Entanglement Entropy In Gauge Theories

Sandip Trivedi

Tata Institute of Fundamental Research, Mumbai, India.

On The Entanglement Entropy For Gauge Theories, arXiv: 1501.2593

Sudip Ghosh, Ronak Soni and SPT

Aspects of Entanglement Entropy For Gauge Theories,
Ronak Soni, SPT, 1510.07455

Entanglement Entropy in U(1) Gauge Theory, Ronak Soni, SPT, in prep.

References :

Casini, Huerta, Rosabal: Phys. Rev. D. 89, 085012 (2014)

(Will refer to this as CHR sometimes).

Casini, Huerta, Phys. Rev. D 90, 105013 (2014)

Cassini, Huerta: arXiv: 1512.06182

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S. Aoki, T. Iritani, M. Nozaki, T. Numasawa, N. Shiba, H. Tasaki, arXiv: 1502.04267

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- K. van Acoleyen, N. Bultnick, J. Haegeman, M. Marien, V. B. Scholz, F. Verstraete, 1511.04369

Outline

- Introduction
- Extended Hilbert Space Definition
- Extractable Part of the Entanglement
- Replica Trick, Continuum Limit And U(1) Case

Outline (Continued)

- Toric Code
- Conclusions

Introduction

Entanglement Entropy is an important measure of quantum correlations.
(the essential ``weirdness' of quantum mechancis)

Gauge Theories are central to our understanding of physics.

Introduction

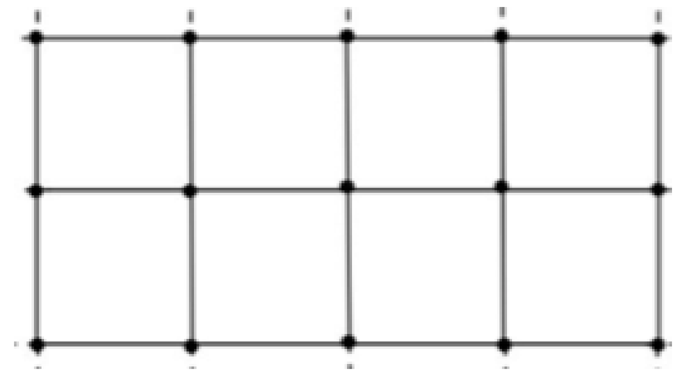
It is therefore worth trying to give a precise definition of entanglement entropy in gauge theories.

Introduction

In a system with local degrees of freedom the entanglement entropy is straightforward to define.

Introduction

E.g. in a spin system



Z_2 Spin System: single qubit at each site $|\pm>$

2- dim Hilbert space at site i: $\mathcal{H}_i$

Full Hilbert space:

$$\mathcal{H} = \bigotimes \mathcal{H}_i$$

Interested in the entanglement between a subset of spins, called the ``inside" with the rest, the ``outside"

$$\rho = Tr_{\mathcal{H}_{out}} |\psi >< \psi|$$

Full Hilbert space:

$$\mathcal{H} = \bigotimes \mathcal{H}_i$$

Admits a tensor product decomposition

$$\mathcal{H} = \mathcal{H}_{in} \otimes \mathcal{H}_{out}$$

$$\rho = Tr_{\mathcal{H}_{out}} |\psi >< \psi|$$

$$\rho = Tr_{\mathcal{H}_{out}} |\psi >< \psi|$$

$$\boxed{S_{EE} = S_{vN} = -Tr_{\mathcal{H}_{in}} \rho \log \rho}$$

Von Neumann entropy

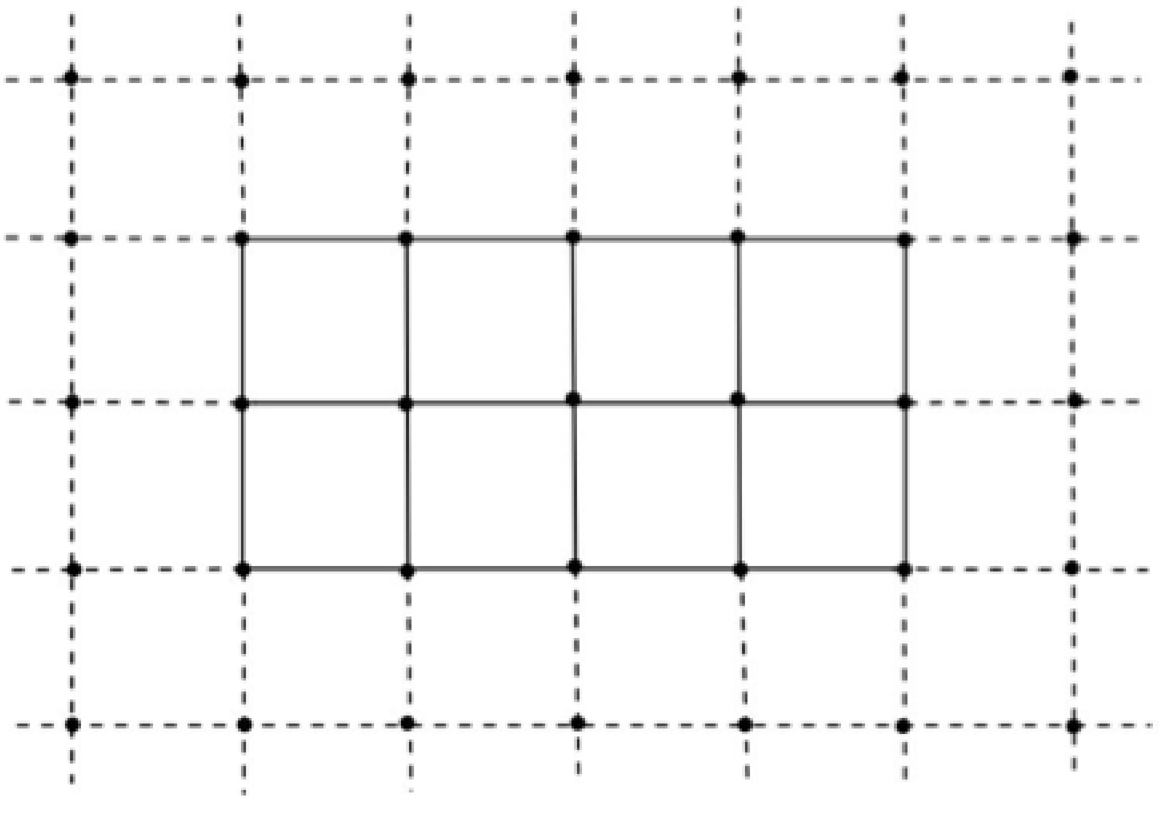
Entanglement In A Gauge Theory

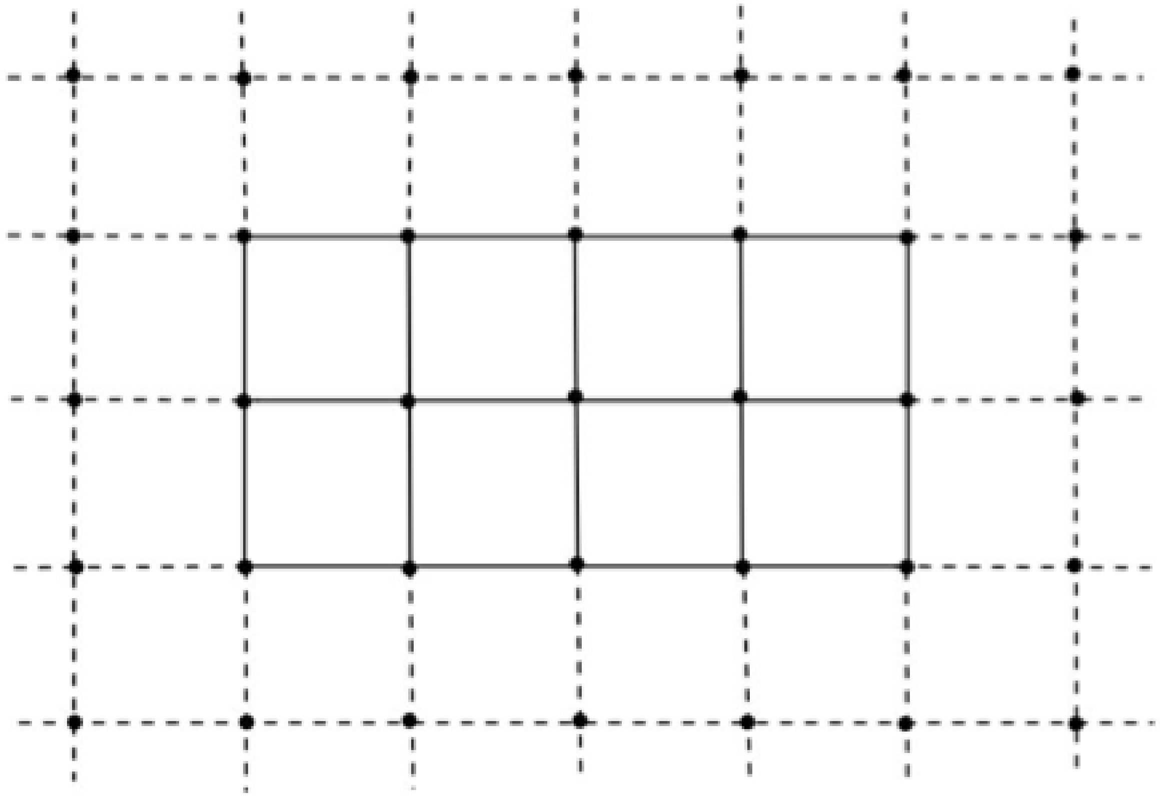
- Not as simple to define.
- Because there are non-local degrees of freedom, e.g., Wilson loops, or loops of electric flux.
- Physical Hilbert space of states does not admit a tensor product decomposition between $\mathcal{H}_{in}, \mathcal{H}_{out}$

Entanglement Entropy In Gauge Theory

- Lattice Gauge Theory
- Hamiltonian Framework: time continuous, Spatial lattice
- Discussion applies in d+1 dimensions.
- (Diagrams in 2 spatial dimensions)

Entanglement Entropy In A Z_2 Gauge Theory :





We are interested in the entanglement of the solid links, the ``inside” links, with the rest the ``outside” links.

Definition will apply to a general set of ``Inside” links. They need not fill out a rectangle, or even be a connected set.

Degrees of freedom live on links.

