Out of Time Ordered

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[1] Chaos in quantum channel, Hosur, Qi, Roberts, BY 1511.04021

[2] Complexity by design, (in prep, with Daniel Roberts)

Scrambling

Out-of-time ordered correlation function

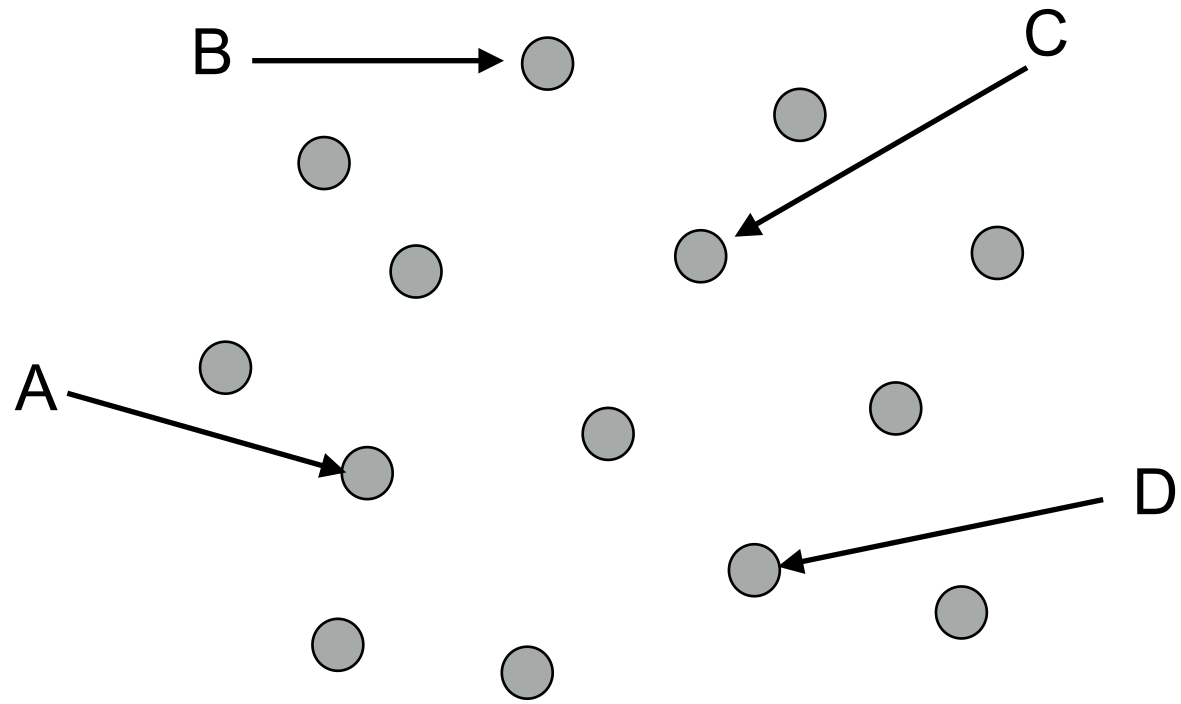
- Scrambling: delocalization of quantum information, hidden into non-local DOF.
- Fast scrambling is the defining feature of black holes.
- Out-of-time ordered (OTO) correlators can detect scrambling.
- Definition

$$\mathrm{OTO} = \langle A(0)B(t)\boxed{C}(0)D(t)\rangle$$

local operators

$$B(t) = e^{-iHt}B(0)e^{iHt}$$

$$D(t) = e^{-iHt}D(0)e^{iHt}$$



system of “qubits”

[Larkin-Ovchinnikov, Hayden-Preskill, Kitaev, Shenker-Stanford-Roberts-Susskind]

Scrambling implies decay of OTO

- Local perturbation to an initial state cannot be detected by any local measurement on an output state.

- Consider $$\mathrm{OTO} = \langle A(0)B(t)A^{\dagger}(0)B^{\dagger}(t)\rangle$$ (group commutator)

$$[A, B] = 0 \qquad \text{then} \qquad \mathrm{OTO} = 1$$

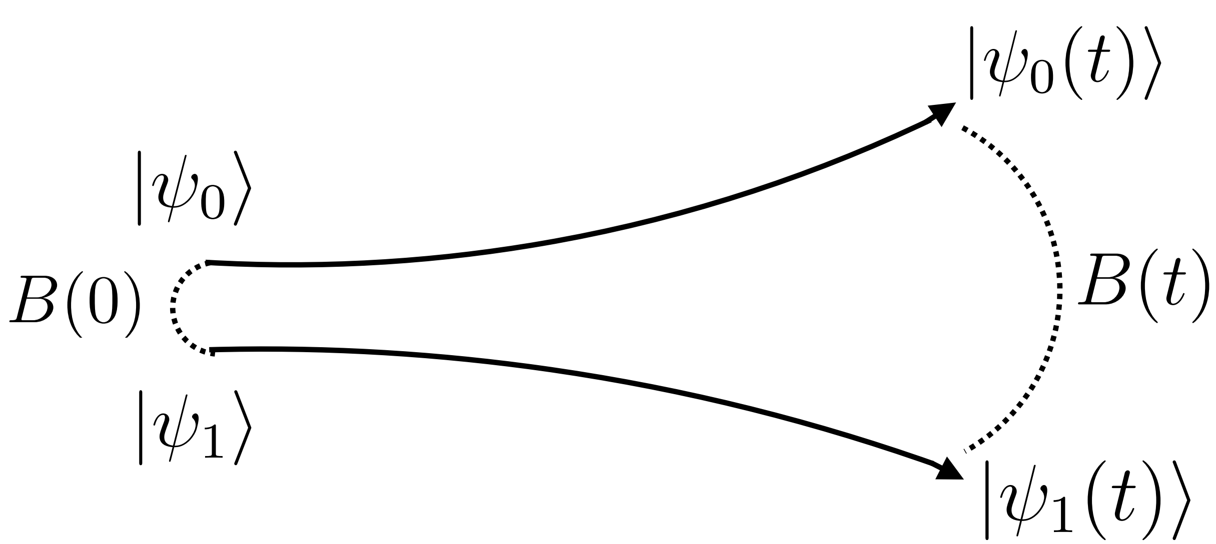
$$\{A, B\} = 0 \qquad \text{then} \qquad \mathrm{OTO} = -1$$

Non-commutativity between A(0) and B(t)

- Expand B(t):

$$B(t) = e^{-iHt}Be^{iHt} = \sum_j \alpha_j \boxed{P_j}$$

high-weight Pauli operators



scrambling/chaos (butterfly effect)

$$\boxed{\text{scrambling}} \longrightarrow \boxed{\mathrm{OTO} \simeq 0}$$

[Roberts-Susskind-Stanford]

Key Questions

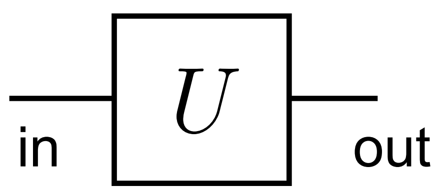
- How do we define scrambling ?
- Quantum information theoretic meaning of OTO ?
- Is the converse true ? $\boxed{\text{scrambling}} \; \underset{?}{\overset{\longrightarrow}{\longleftarrow}} \; \boxed{\mathrm{OTO} \simeq 0}$
- Relation to entanglement entropy (and geometric quantities) ?

