seminara, E.T., (2015)]



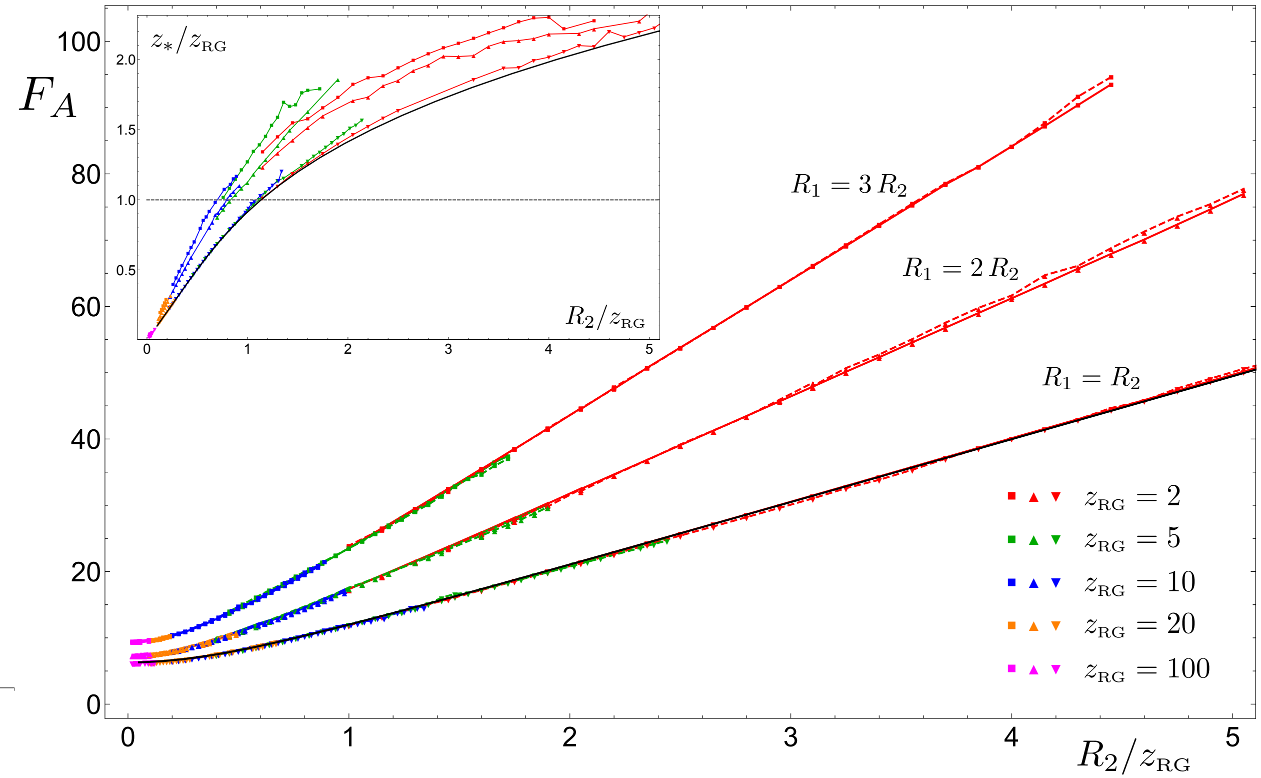


$$4G_N\, C_{\text{UV}} = 2\pi \qquad 4G_N\, C_{\text{IR}} = \frac{2\pi}{(1+\alpha\gamma)^2} < 2\pi$$

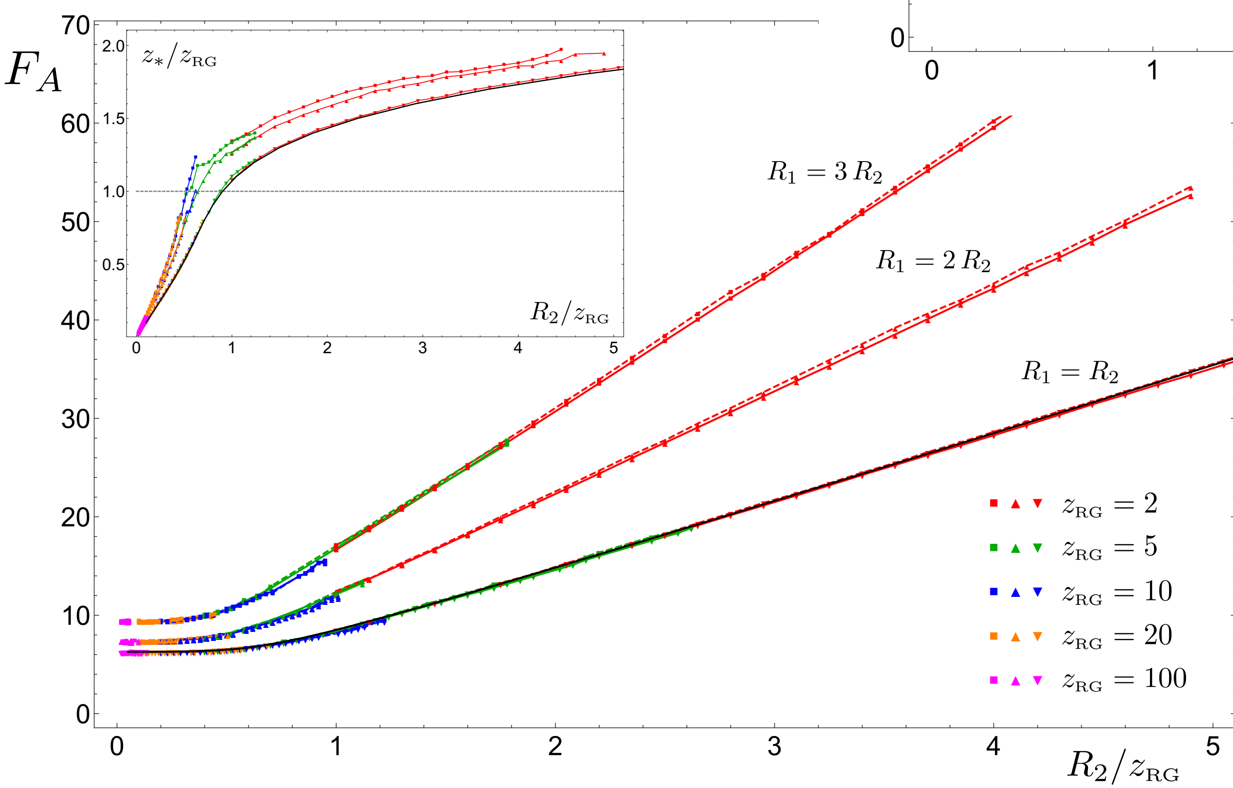
The slope of F_A in the IR depends on α and γ separately