Z_2 case:

1 qubit : $|\pm>$

$$\sigma_3|\pm 1\rangle = \pm 1|\pm 1\rangle$$

``Electric field" operator

$$\sigma_1|\pm> = |\mp>$$

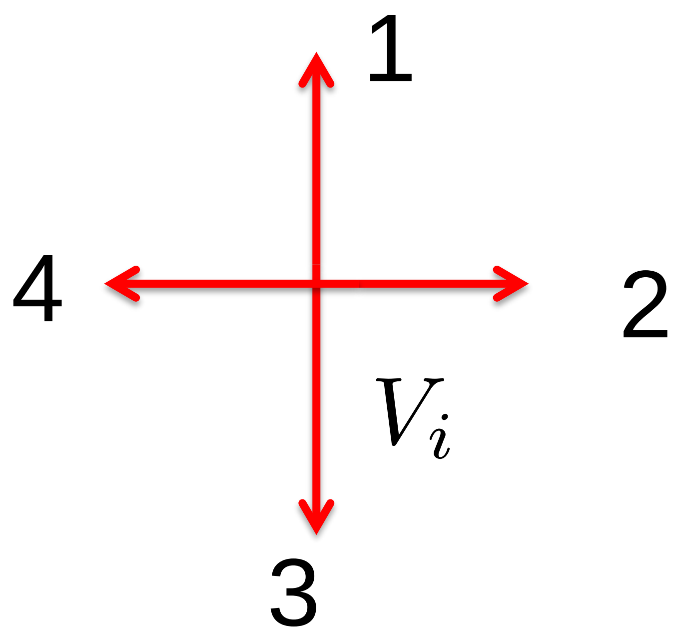
Gauge Transformations defined on vertices

$$G_{V_i} = \prod_{(ij)} \sigma_{1(ij)}$$

Physical states must be Gauge Invariant :

$$G_{V_i}|\psi> = |\psi> \forall i$$

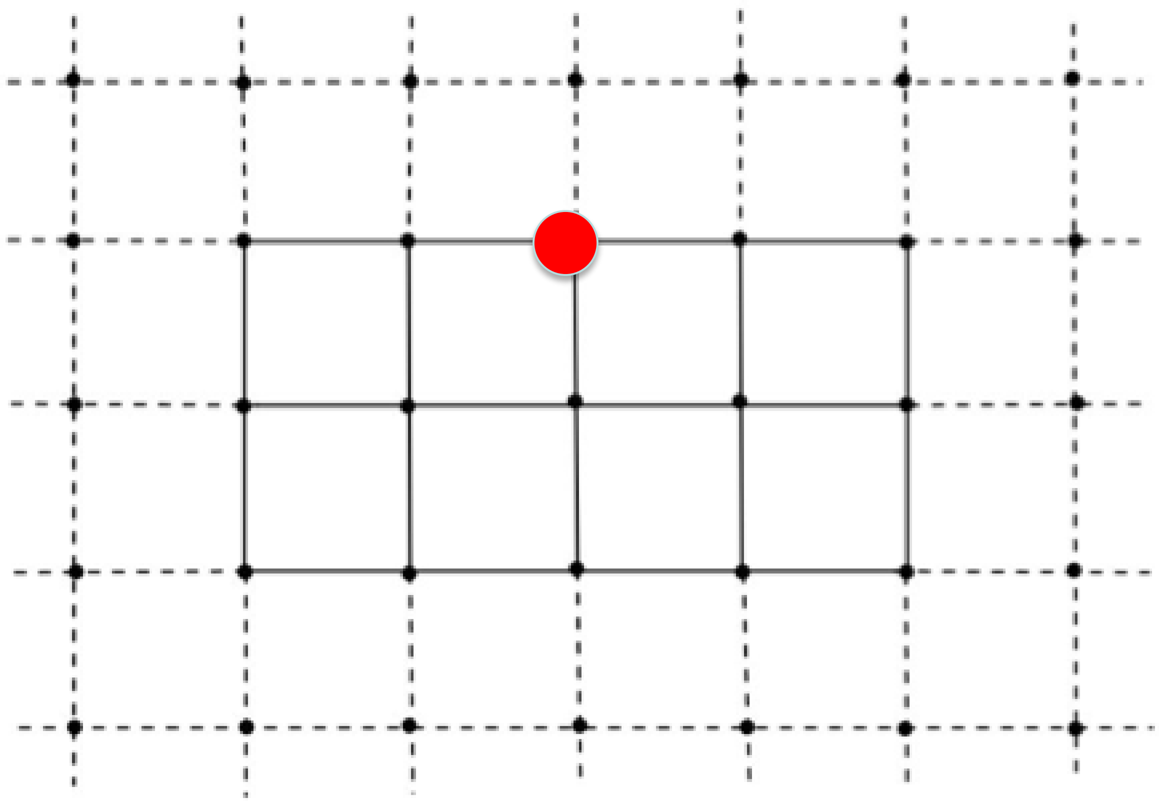
(Gauss' Law)


1
4
2
V_i
3

Gauss' Law:

$$\sigma_1 \sigma_2 \sigma_3 \sigma_4 |\psi >= |\psi >$$

This means the state of the links on the inside and outside are not independent. Thus the Hilbert space of gauge invariant states does not admit a tensor product decomposition.



The Gauss law constraint must apply at every boundary vertex, e.g. the red one in this figure.

This is the essential obstruction or complication in defining the entanglement entropy for gauge theories.

Gauge Fixing in different ways leads to dependent answers which are therefore gauge dependent.