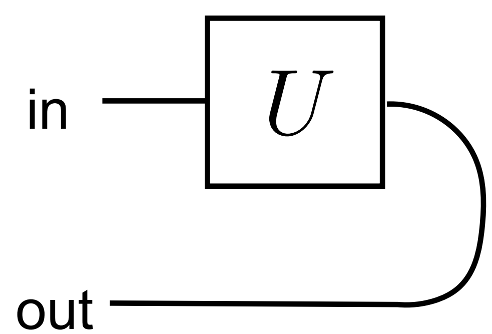
State-Channel duality

- Quantum channel on n qubits can be viewed as a state on 2n qubits.

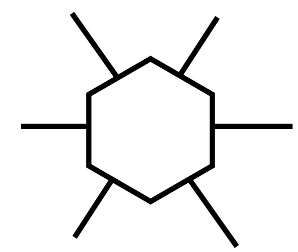
unitary operator as a state (T=infty)

$$U = \sum_{i,j} U_{i,j} |i\rangle\langle j| \qquad |U\rangle = \sum_{i,j} U_{i,j} |i\rangle \otimes |j\rangle$$

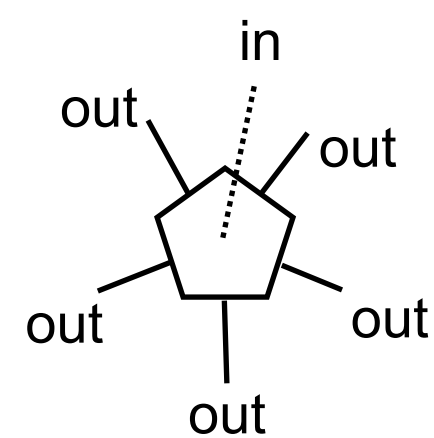




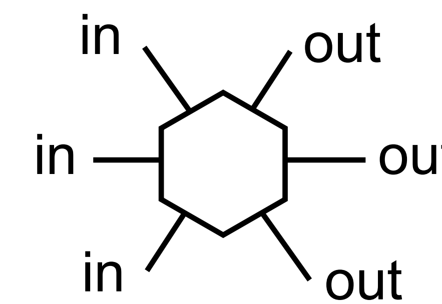
cf) perfect tensor



state

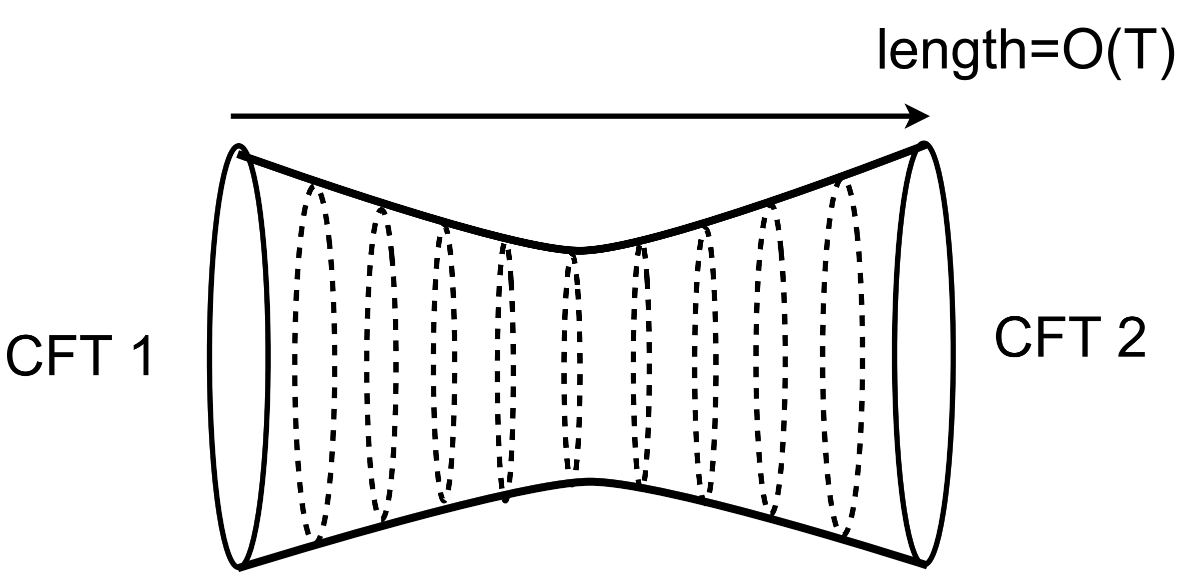


code



unitary

- Thermofield double state (finite T)



ER bridge = quantum channel

$$\rho_{in} = \frac{e^{-\beta H}}{\mathrm{Tr}\ e^{-\beta H}} \qquad \rho_{out} = \frac{e^{-\beta H}}{\mathrm{Tr}\ e^{-\beta H}}$$

$$|\Psi\rangle = \sum_j e^{-\beta E_j/2} e^{-iE_j t} |\psi_j\rangle \otimes |\psi_j\rangle$$

[Choi, Jamilkowski, Hayden-Preskill]

Average value of OTO

- Average of OTO over local operators A and D at T=infty

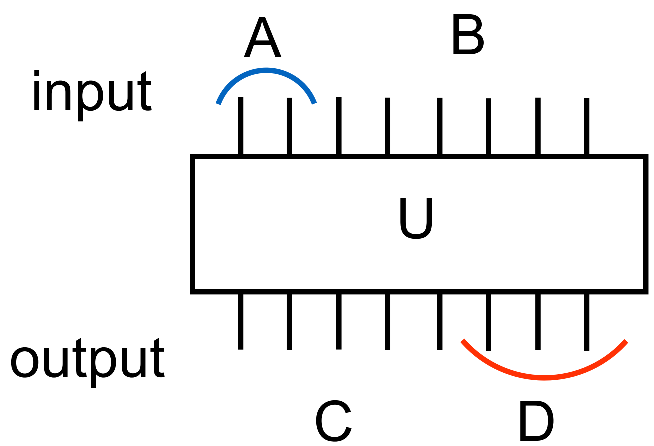
$$\left|\langle A(0)D(t)A^\dagger(0)D^\dagger(t)\rangle\right| = \frac{1}{4^{a+d}}\sum_{\boxed{A,D}}\langle A(0)D(t)A^\dagger(0)D^\dagger(t)\rangle$$