Domain wall geometries: ellipses

- Domains A delimited by ellipses

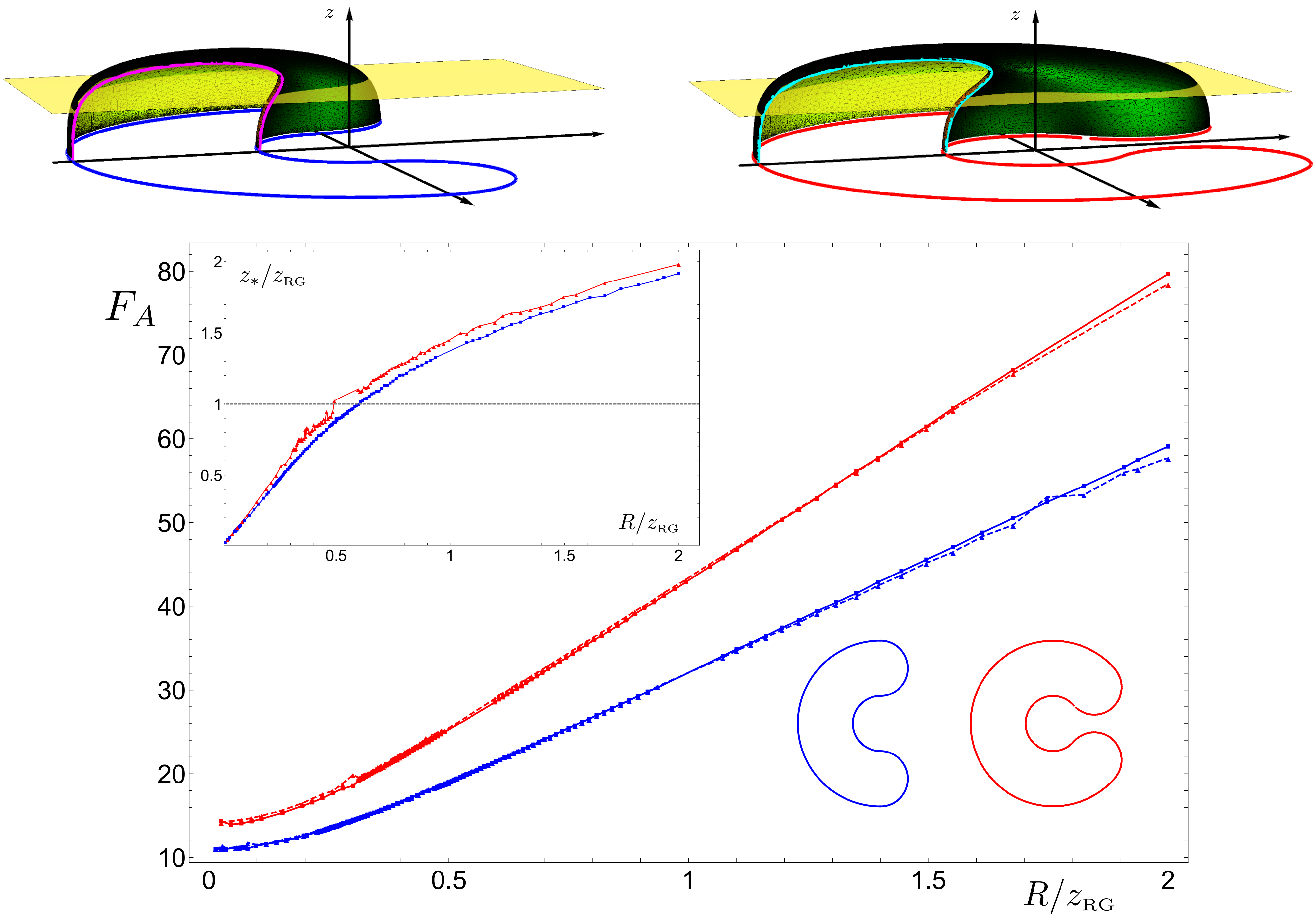


$\alpha = 2$
$\gamma = 1$



$\alpha = 4$
$\gamma = 1$

Domain wall geometries: other smooth domains

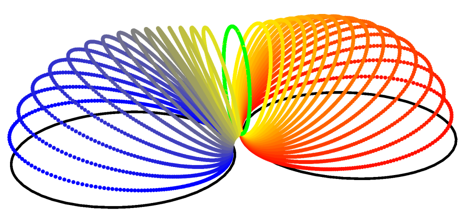


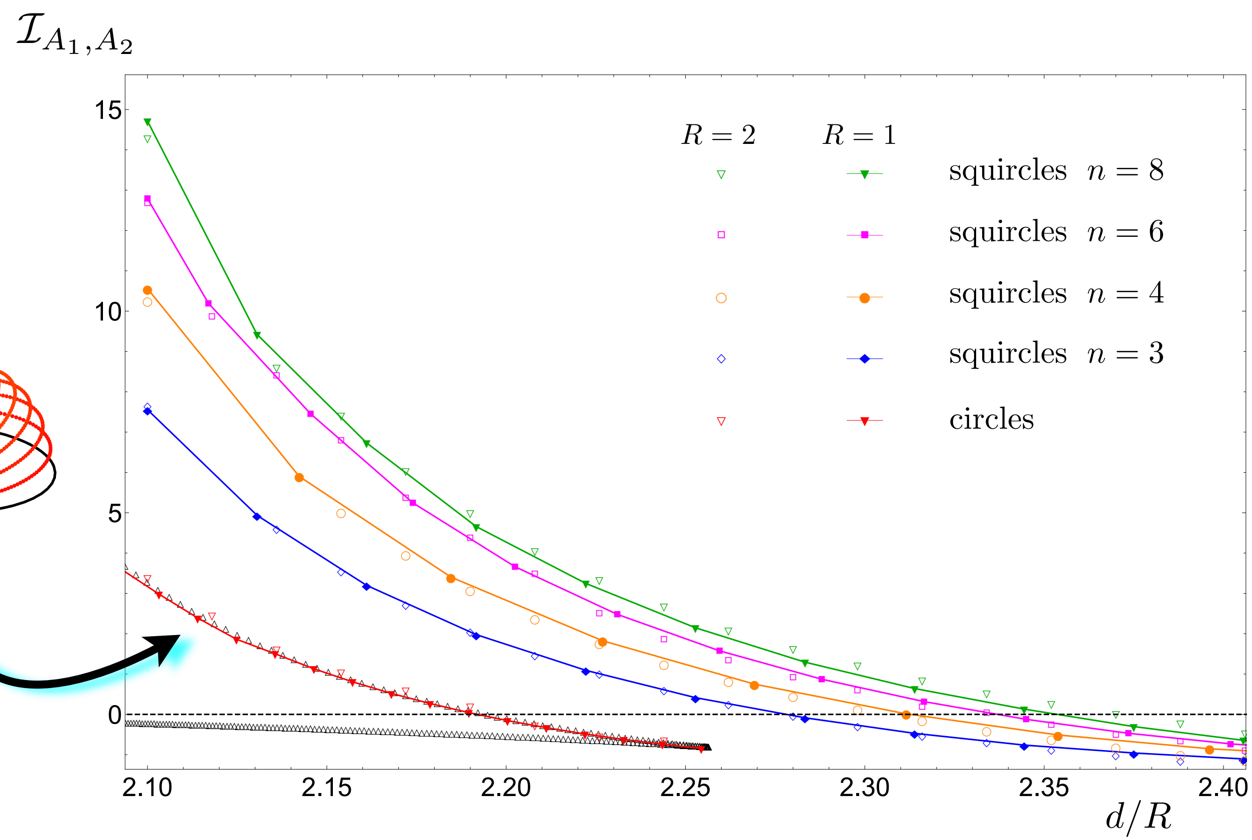
Holographic mutual information in AdS(4)

$$\boxed{I_{A_1,A_2} \equiv S_{A_1} + S_{A_2} - S_{A_1 \cup A_2} \equiv \frac{\mathcal{I}_{A_1,A_2}}{4G_N}} \qquad \mathcal{I}_{A_1,A_2} = F_{A_1 \cup A_2} - F_{A_1} - F_{A_2} + o(1)$$

- Beyond a critical distance $\mathcal{I}_{A_1,A_2} = 0$ and the disconnected configuration is the minimal one

[Gross, Ooguri, (1998)] [Zarembo, (1999)]
[Drukker, Fiol, (2005)]
[Fonda, Giomi, Salvio, E.T., (2014)]





- The Clifford torus minimises the Willmore energy among the genus one surfaces: $\mathcal{W}[\Sigma_1] \geqslant 2\pi^2$ [Willmore, (1965)] [Marques, Neves, (2012)]

→ It cannot be found in this holographic context [Fonda, Seminara, E.T., (2015)]

Time dependent backgrounds & Vaidya-AdS metrics

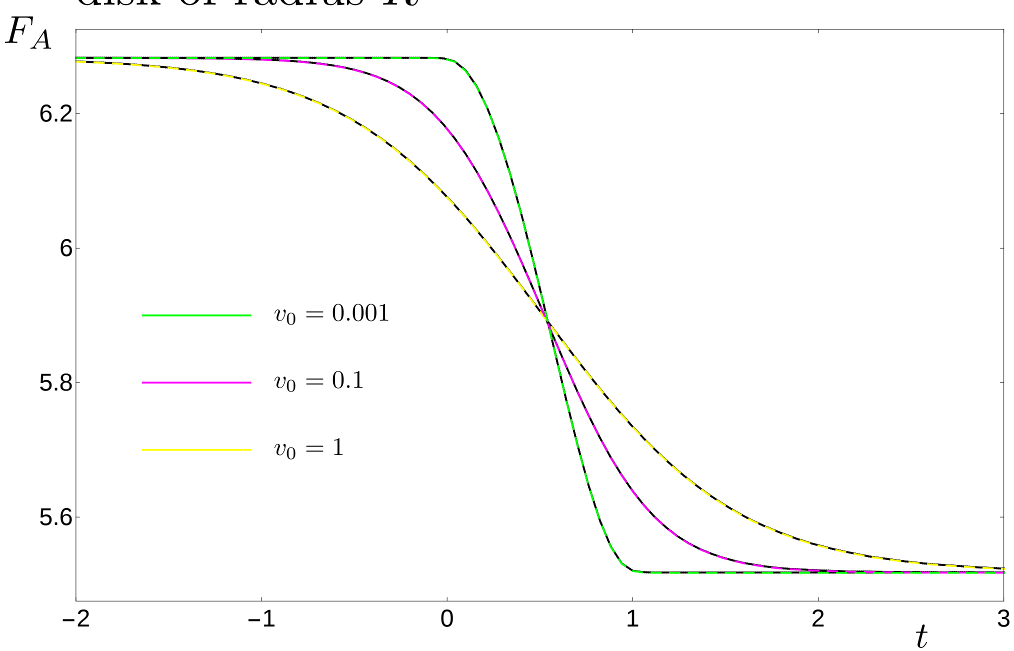
- HEE for time dependent metrics [Hubeny, Rangamani, Takayanagi, (2007)] [many others]
- The previous analysis can be generalised [Fonda, Seminara, E.T., (2015)]