Extended Hilbert Space Definition

Work in an extended Hilbert Space $\mathcal{H}$
Obtained by taking the tensor product of the Hilbert spaces on each link.

$$\mathcal{H} = \bigotimes \mathcal{H}_{ij}$$

$\mathcal{H}$ admits a tensor product decomposition.

$$\mathcal{H}_{in} = \bigotimes_{(ij)\in in} \mathcal{H}_{ij}$$

$$\mathcal{H}_{out} = \bigotimes_{(ij)\in out} \mathcal{H}_{ij}$$

$$\mathcal{H} = \mathcal{H}_{in} \bigotimes \mathcal{H}_{out}$$

Extended Hilbert Space Definition

Embed $|\psi>$ in $\mathcal{H}$

$$|\psi> \in \mathcal{H}_{ginv} \subset \mathcal{H}$$

Construct $\rho_{in} = Tr_{\mathcal{H}_{out}} |\psi >< \psi|$

$$S_{EE} = -Tr_{\mathcal{H}_{in}} (\rho_{in} \log \rho_{in})$$

Ghosh, Soni, SPT, 1501.02593
S. Aoki, T. Iritani, M. Nozaki, T. Numasawa, N. Shiba, H. Tasaki, arXiv: 1502.04267

Renyi Entropies Can be Defined Similarly.

Properties of Extended Hilbert Space Definition

Definition Unambiguous.

Gauge Invariant

Meets Strong Subadditivity property

Properties of Extended Hilbert Space Definition

Simple, can be easily generalised:

- Discrete and Continuous,
- Abelian, Non-Abelian Groups.
- Also, with matter.

Properties of Extended Hilbert Space
Definition

Definition unambiguous.

$|\psi>$ Gives rise to a unique state in $\mathcal{H}$

And after tracing out over $\mathcal{H}_{out}$ a unique

ρ

A gauge invariant state has a unique embedding in $\mathcal{H}$

$$\mathcal{H}_{ginv} \subset \mathcal{H}$$

$$\mathcal{H} = \mathcal{H}_{ginv} \bigoplus \mathcal{H}_{ginv}^{\perp}$$

Orthogonal complement

$$|\psi> \in \mathcal{H}_{ginv}$$

Properties:

$\mathcal{H}$ Endowed with a natural inner product from that on $\mathcal{H}_{ij}$. Meets positivity condition.

Properties

Thus S_{EE} meets the strong subadditivity condition

A, B, C three sets of links that do not share any links in common

$$S_{A\cup B} + S_{B\cup C} \geq S_{A\cup B\cup c} + S_B$$

Gauge Invariant Characterisation Of The Extended Hilbert Space Definition

- Let us understand the gauge invariance of the definition some more.

- How can the resulting answer be expressed in terms of gauge invariant data?

Gauge Invariant Characterisation

For simplicity we consider a theory without matter.

Any state can be decomposed into different sectors.

Each sector has a fixed amount of electric flux coming into the inside region.

Let the boundary vertices be labelled as

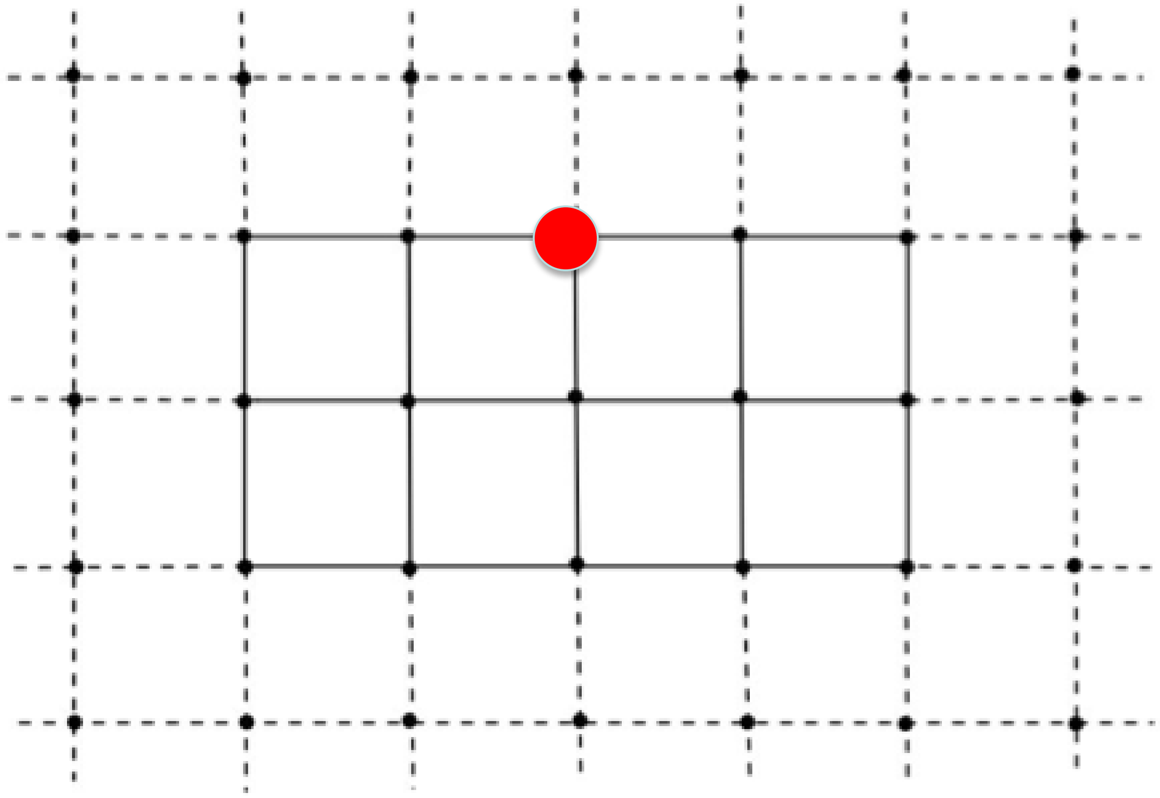
$$V_1, V_2, \cdots V_N$$

And let the Electric flux coming in be

$$\mathbf{k} = (k_1, k_2, \cdots k_N)$$

Then each sector is specified by
A choice of the electric flux vector $\mathbf{k}$.