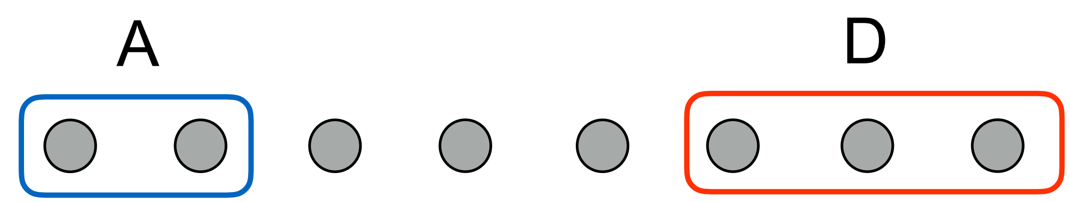
Pauli operators (unitary 1-design)

average over A, D

$$= 2^{n-a-d-\boxed{S^{(2)}_{BD}}}$$

Renyi-2 entropy





- If $\mathrm{OTO} \simeq 0$ then, $S^{(2)}_{BD}$ is large

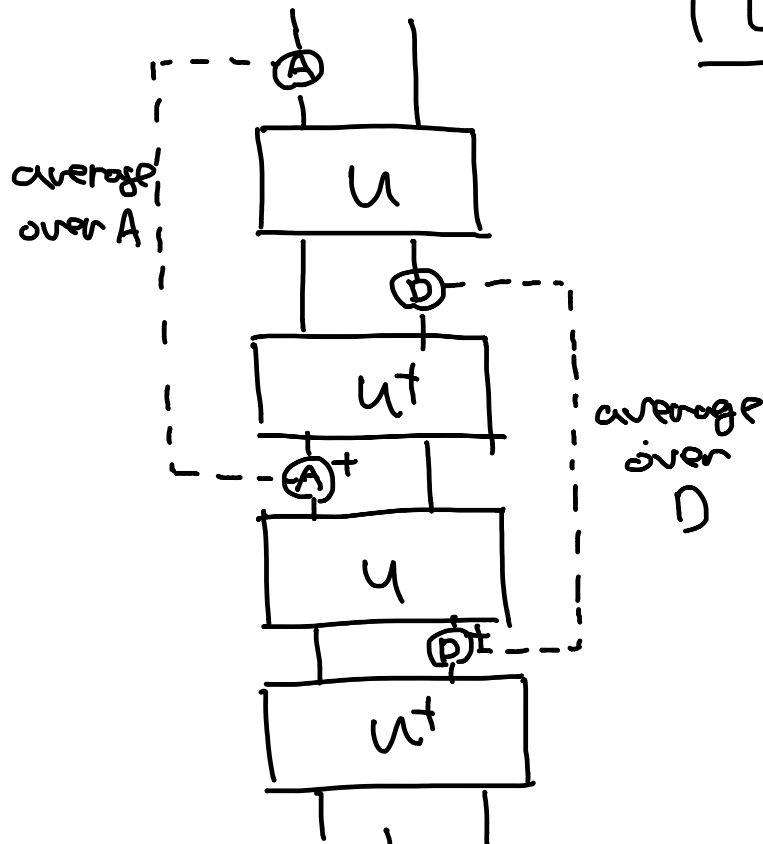
  This implies the mutual information $I^{(2)}_{BD} = S^{(2)}_{B} + S^{(2)}_{D} - S^{(2)}_{BD}$ is small

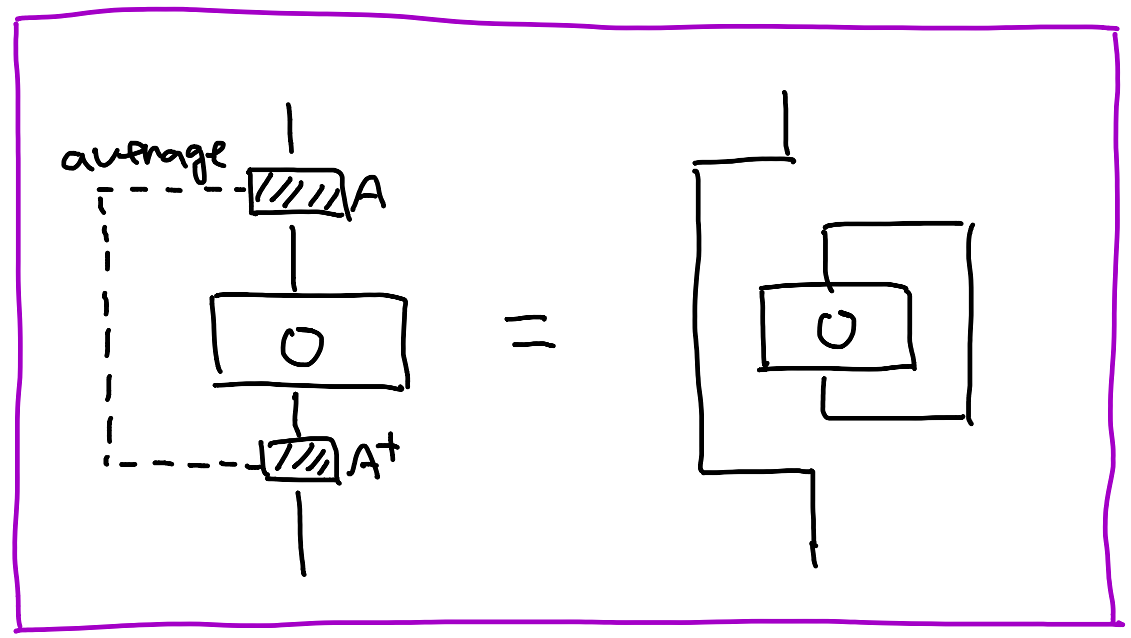
  B and D are not correlated, so the system is scrambling.

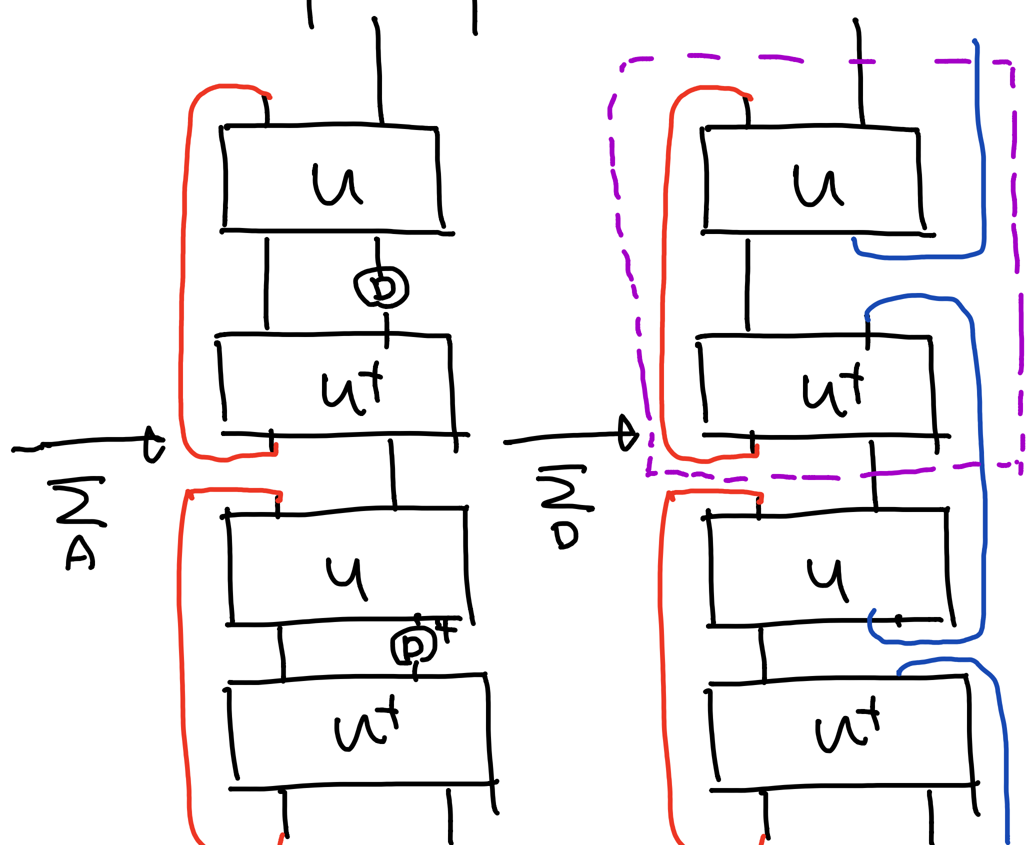
- For finite T, we consider TFD state.