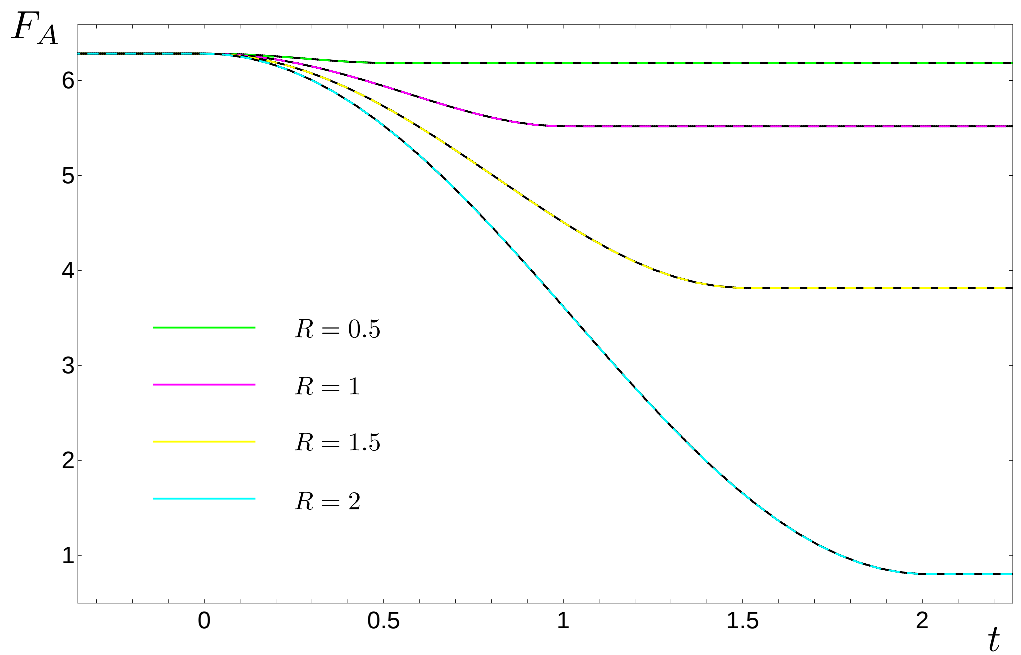
$$\begin{aligned}F_A &= \int_{\hat\gamma_A}\left(\frac{1}{2}\sum_{i=1}^{2}\epsilon_i\left(\mathrm{Tr}\widetilde K^{(i)}\right)^2+\widetilde D^2\varphi-e^{2\varphi}-\sum_{i=1}^{2}\epsilon_i\,\tilde n^{(i)M}\tilde n^{(i)N}\widetilde D_M\widetilde D_N\varphi\right)d\tilde{\mathcal A}\\ &= \int_{\hat\gamma_A}\left(\frac{1}{4}\sum_{i=1}^{2}\epsilon_i\left(\mathrm{Tr}\widetilde K^{(i)}\right)^2-\frac{1}{2}\sum_{i=1}^{2}\epsilon_i\,\widetilde G(\tilde n^{(i)},\tilde n^{(i)})-\frac{1}{6}\,\widetilde R\right)d\tilde{\mathcal A}+\int_{\hat\gamma_A}\left(\frac{1}{2}\sum_{i=1}^{2}\epsilon_i\,T(n^{(i)},n^{(i)})-\frac{1}{6}\,T\right)d\mathcal A\end{aligned}$$

- Vaidya-AdS$_4$ $\quad ds^2=\dfrac{1}{z^2}\Big(-f(v,z)\,dv^2-2\,dv\,dz+d\boldsymbol{x}^2\Big)$

$$f(v,z)=1-M(v)z^3$$
$$M(v)=\frac{M}{2}\left(1+\tanh(v/v_0)\right)$$

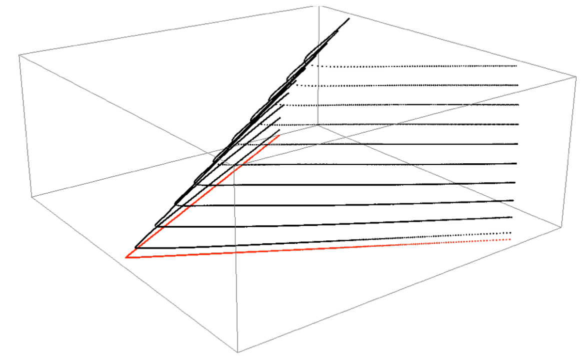
disk of radius R





HEE in AdS(4). Polygons (I)

- Infinite wedge with opening angle α ($|\phi| \leqslant \alpha/2$)
  [Drukker, Gross, Ooguri, (1999)] [Hirata, Takayanagi, (2006)]

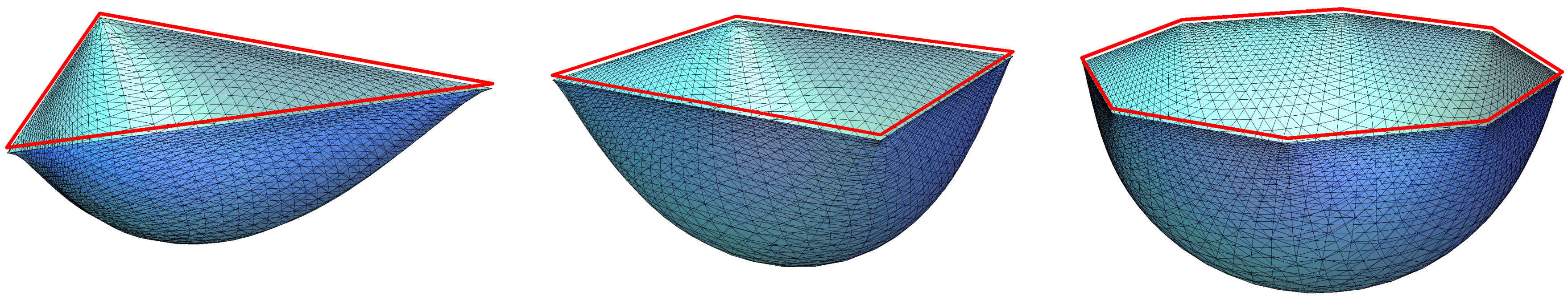


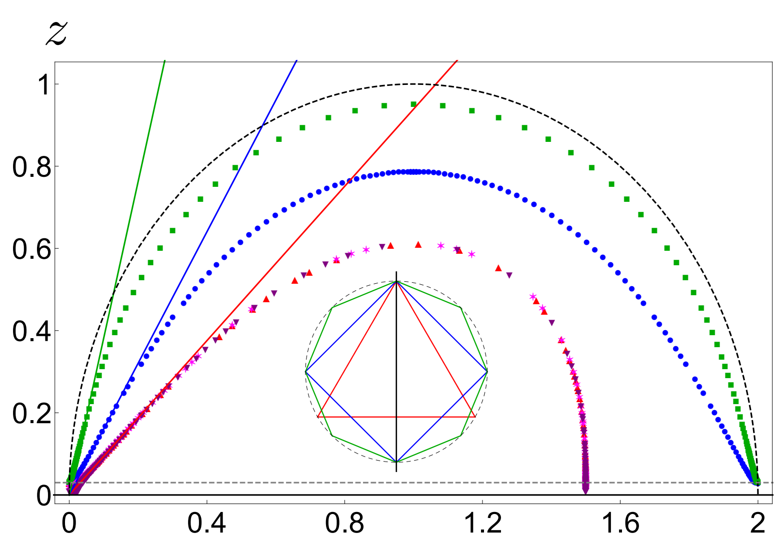
$$\boxed{z = \frac{\rho}{f(\phi)}}$$

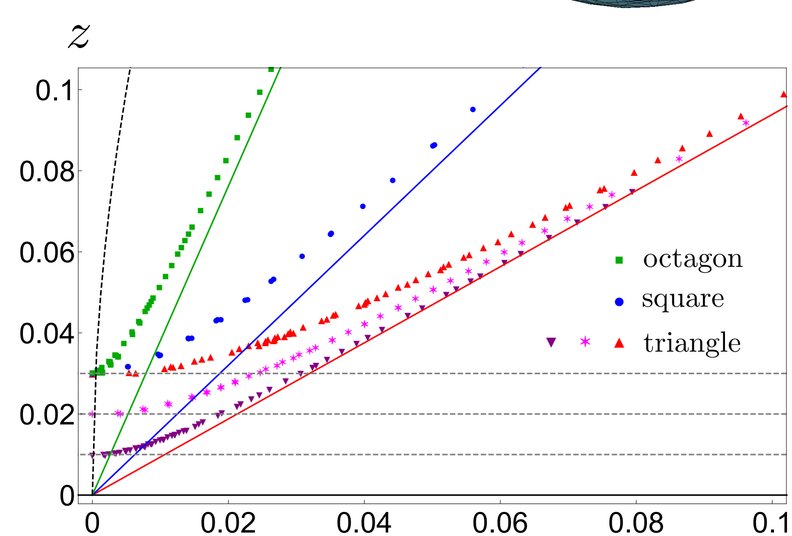
$$\phi = \int_{f_0}^{f} \frac{1}{\zeta}\left[(\zeta^2+1)\left(\frac{\zeta^2(\zeta^2+1)}{f_0^2(f_0^2+1)} - 1\right)\right]^{-\frac{1}{2}} d\zeta \qquad f_0 \equiv f(0)$$

$$f \to \infty \quad \text{then} \quad \phi \to \alpha/2$$

- Minimal surfaces anchored on finite polygons can be studied numerically [Fonda, Giomi, Salvio, E.T., (2014)]







HEE in AdS(4). Polygons (II)