$k_a, a = 1, \cdots N$ integer in the Abelian case (compact U(1)).



We are interested in the entanglement of the solid links, the ``inside” links, with the rest the ``outside” links.

Result (Abelian Case)

$$S_{EE} = -\sum_i p_i \log p_i - \sum_i p_i (\bar{\rho}_i \log \bar{\rho}_i)$$

`i’: labels sectors of different electric flux

p_i : probability to be in sector i

$\bar{\rho}_i$ normalised density matrix in sector `i’

$$Tr_i(\bar{\rho}_i) = 1$$

Result

$$S_{EE} = \underbrace{-\sum_i p_i \log p_i}_{\text{``Classical Term''}} \underbrace{- \sum_i p_i (\bar{\rho} \log \bar{\rho}_i)}_{\text{``Quantum entanglement''}}$$

The Extended Hilbert Space definition, in Abelian Case can be shown to be equivalent to the electric center prescription of Cassini, Huerta and Rosabal.

Non-Abelian Theory

R. Soni, SPT, JHEP 1601 (2016), 136

An extra contribution arises in the Non-Abelian case.

Tied to the fact that irreducible representations have dimension greater than unity.
(Donnelley)

Non-Abelian Theory

Let the total flux going inside at a boundary vertex transform in an irreducible representation R_a of the group.

Then the total flux going out at this vertex must also be in the same R_a irreducible representation.

And together they must pair to form a singlet under the gauge group (Gauss' Law)

Result (Non-Abelian Case)

The extra entanglement arises due to this pairing.

In the `i'th superselection sector let the dimensions of the representations at the boundary vertices $V_1, V_2, \cdots V_N$ be

$$(d^i_1, d^i_2 \cdots d^i_N)$$

Result (Non-Abelian Case)

Then

$$S_{EE} = -\sum_i p_i \log p_i - \sum_i Tr_i p_i \bar{\rho}_i \log \bar{\rho}_i + \underbrace{\sum_i p_i (\sum_a \log d_a^i)}_{\text{Extra piece}}$$

The extra contribution can be large for a big gauge group.

Superselection Sectors

The different sectors are actually different superselection sectors.

Measurements, or Gauge invariant operations in the inside or outside cannot change these sectors, or the probabilities p_i and dimensions d_a^i

Superselection Sectors

- This prevents some of the entanglement entropy from being ``extracted” through local operations acting only on the inside or outside, as we will see.

Operational Definition

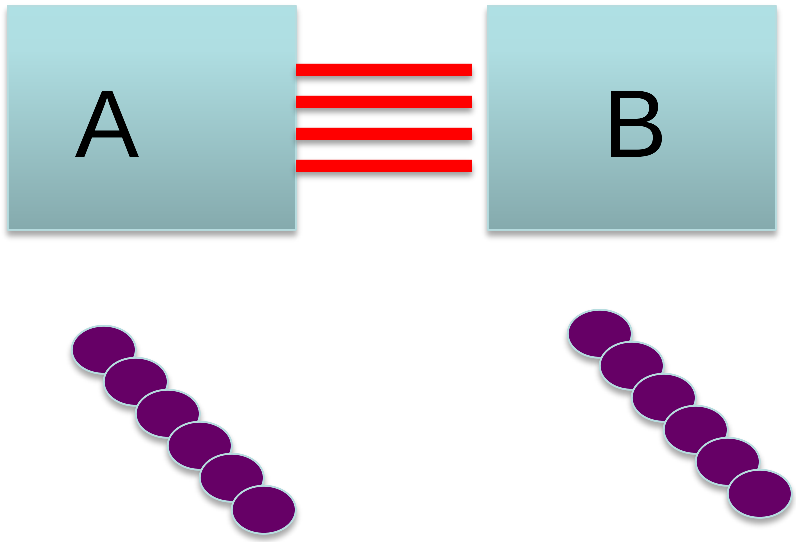
How much of the entanglement we have defined can be actually used for quantum information processing?

Quantum Information Theory

- Entanglement quantified by comparison with a reference system of N Bell pairs.

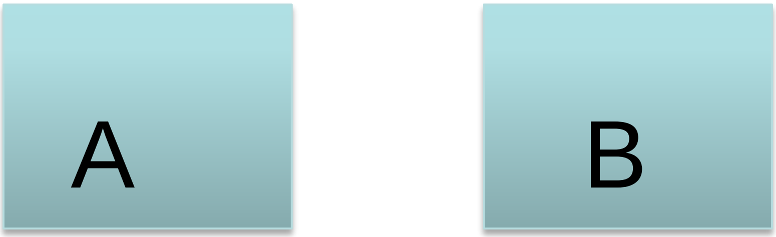
- Comparison done using entanglement distillation or entanglement dilution.