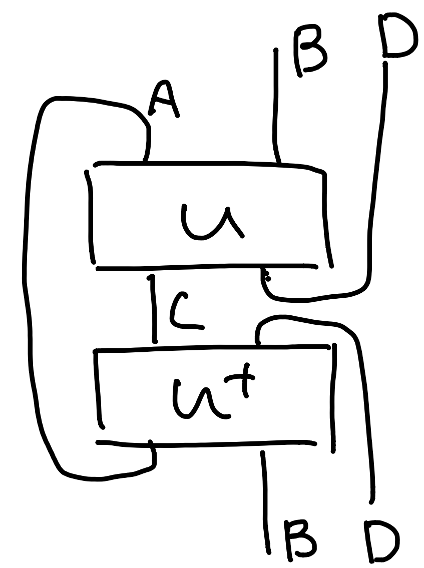
[Hosur, Qi, Roberts, BY]

Derivation









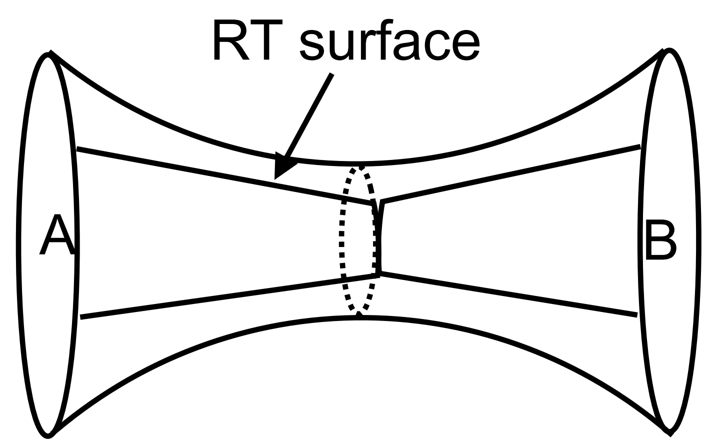
$= \rho_{BD}$

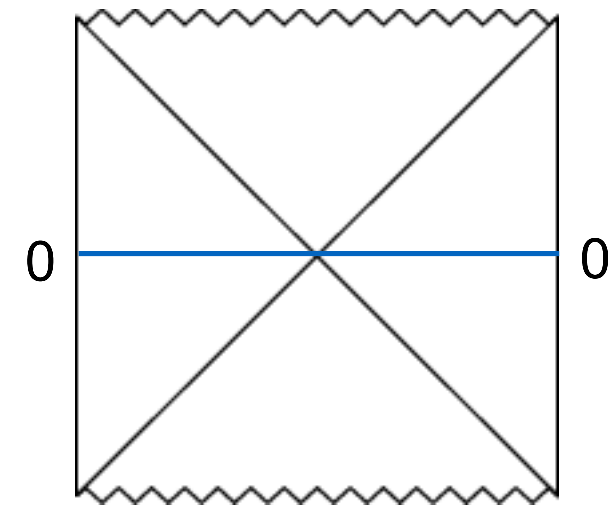
$= \mathrm{Tr}[\rho_{BD}{}^2] \longrightarrow$ Renyi-2.

[Hosur, Qi, Roberts, BY]

Scrambling in AdS black hole

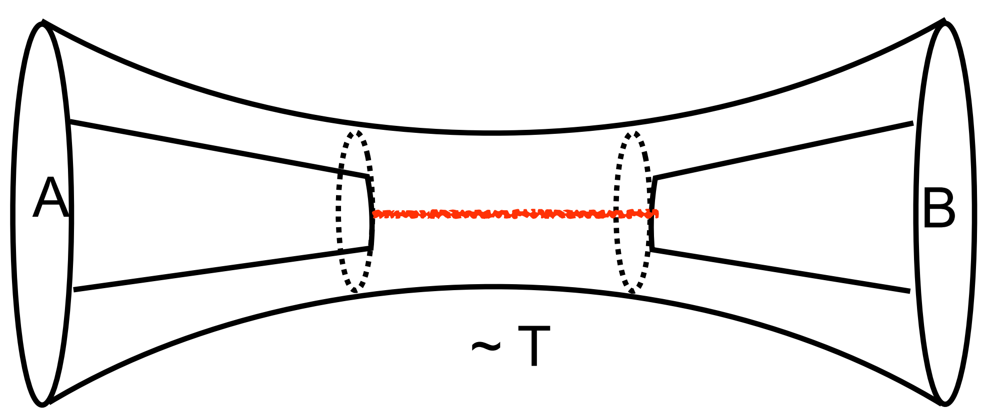
TFD(0)

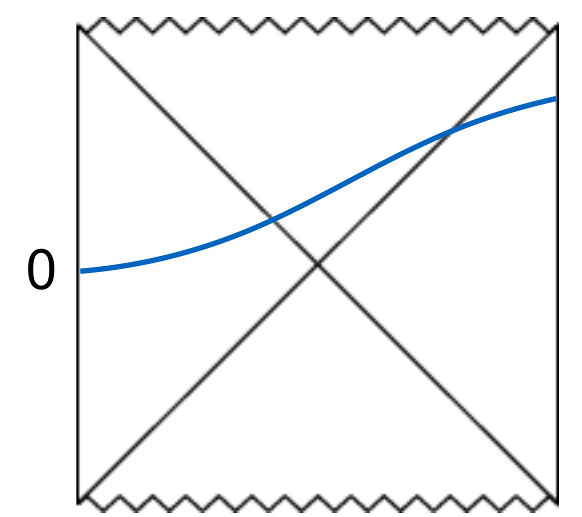




mutual information $I(A, B)$ is large

TFD(T)