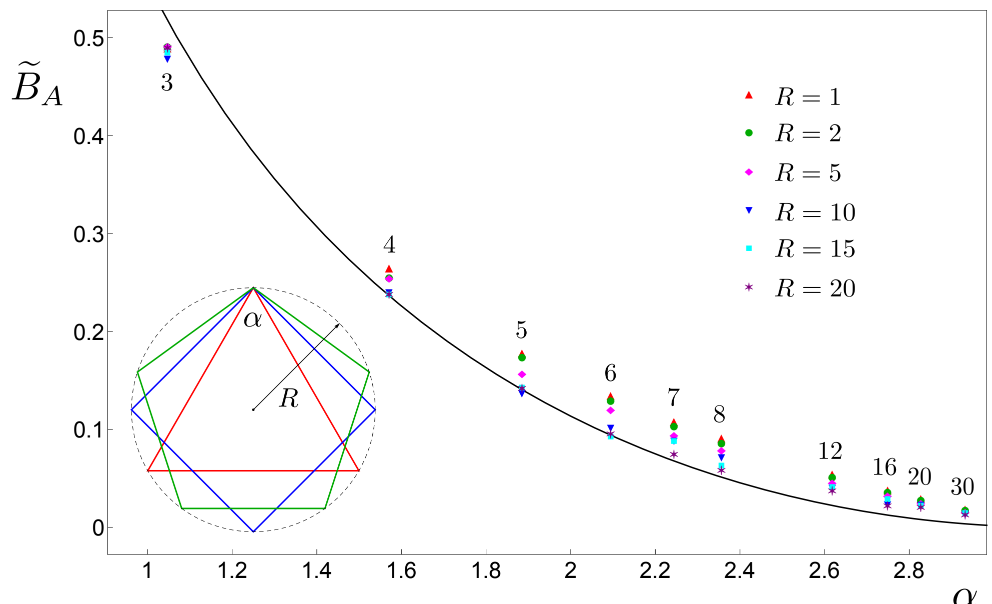
- Area of the minimal surfaces anchored on polygons [Drukker, Gross, Ooguri, (1999)]

$$\boxed{\mathcal{A}_A = \frac{P_A}{\varepsilon} - B_A \log(P_A/\varepsilon) - W_A + o(1) \equiv \frac{P_A}{\varepsilon} - \widetilde{B}_A \log(P_A/\varepsilon)}$$

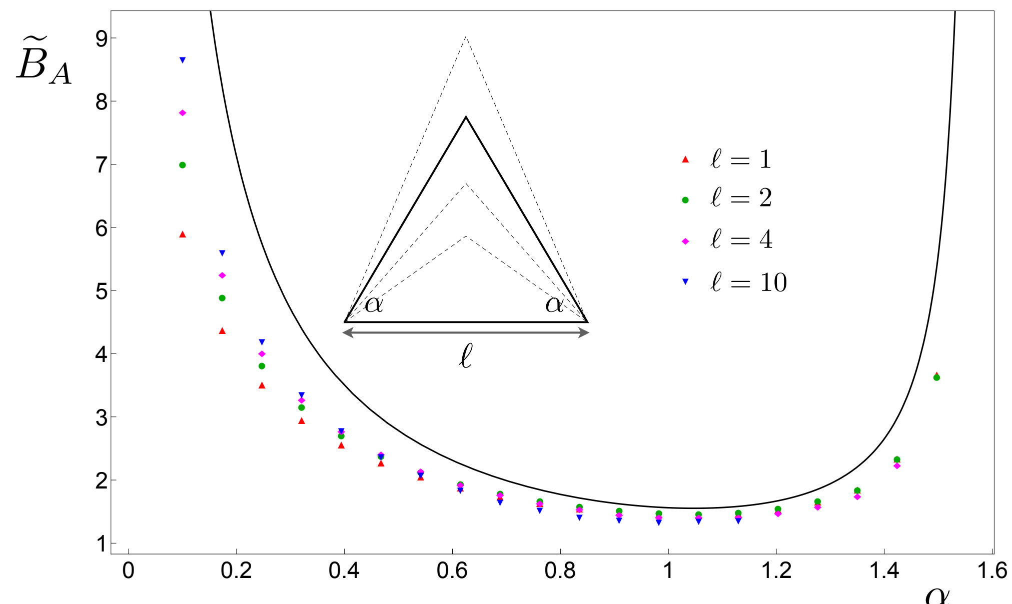
W_A influenced
by the regularization

$$B_A \equiv 2\sum_{i=1}^{N} b(\alpha_i) \qquad b(\alpha) \equiv \int_0^\infty \left(1 - \sqrt{\frac{\zeta^2 + f_0^2 + 1}{\zeta^2 + 2f_0^2 + 1}}\right) d\zeta$$

- Numerical checks with *Surface Evolver* [Fonda, Giomi, Salvio, E.T., (2014)]



Regular polygons

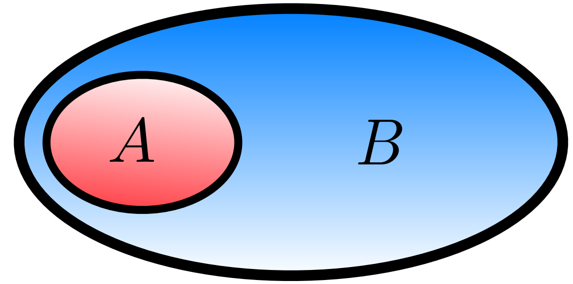


Isosceles triangles

- Log term obtained also through the Willmore energy [Fonda, Seminara, E.T., (2015)]

Mutual Information & Entanglement Negativity

- Ground state $\rho = |\Psi\rangle\langle\Psi|$ and *bipartite* system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$



Reduced density matrix $\boxed{\rho_A = \mathrm{Tr}_B \rho}$

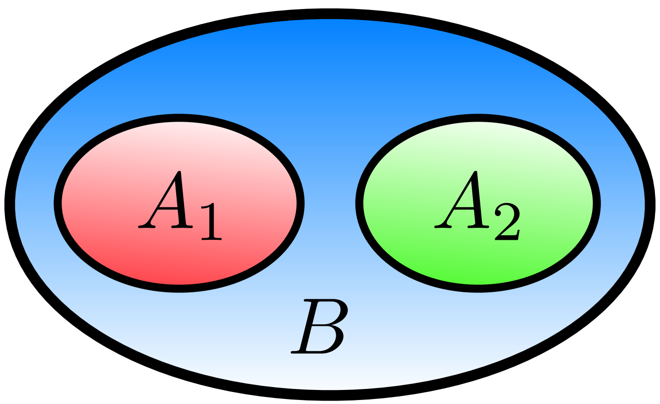
Rényi entropies

Entanglement entropy

$$S_A \equiv -\mathrm{Tr}(\rho_A \log \rho_A) = \lim_{n\to 1} \boxed{\frac{\log(\mathrm{Tr}\rho_A^n)}{1-n}} = -\lim_{n\to 1} \frac{\partial}{\partial n} \mathrm{Tr}\, \rho_A^n$$

- $S_A = S_B$ for pure states

- *Tripartite* system $\mathcal{H} = \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \mathcal{H}_B$ $\boxed{\rho_{A_1\cup A_2} \text{ is mixed}}$

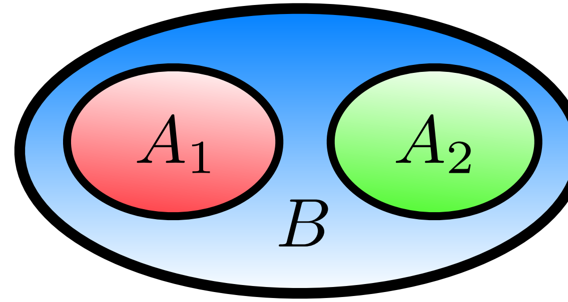


$\boxed{\textit{Entanglement between } A_1 \textit{ and } A_2\textit{?}}$

- $S_{A_1\cup A_2}$: entanglement between $A_1 \cup A_2$ and B
  The mutual information $S_{A_1} + S_{A_2} - S_{A_1\cup A_2}$ gives an upper bound
- A *computable* measure of the entanglement is the *logarithmic negativity*

Entanglement between disjoint regions: Negativity

- $\rho = \rho_{A_1 \cup A_2}$ is a mixed state



ρ^{T_2} is the *partial transpose* of ρ

$$\boxed{\langle e_i^{(1)} e_j^{(2)} | \, \rho^{T_2} \, | e_k^{(1)} e_l^{(2)} \rangle \; = \; \langle e_i^{(1)} e_l^{(2)} | \, \rho \, | e_k^{(1)} e_j^{(2)} \rangle}$$

$(|e_i^{(k)}\rangle$ base of $\mathcal{H}_{A_k})$

[Peres, (1996)] [Zyczkowski, Horodecki, Sanpera, Lewenstein, (1998)] [Lee, Kim, Park, Lee, (2000)] [Eisert, (2001)] [Vidal, Werner, (2002)] [Plenio, (2005)]

- *Trace norm*

$$\boxed{||\rho^{T_2}|| = \mathrm{Tr}|\rho^{T_2}| = \sum_i |\lambda_i| = 1 - 2 \sum_{\lambda_i < 0} \lambda_i}$$