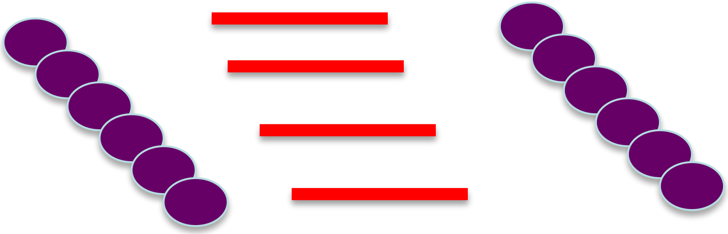
Entanglement Distillation :



2N unentangled qubits

To finally arrive at the situation:





N entangled Bell pairs

Entanglement Distillation

Carry Out Transformations involving Local Operations and Classical Communication

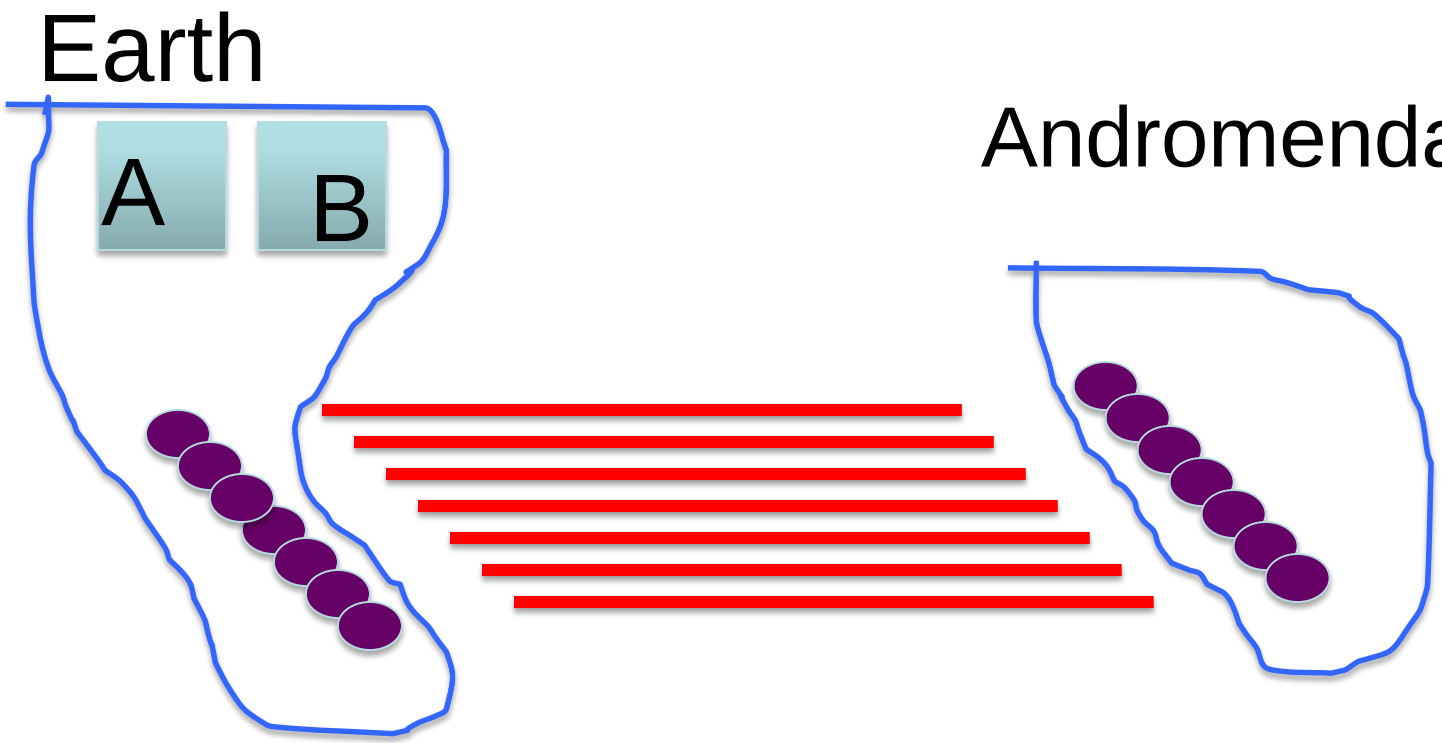
Local operations act only in A and one set of N qubits. Or B and the other set.

Let N be the maximum number of Bell pairs we can produce.

Then $S_{EE} = N \log(2)$

(Actually in an asymptotic sense with $\mathcal{N}$ copies of the system in the $\mathcal{N} \to \infty$ Limit.)

Entanglement Dilution



N entangled Bell pairs

Entanglement Dilution

Measure of Entanglement: Minimum number of Bell pairs needed to Teleport the state of B to Andromeda.

i.e., Transfer the entanglement on earth between A and B to now lie between Earth and Andromeda?

(Again in an asymptotic sense).

For a system with local degrees, like a spin system, the entanglement which can be operationally defined in these two ways agree with the von Neumann entropy of the density matrix:

$$S_{EE} = -Tr\rho\log(\rho)$$

But for a gauge theory it is different.

This is due to the superselection sectors which cannot be changed using gauge invariant operations on only the inside or outside.

Hence, we cannot extract the full entanglement, in the extended Hilbert space definition, through distillation or dilution.

This is a general difference between gauge theories and say spin systems.

Extracting The Entanglement

So how well can we do?

The maximum entanglement which can be extracted is

$$\Delta S_{EE} = -\sum_i Tr_i \; p_i \bar{\rho}_i \log(\bar{\rho}_i)$$

R. Soni and SPT, 1510.07455

K. van Acoleyen, N. Bultnick, J. Haegeman, M. Marien, V. B. Scholz, F. Verstraete, 1511.04369

Only this term can be extracted

$$S_{EE} = -\sum_i p_i \log p_i \overbrace{- \sum_i Tr_i p_i \bar{\rho}_i \log \bar{\rho}_i}$$

$$+ \sum_i p_i \left(\sum_a \log d_a^i\right)$$

Extracting The Entanglement

We have given explicit protocols showing that this is the maximum bound on the extractable entanglement.