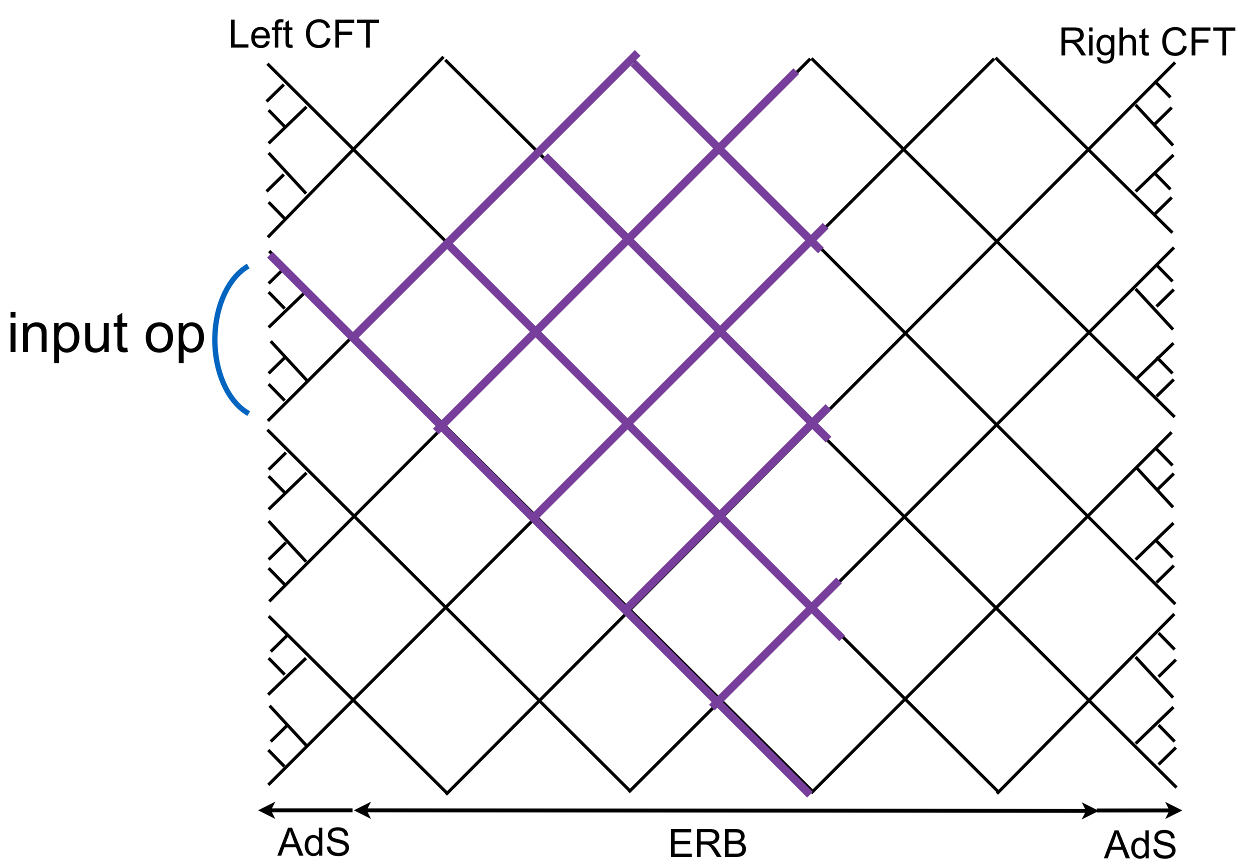




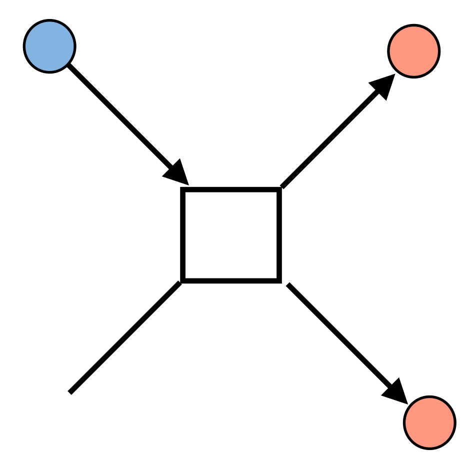
mutual information $I(A, B)$ is (almost) zero

Toy model of the ER bridge

- Operators grow ballistically, leading to decay of OTO correlators.



network of perfect
(or random) tensors



[Hosur, Qi, Roberts, BY]

OTO average for Alice and Bob

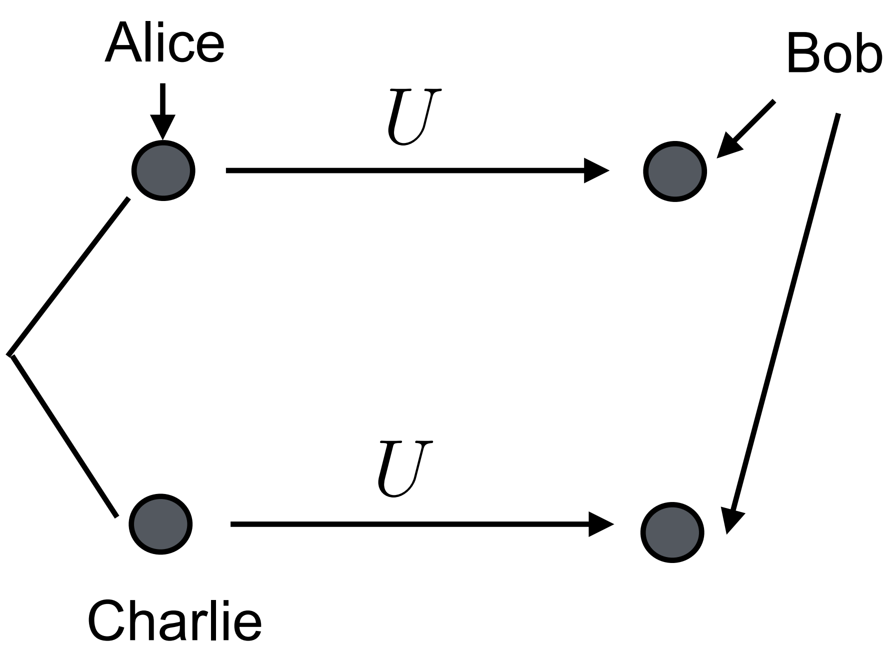
- Alice and Bob are playing a catch ball. Alice may apply some perturbation
- Bob asks Charlie to throw the same ball, and compare it with Alice's ball.

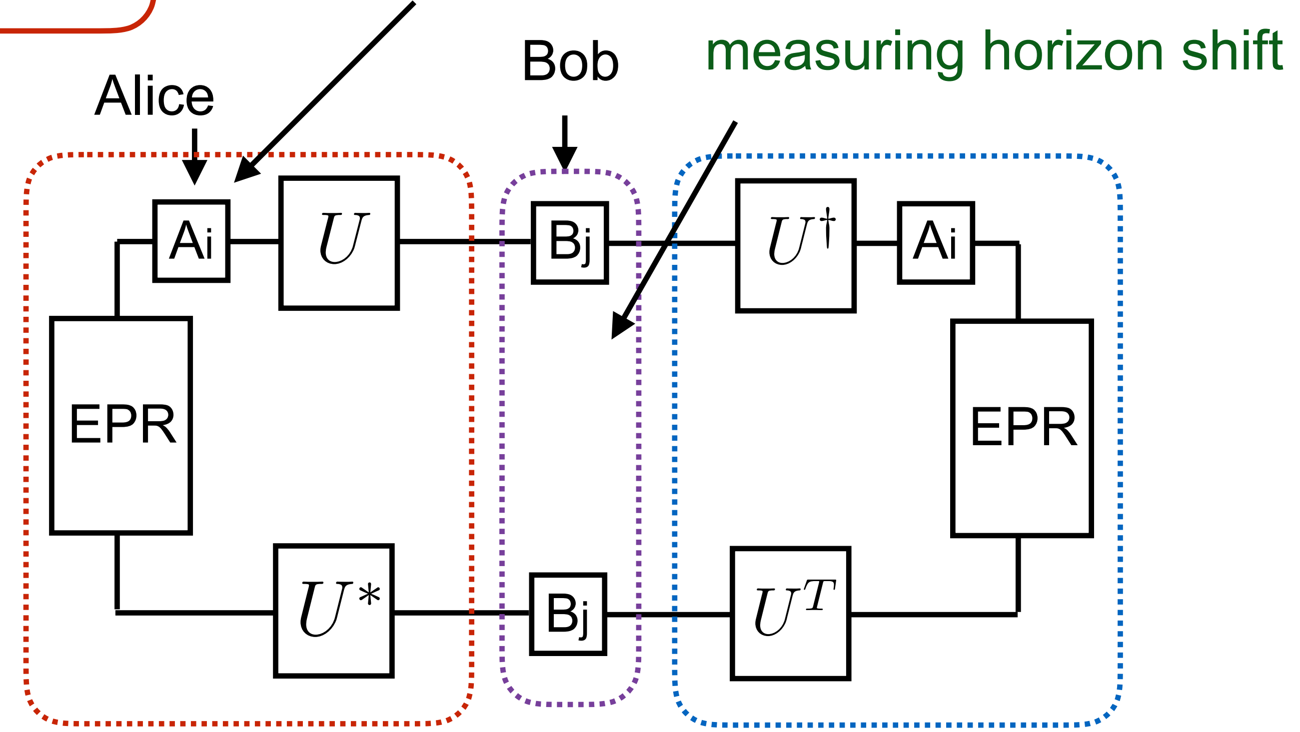
- Alice applies Ai with equal probability. $\longrightarrow$ superoperator $\sum_i A_i(\cdot)A_i^\dagger$
- Bob performs a joint measurement on two systems. $\longrightarrow$ $\sum_j B_j \otimes B_j^\dagger$