λ_j eigenvalues of ρ^{T_2}

$\mathrm{Tr}\, \rho^{T_2} = 1$

Logarithmic negativity

$$\boxed{\mathcal{E}_{A_2} \; = \; \ln ||\rho^{T_2}|| \; = \; \ln \mathrm{Tr}|\rho^{T_2}|}$$

- *Bipartite* system $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ in *any* state ρ $\longrightarrow$ $\mathcal{E}_1 = \mathcal{E}_2$

Replica approach to Negativity

[Calabrese, Cardy, E.T., (2012)]

- A *parity effect* for $\mathrm{Tr}(\rho^{T_2})^n$

$$\mathrm{Tr}(\rho^{T_2})^{n_e} = \sum_i \lambda_i^{n_e} = \sum_{\lambda_i>0} |\lambda_i|^{n_e} + \sum_{\lambda_i<0} |\lambda_i|^{n_e}$$

$$\mathrm{Tr}(\rho^{T_2})^{n_o} = \sum_i \lambda_i^{n_o} = \sum_{\lambda_i>0} |\lambda_i|^{n_o} - \sum_{\lambda_i<0} |\lambda_i|^{n_o}$$

- Analytic continuation on the *even* sequence $\mathrm{Tr}(\rho^{T_2})^{n_e}$ (*make 1 an even number*)

$$\boxed{\mathcal{E} = \lim_{n_e \to 1} \log \left[\mathrm{Tr}(\rho^{T_2})^{n_e} \right]} \qquad \lim_{n_o \to 1} \mathrm{Tr}(\rho^{T_2})^{n_o} = \mathrm{Tr}\, \rho^{T_2} = 1$$

- **Pure states** $\rho = |\Psi\rangle\langle\Psi|$ and *bipartite* system ($\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$)

Schmidt decomposition

$$\mathrm{Tr}(\rho^{T_2})^n = \begin{cases} \mathrm{Tr}\,\rho_2^n & n = n_o \quad \text{odd} \\ \left(\mathrm{Tr}\,\rho_2^{n/2}\right)^2 & n = n_e \quad \text{even} \end{cases}$$

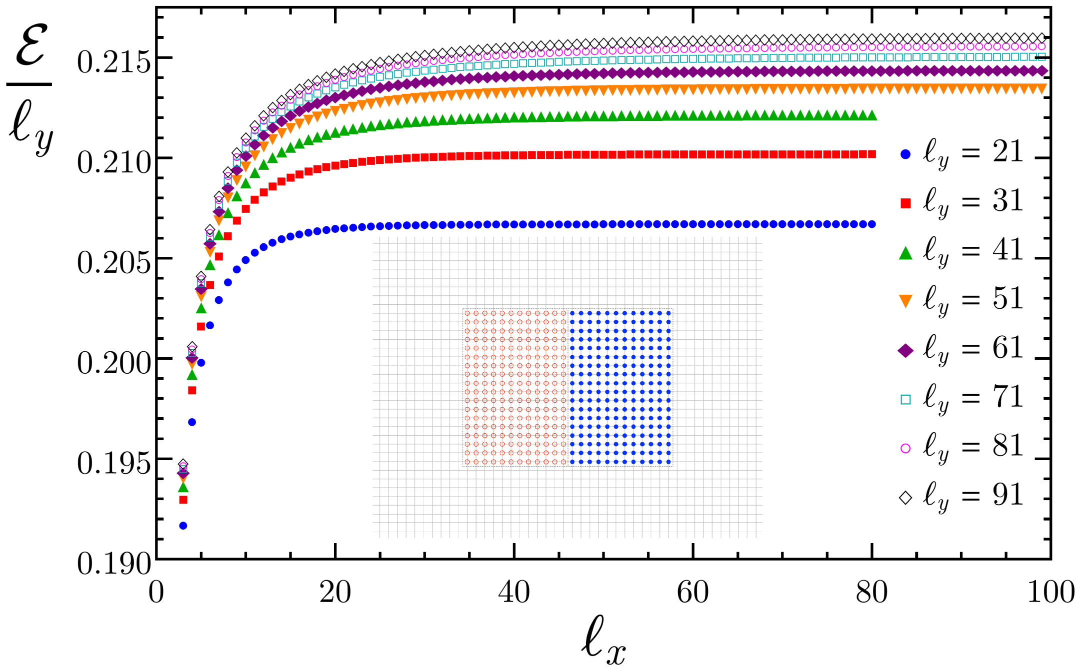
- Taking $n_e \to 1$ we have $\boxed{\mathcal{E} = 2\log \mathrm{Tr}\rho_2^{1/2}}$ (Renyi entropy 1/2)

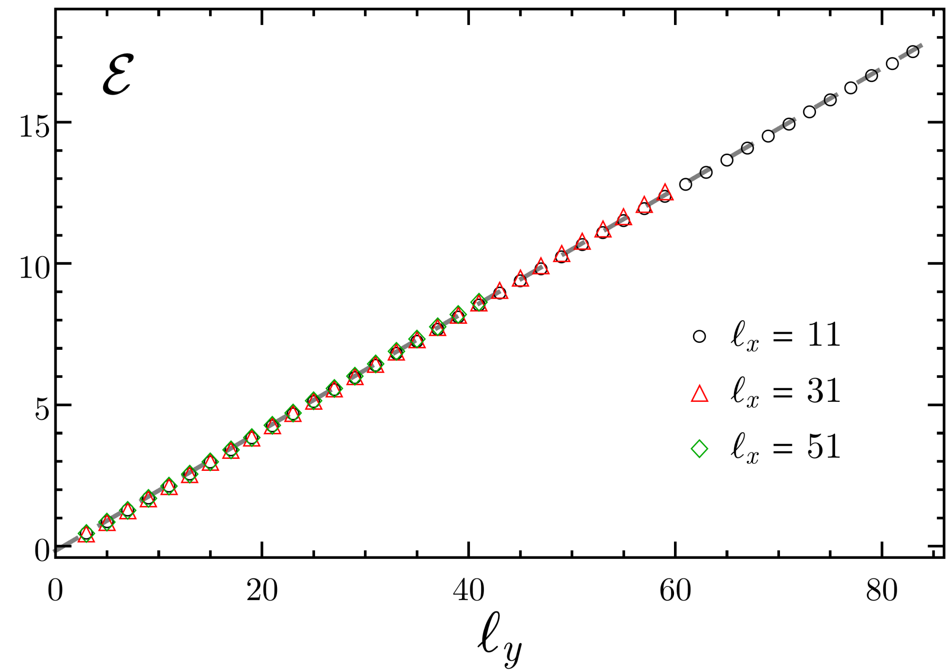
Negativity in a 2D harmonic lattice: Adjacent regions

- $$H = \sum_{\substack{1\leqslant i\leqslant L_x\\ 1\leqslant j\leqslant L_y}} \left\{ \frac{p_{i,j}^2}{2M} + \frac{M\omega^2}{2}\, q_{i,j}^2 + \frac{K}{2}\Big[\big(q_{i+1,j}-q_{i,j}\big)^2 + \big(q_{i,j+1}-q_{i,j}\big)^2 \Big] \right\}$$

- The partial transpose w.r.t. A_2 is obtained by sending $p_i \to -p_i$ in A_2
  [Simon, (2000)] [Audenaert, Eisert, Plenio, Werner, (2002)]

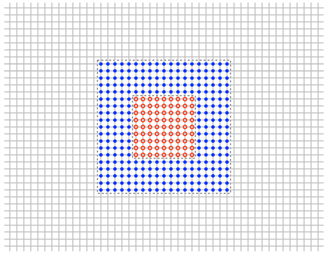
- We consider the massless case in the thermodynamic limit.
  Adjacent regions: e.g. two adjacent rectangles [Eisler, Zimboras, (2015)] [De Nobili, Coser, E.T., (2016)]





Negativity in a 2D harmonic lattice: Area law (I)

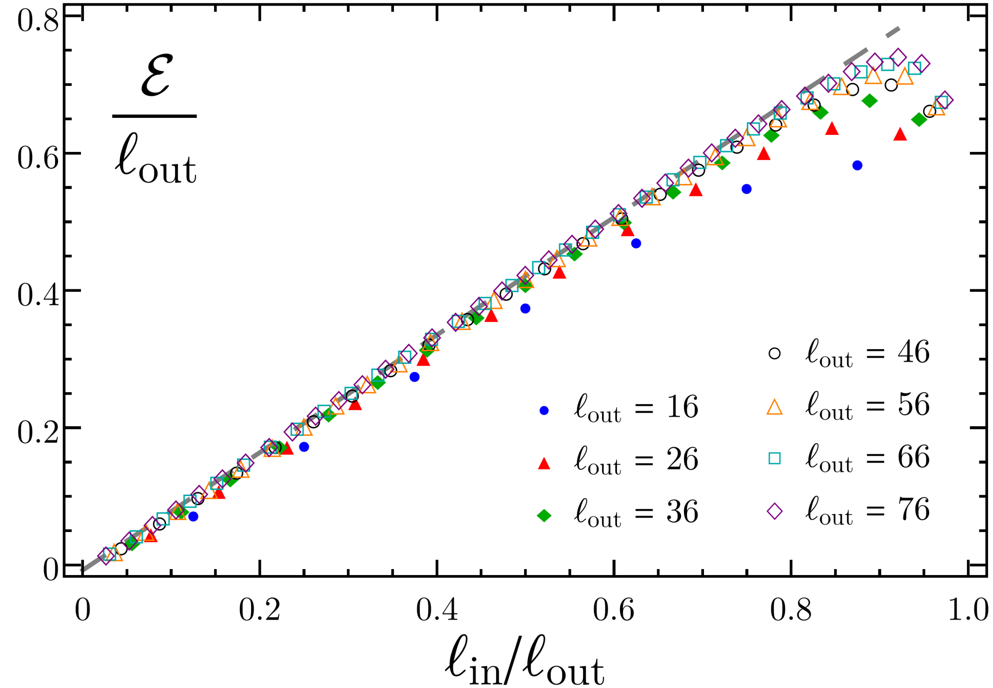
[De Nobili, Coser, E.T., (2016)]