R. Soni and SPT, 1510.07455

K. van Acoleyen, N. Bultnick, J. Haegeman, M. Marien, V. B. Scholz, F. Verstraete, 1511.04369

Two more features of the Extended Hilbert Space Definition.

1) Agrees with Replica Trick

1) Often gives results which agree with other physical expectations and are useful. E.g., Toric code.

Connection to the Replica Trick

Replica Trick: Euclidean Path Integral method to calculate entanglement.

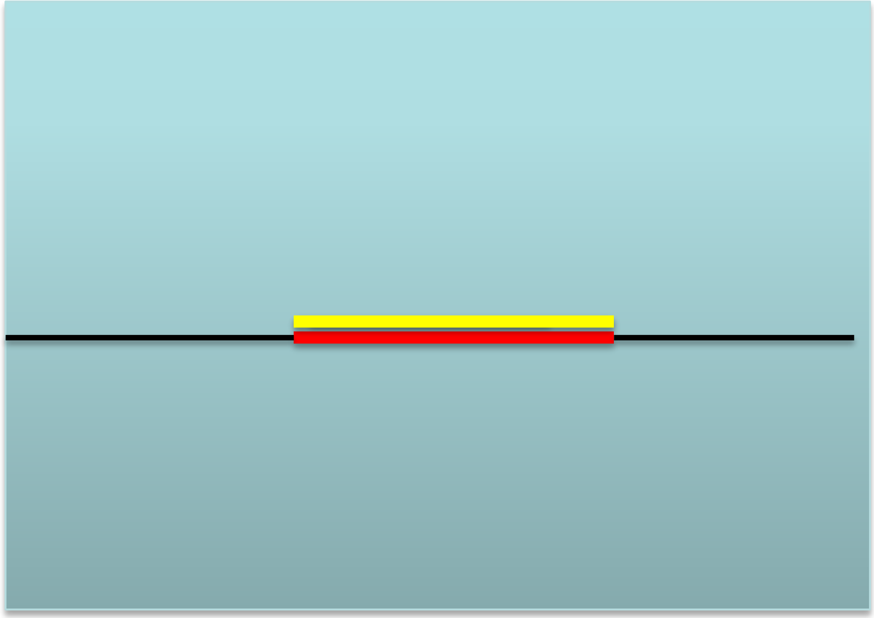
First calculate $Tr\rho^n$

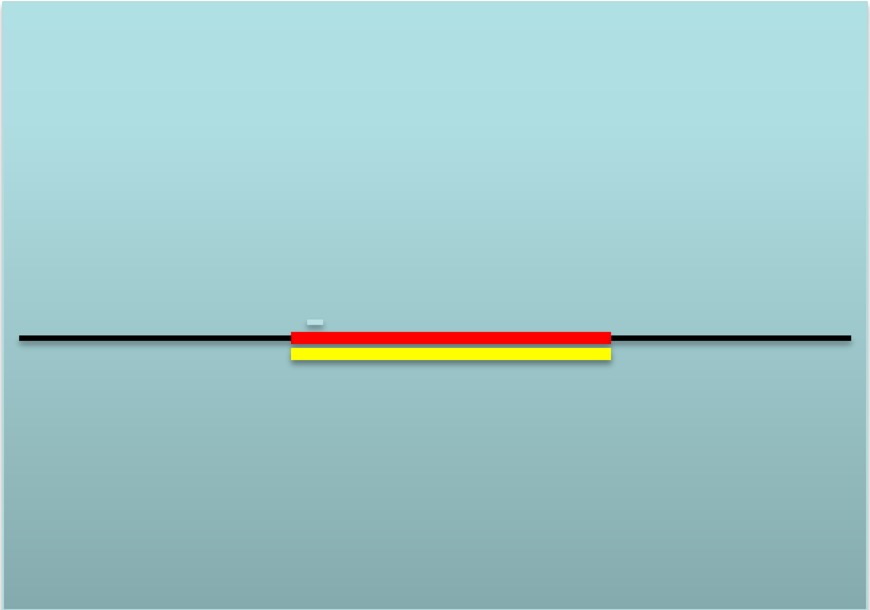
$$S = -[n\partial_n - 1] \log(Tr\rho^n)|_{n\to 1}$$

- To calculate R_n in d+1 dim. We work on an n-fold cover of R^{d+1}

- Obtained by introducing a branch cut at a particular instant of ``time'' along the spatial region of interest

- And identifying values of fields at bottom of branch cut in one copy with their values above the cut in the next.





$Tr(\rho^n)$ is obtained by carrying out the path integral on this n-fold cover.

(Minor point: Normalisation).

- This can be done also for a lattice gauge theory.

- In the Hamiltonian formulation time is continuous, and for each instance of time we have a spatial lattice.

- Each link U_{ij} an independent variable.

$$|\psi > = \int_{-\infty}^{t=0^-} DU_{ij}\ e^{-S}$$

$$< \psi| = \int_{0^+}^{t=\infty} DU_{ij}\ e^{-S}$$

- The path integral automatically gives $|\psi >$ embedded in the extended Hilbert space

- Since each link variable is an independent degree of freedom.