SWAP operator

The outcome : $\sum_{i,j} \langle A_i(0)B_j(t)A_i^\dagger(0)B_j^\dagger(t)\rangle$

gravitational shockwave !





[Kitaev, Roberts-BY]

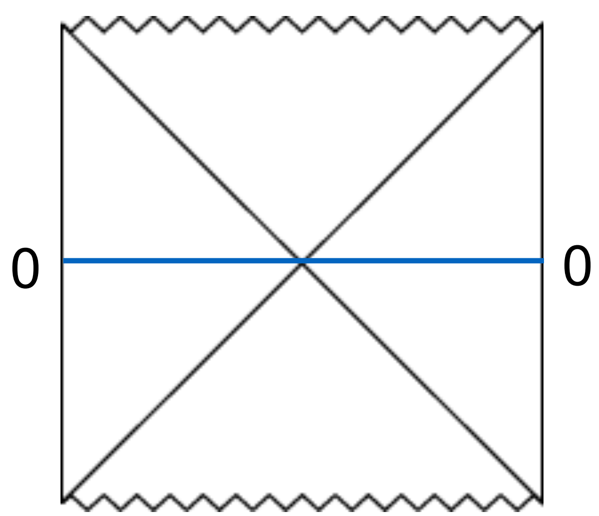
Complexity

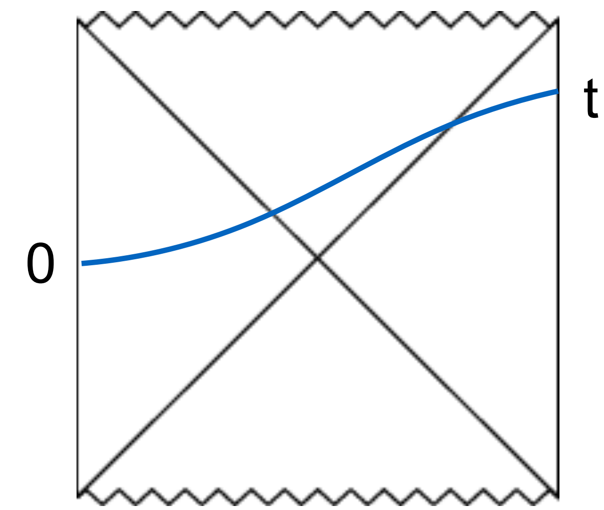
Can we detect complexity growth ?

- The complexity of the TFD state still keeps growing ?

Reference state $\longrightarrow$ Target state

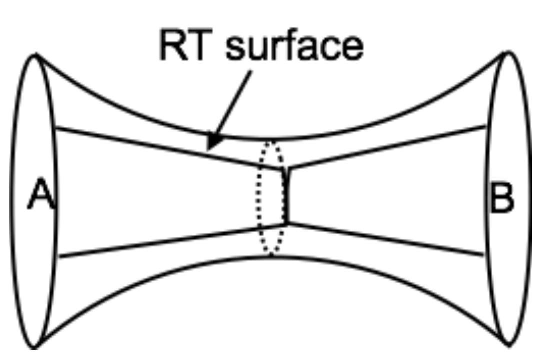
How many quantum gates do we need ?

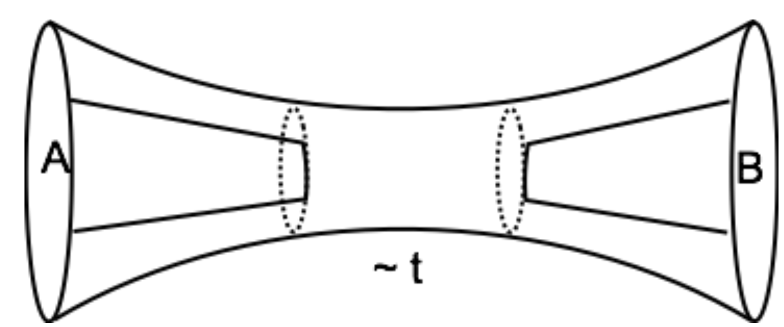




$$|\Psi(0)\rangle = \sum_j e^{-\beta E_j/2} |\psi_j\rangle \otimes |\psi_j\rangle$$

$$|\Psi(t)\rangle = \sum_j e^{-\beta E_j/2} e^{-iE_j t} |\psi_j\rangle \otimes |\psi_j\rangle$$