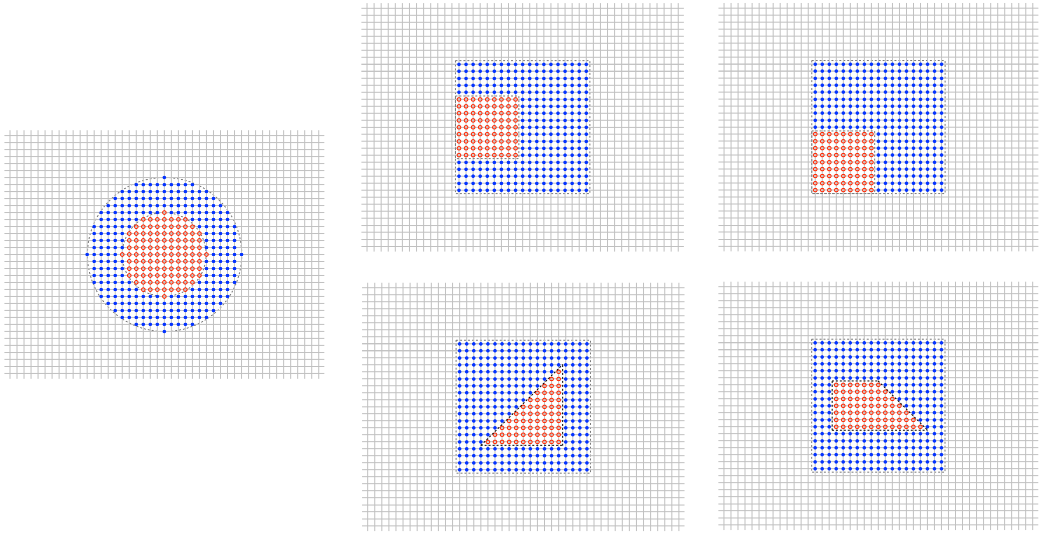
- Area law in terms of
  $P_{\text{shared}} \equiv \text{length}(\partial A_1 \cap \partial A_2)$

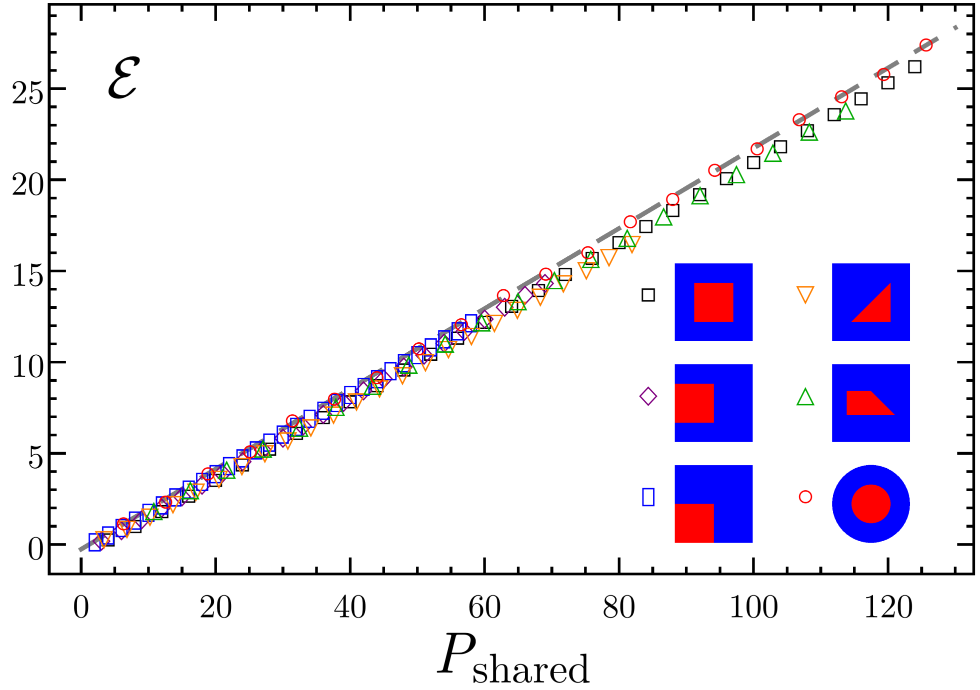
$$\boxed{\mathcal{E} = a\, P_{\text{shared}} + \ldots}$$

$\mathcal{E}$ gives information about the entanglement between A_1 and A_2



- Other configurations of adjacent domains in the scaling limit with $\ell_{\text{in}}/\ell_{\text{out}} \sim 1/3$





Negativity in a 2D harmonic lattice: Area law (II)

- Moments of the partial transpose

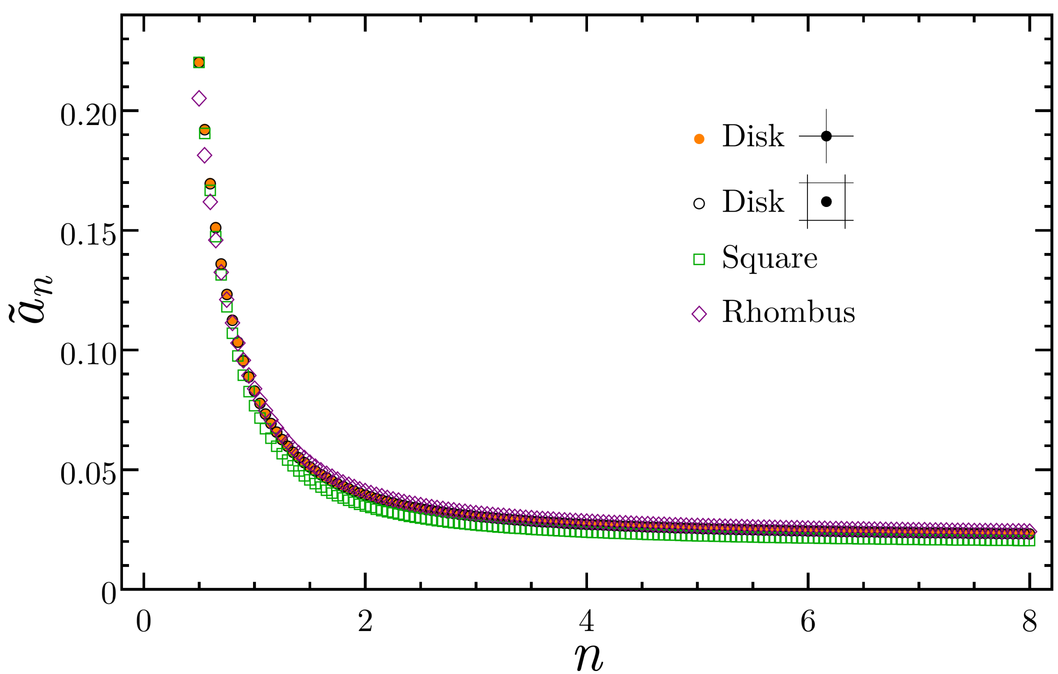
$$\boxed{\mathcal{E}_n \equiv \log\left(\frac{\mathrm{Tr}(\rho_A^{T_2})^n}{\mathrm{Tr}\rho_A^n}\right) = a_n\, P_{\text{shared}} + \dots} \quad \xrightarrow[n_e \to 1]{} \mathcal{E}$$