Starting with a gauge invariant state in the far past (or in effect the vacuum) we get a gauge invariant state at $t = 0$

The further identification at t=0 of link variables outside the branch cut (black line) gives the density matrix

$$\rho = Tr_{\mathcal{H}_{out}}|\psi >< \psi|$$

And the Path Integral over the N-fold cover then gives $R_n = Tr_{\mathcal{H}_{in}}(\rho^n)$

So that $$S = -Tr_{\mathcal{H}_{in}}\rho\log(\rho)$$

So in the end what the path integral calculates from the replica trick is

$$S = -Tr_{\mathcal{H}_{in}} \rho \log(\rho)$$

Which is exactly the entanglement entropy in the Extended Hilbert Space definition.

Connection With Replica Trick

Caveat **:** The path integral is over a singular geometry.

The lattice gives a a particular way to reguliarise this integral.

Cut-off independent quantities should be independent of regularisation procedure.

Connection With Replica Trick

Caveat: In the continuum limit the replica trick gives a divergent answer. The Lattice provides a regularisation.

However, physical results in continuum should be independent of precise choice of regularisation.

One should therefore be able to obtain them from the lattice replica path integral, or equivalently our definition.

Continuum Limit: U(1)

(R. Soni and S. P. T. in prep.)

Start with compact U(1) (without matter) at weak coupling on the lattice.

And take the continuum limit.

This gives the replica trick integral with the standard Fadeev Popov Gauge fixing:

U(1) Theory

$$\int_M [DA] e^{-S} \delta(\nabla \cdot \vec{A}) det'(\nabla^2)$$

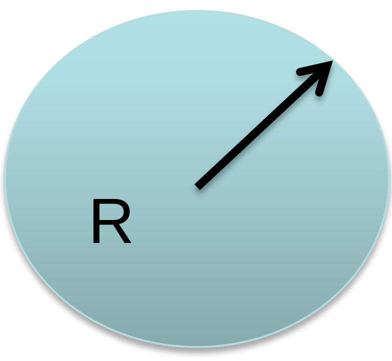
$$S = \int d^4x F_{\mu\nu} F^{\mu\nu}$$

M is the singular n-fold cover of R^{d+1}
Obtained by introducing a branch cut along the boundary

U(1) Theory 3+1 dimensions

A cutoff needs to be introduced to make the Path Integral well defined.

Take 3+1 dim and the spatial region whose entanglement we seek to be a sphere of radius R



U(1) In 3+1 Dim

On general grounds we expect:

$$S_{EE} = A/\epsilon^2 + C\log(R/\epsilon) + \text{finite}$$

Cut-off independent

U(1) Theory in 3+1 Dim

C gets related to the A anomaly coefficient

$$C = -\frac{31}{45}$$

$$T^{\mu}_{\mu} = aE_4 + bW^2$$

$$C = a$$

Fursayev, Patrushev, Solodukhin, 1306.400, …

U(1) Theory in 3+1 Dim

How much of this is extractable?

The ``classical piece” turns out to be:

$$\Delta S = -\sum_i p_i \log(p_i)$$
$$= -\frac{1}{3}\log(R/\epsilon)$$

Related to 1+1 Free scalar CFT
(R. Soni and SPT, In prep.)

Donnelley and Wall, PRL, 2015, 114, 111603.

U(1) in 3+1 Dim

The resulting extractable piece is then

$$\Delta S_{extractable} = -\frac{16}{45}\log(R/\epsilon)$$

U(1) in 3+1 Dim

The extractable part agrees with the result for the full entanglement recently obtained by Casini and Huerta, using a trivial center definition.

And also earlier by D'Howker.

The confusion about why this answers did not agree with the anomaly coefficient can now be understood.

U(1) in 3+1 Dim

Note the full answer and the extractable piece are both different from that for two scalar fields:

C= $2 \times -(1/90) = -1/45$

Implications For the Ryu-Takayanagi Entanglement

Since Extended Hilbert Space Definition gives rise to the Replica Trick,

And in turn the Replica trick in the boundary gauge theory corresponds to the Ryu-Takayanagi Proposal for the bulk in asymptotically AdS spacetimes. (Lewkowycz, Maldacena)

We learn that our definition agrees with the RT proposal on the gravity side.

And implies that the entanglement given by the minimal area in gravity does not correspond to the maximum extractable entanglement

In fact, it is usually smaller.

The full implications of this for gravity and holography remain to be explored.

Toric Code (Kitaev)

For discrete group G (can be non-Abelian) with dimension |G|

Extended Hilbert Space Definition in turn gives rise to a topological entanglement

$$S_{top} = -\log |G|$$

Topological Entanglement: Kitaev, Preskill; Levin, Wen

The non-extractable parts of entanglement entropy play an important role in this answer.

The answer agrees with the expected result based on the quantum dimensions of various particles.

Conclusions

- We have proposed a definition for entanglement entropy in gauge theories.

- The definition is applicable to Abelian and Non-Abelian Theories, and also to theories with matter.

Conclusions

- It has many nice properties.
  i) It is gauge invariant.
  ii) Agrees with the Replica trick,
etc.

- But, in general, it does not agree with an operational definition based on entanglement distillation or dilution.

Conclusions

- Implications for gravity remain to be fully understood.

1. The Ryu-Takayanagi proposal agrees with our definition. It therefore does not give the extractable entanglement.

2. Gravity is also a gauge theory. What can we learn by applying this understanding directly in the bulk?

Banyan Tree In TIFR

Thank you!