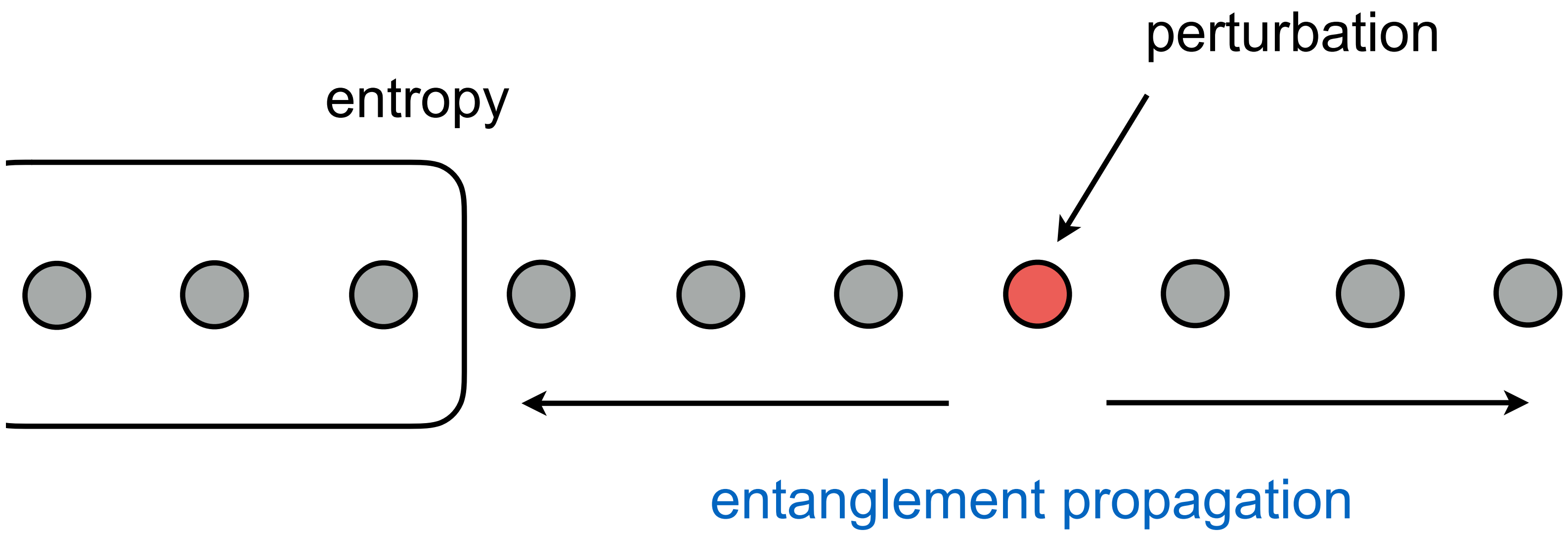




[Hayden-Harlow, Susskind, Brown et al]

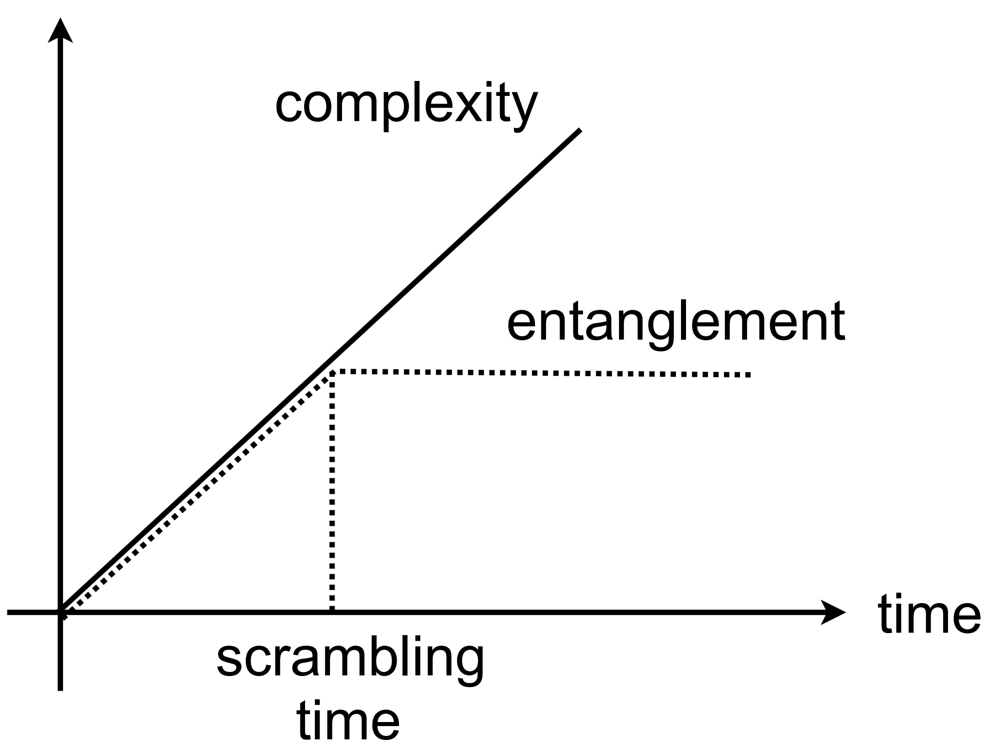
Entanglement can detect complexity ?

- Entropy grows as the complexity grows.



quench dynamics of
local perturbation

- After the scrambling time, entanglement entropies get saturated.



Entanglement is not enough !

[Susskind]

Unitary k-design (complexity of randomness)

- Imagine an ensemble of Haar (uniformly distributed) random unitary (on n qubits)

  A typical operator in the Haar ensemble has exp(n) complexity.

- Consider k copies of the system (kn qubits in total) and consider "k-fold twirl" :

$$\rho \quad \rightarrow \quad K(\rho) = \int dU \; \underbrace{(U \otimes \cdots \otimes U)}_{\text{k copies}} \underbrace{\boxed{\rho}}_{\text{kn qubits}} (U^{\dagger} \otimes \cdots \otimes \underbrace{\boxed{U^{\dagger}}}_{\text{n qubits}})$$

- We think of approximating the Haar random ensemble by some ensembles which are easier to generate.

$$\{p_j, U_j\} \qquad \rho \quad \rightarrow \quad \Phi(\rho) = \sum_j p_j \; (U_j \otimes \cdots \otimes U_j) \; \rho \; (U_j^{\dagger} \otimes \cdots \otimes U_j^{\dagger})$$

- If $\Phi = K$, then $\{p_j, U_j\}$ is said to form a unitary k-design,

  i.e. it is <u>as good as Haar</u> up to k-th moment.

[For an easy introduction, see a recent paper by Zak Webb]

Examples of k-design

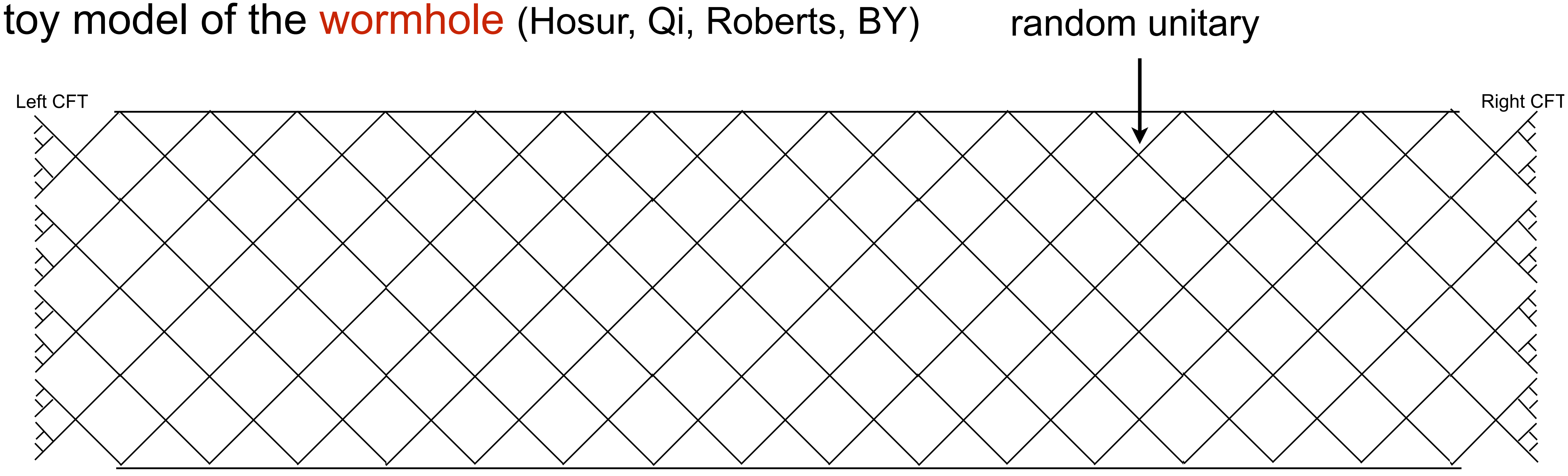
- A group of Pauli operators is 1-design.