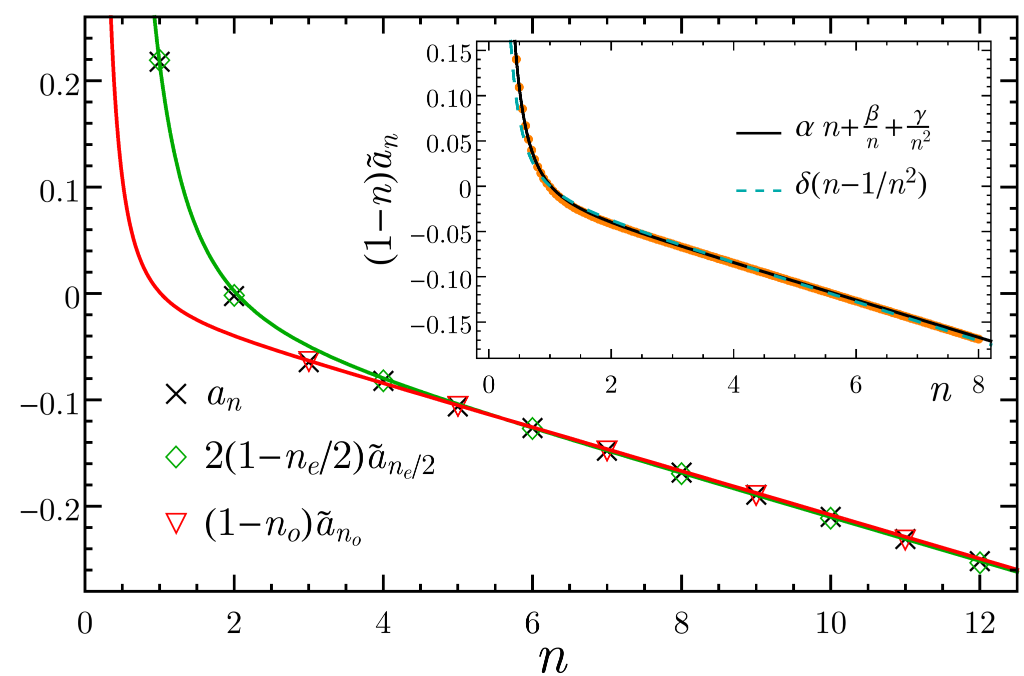
- Area law behaviour due to local effects close to $\partial A_1 \cap \partial A_2$

$\Downarrow$

The coefficient a_n is related to the coefficient of the area law of the Rényi entropies $S_A^{(n)} = \tilde{a}_n P_A + \dots$

$$a_n = \begin{cases} (1-n_o)\,\tilde{a}_{n_o} & \text{odd } n = n_o \\ 2(1-n_e/2)\,\tilde{a}_{n_e/2} & \text{even } n = n_e \end{cases}$$



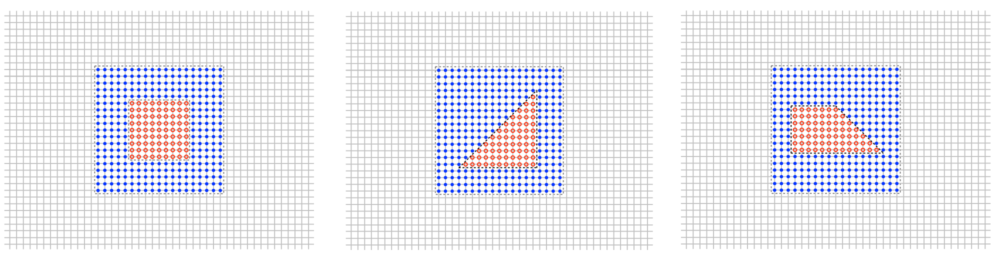


Negativity in a 2D harmonic lattice: Corner contributions

- Only the vertices of $\partial A_1 \cap \partial A_2$ contribute to a (universal) logarithmic term
- When only vertices corresponding to bipartitions or tripartitions of 2π occur

$$\boxed{\mathcal{E} = a\,P_{\text{shared}} - \Bigg(\sum_{\substack{\text{vertices of} \\ \partial A_1 \cap \partial A_2}} b(\theta_i^{(1)}, \theta_i^{(2)}) \Bigg) \log P_{\text{shared}} + \dots}$$

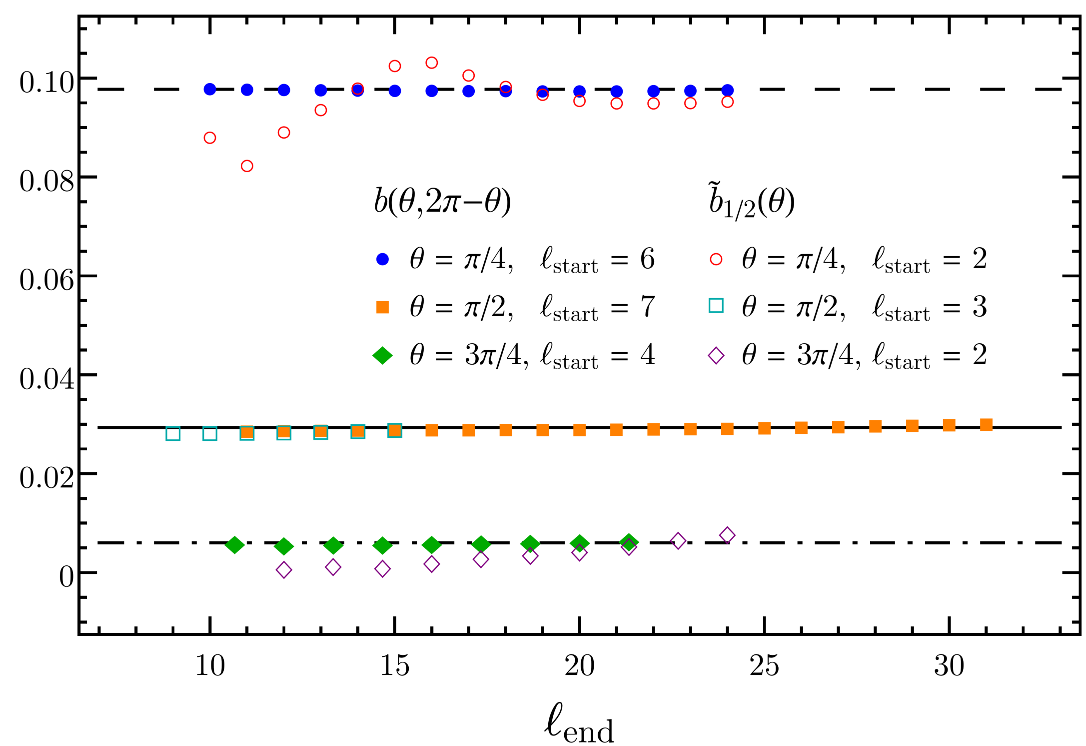
- Similar expression for $\mathcal{E}_n$: corner function $b_n(\theta_i^{(1)}, \theta_i^{(2)})$
- Vertices corresponding to explementary angles



$\tilde{b}_n(\theta)$ corner function of $S_A^{(n)}$

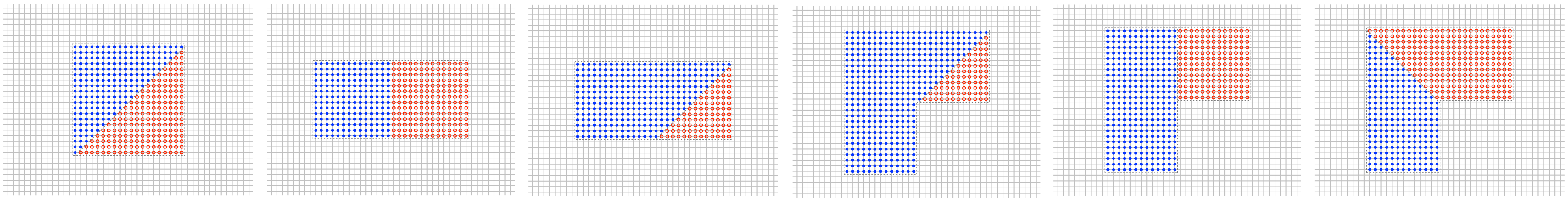
$$b_n(\theta, 2\pi - \theta) = \begin{cases} (1 - n_o)\,\tilde{b}_{n_o}(\theta) \\ 2(1 - n_e/2)\,\tilde{b}_{n_e/2}(\theta) \end{cases}$$

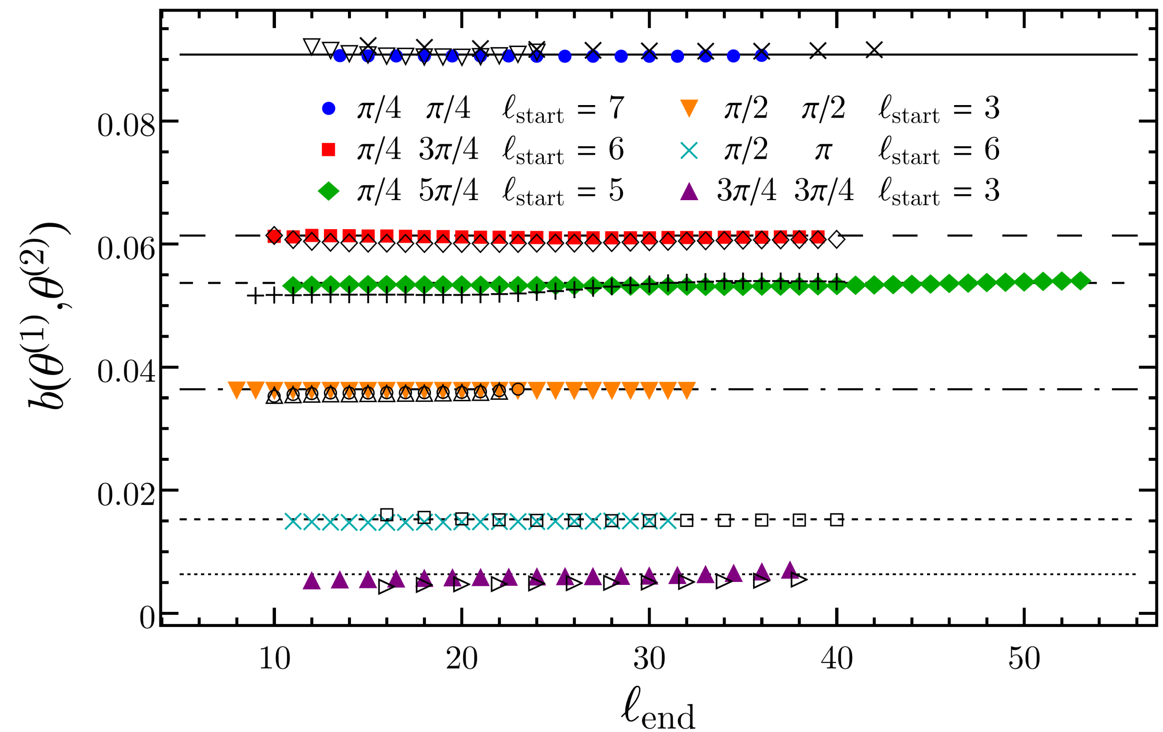
$$\Rightarrow \quad \boxed{b(\theta, 2\pi - \theta) = \tilde{b}_{1/2}(\theta)}$$

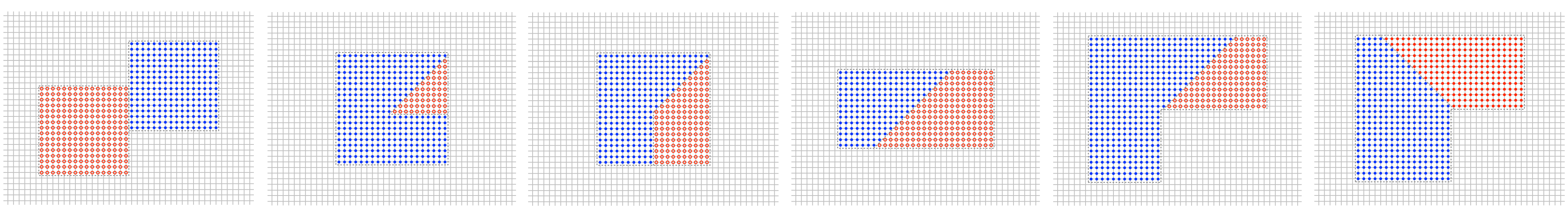


Negativity in a 2D harmonic lattice: Corner contributions

Vertices corresponding to tripartitions of 2π





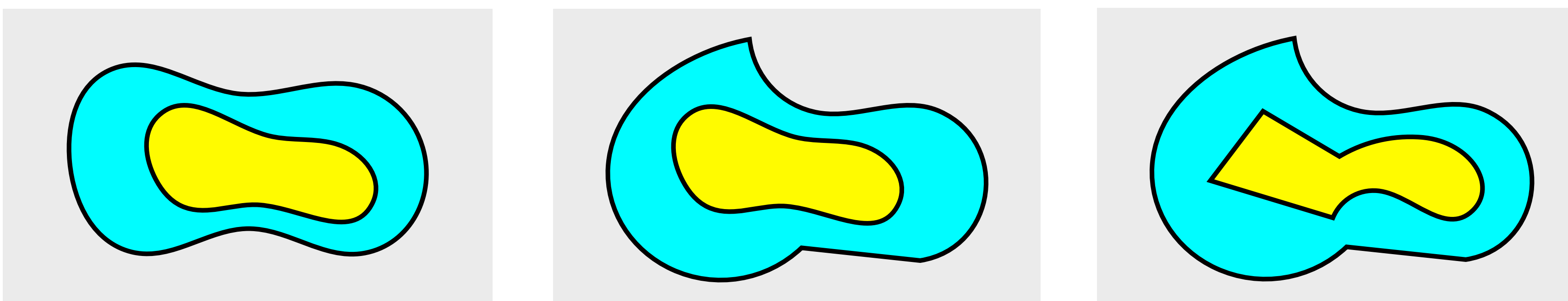


On the corner contributions to negativity in the continuum

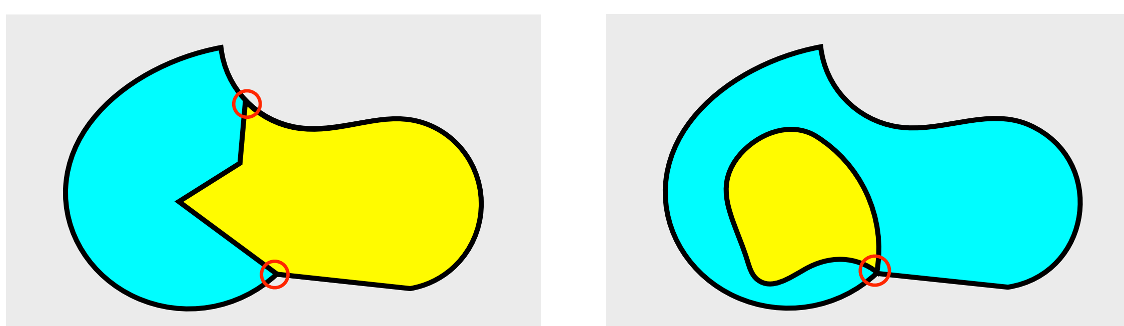
- $$\mathcal{E}_n = \alpha_n \frac{P_{\text{shared}}}{\varepsilon} - \left(\sum_{\substack{\text{vertices of} \\ \partial A_1 \cap \partial A_2}} b_n(\theta_i^{(1)}, \theta_i^{(2)}) \right) \log(P_{\text{shared}}/\varepsilon) + \dots$$

- The combination $\mathcal{E} - I^{(1/2)}_{A_1,A_2}/2$ and its generalisation $\begin{cases} \mathcal{E}_{n_o} - (1-n_o) I^{(n_o)}_{A_1,A_2}/2 \\ \mathcal{E}_{n_e} - (1-n_e/2) I^{(n_e/2)}_{A_1,A_2} \end{cases}$

  are UV finite for these kinds of configurations



  are UV divergent for these kinds of configurations



Some other results on entanglement negativity

- finite temperature in CFTs [Calabrese, Cardy, E.T., (2014)]
- massive field theory [Calabrese, Cardy, E.T., (2012)] [Blondeau-Fournier, Castro-Alvaredo, Doyon (2015)]
- critical Ising model [Calabrese, Tagliacozzo, E.T., (2013)] [Alba, (2013)]
- moments of the partial transpose on the lattice via correlators [Eisler, Zimboras, (2015)] [Coser, E.T., Calabrese, (2015)]
- free fermions: moments of the partial transpose [Coser, E.T., Calabrese, (2015)] [Herzog, Wang, (2016)]
- Kondo model [Bayat, Sodano, Bose, (2010)] [Bayat, Bose, Sodano, Johannesson, (2012)]
- CFTs with boundaries [Calabrese, Cardy, E.T., (2012)]
- results for holographic models [Rangamani, Rota, (2014)] [Kulaxizi, Parnachev, Policastro, (2014)]
- evolution after a quantum quench [Eisler, Zimboras, (2014)] [Coser, E.T., Calabrese, (2014)] [Hoogeveen, Doyon, (2014)] [Wen, Chang, Ryu, (2015)]
- topological systems (toric code) [Lee, Vidal, (2013)] [Castelnovo, (2013)]
- two dimensional lattice models [Eisler, Zimboras, (2015)] [Sherman, Devakul, Hastings, Singh, (2015)]

Conclusions & open issues

- Holographic entanglement entropy in $\mathrm{AdS}_4/\mathrm{CFT}_3$:
  a generalized Willmore functional occurs at $O(1)$ of S_A as $\varepsilon \to 0$

- Entanglement negativity of adjacent domains
  in a 2D massless harmonic lattice: area law & corner contributions

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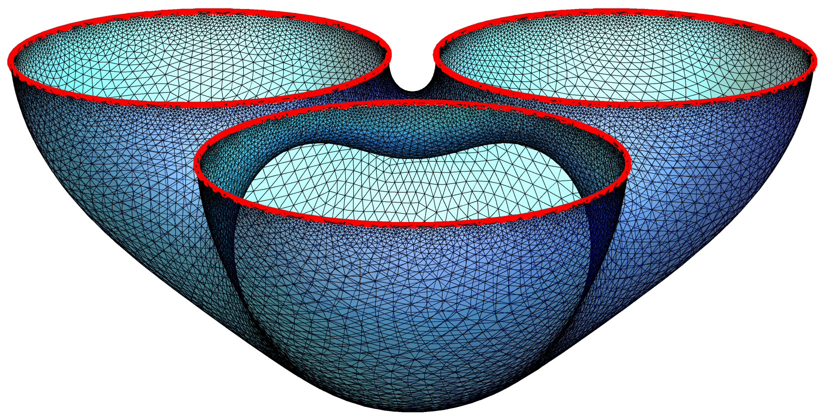
- Some open issues:

  → Higher dimensions

  → QFT analysis of negativity
  in $2+1$ and higher dimensional CFT

  → Interactions

  → Negativity in AdS/CFT



Thank you!