  I, X, Y, Z, ZZ, YYZY, ….

- A group of Clifford operators is 2-design.

  Clifford operators can prepare arbitrary stabilizer states (eg perfect tensors).

- A toy model of the wormhole (Hosur, Qi, Roberts, BY)



this object forms an approximate k-design.

[Brandao-Harlow-Horodecki, Hosur-Qi-Roberts-BY, Roberts-BY]

Lower bound on complexity

- Imagine a system of n qubits. (d=2^n states in the Hilbert space).

- If an ensemble of unitary operators formed a k-design, the ensemble must contain at least

$$|\mathrm{Supp}(\mathcal{E})| \geq \begin{pmatrix} d+k-1 \\ k \end{pmatrix}^2 .$$

(due to the Schur-Weyl duality)

- At each step, the number of implementable quantum gates is

$$\simeq gn^2$$

(g: number of different 2-qubit gates)

- In T step, the number of implementable quantum gates is $(gn^2)^T$

- A typical operator in k-design has a complexity of at least

$$\boxed{\frac{2kn\log(2)}{\log(gn^2)}}$$

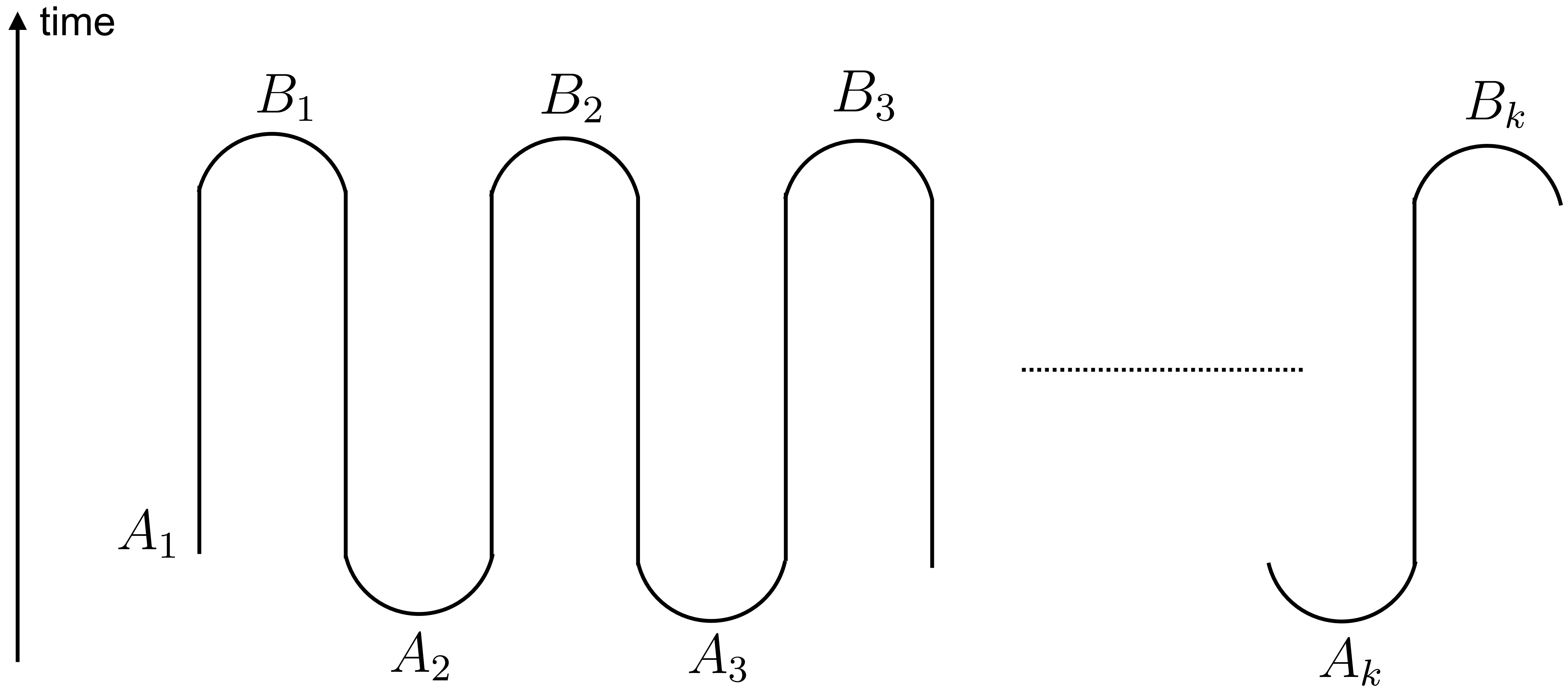
roughly linear in k and n !

[Roberts-BY]

How do we detect the design ?

- Answer : Out-of-time ordered correlation functions
- 2k-point OTO correlators can detect k-design.

$$\langle\ A_1(0)B_1(t)A_2(0)B_2(t)\cdots A_k(0)B_k(t)\ \rangle$$



OTO determines k-fold channel

Consider a k-fold twirl over an arbitrary ensemble $\mathcal{E} = \{p_j, U_j\}$

$$\Phi_{\mathcal{E}}(\rho) = \sum_j p_j (U_j \otimes \cdots \otimes U_j)\rho\underbrace{(U_j^\dagger \otimes \cdots \otimes U_j^\dagger)}_{\text{k copies}}.$$ (quantum channel)

The density matrix can be expanded by Pauli operators, so we are interested in

$$\Phi_{\mathcal{E}}(\boxed{B_1 \otimes \cdots \otimes B_k}) = \sum_{C_1,\ldots,C_k} \boxed{\gamma_{C_1,\ldots,C_k}}(C_1 \otimes \cdots \otimes C_k)$$

Pauli op

knowing these numbers give complete characterization of the channel.

Assume that we know averages of 2k-point OTO correlators for Pauli operators

$$\boxed{\alpha_{A_1,\ldots,A_k}} = \left|\left\langle A_1(0)B_1(t)\cdots A_k(0)B_k(t)\right\rangle_{T=\infty}\right|_{\mathcal{E}} \qquad B_i(t) = UB_jU^\dagger$$

We know these numbers.

<u>Question</u>

Can we determine $\gamma_{C_1,\ldots,C_n}$ from $\alpha_{A_1,\ldots,A_k}$?

Yes

Theorem : OTO and k-fold twirl

Goal : $\alpha_{A_1,\dots,A_k} \to \gamma_{C_1,\dots,C_k}$

$$\alpha_{A_1,\dots,A_k} = \left|\left\langle A_1(0)B_1(t)\cdots A_k(0)B_k(t)\right\rangle_{T=\infty}\right|_{\mathcal{E}}$$

$$\Phi_{\mathcal{E}}(B_1\otimes\cdots\otimes B_k) = \sum_{C_1,\dots,C_k}\gamma_{C_1,\dots,C_k}(C_1\otimes\cdots\otimes C_k)$$

Define :

$$M_{A_1,\dots,A_k}^{C_1,\dots,C_k} = \mathrm{Tr}[A_1C_1\cdots A_kC_k]$$

$$M^{\dagger}{}_{C_1,\dots,C_k}^{A_1,\dots,A_k} = \mathrm{Tr}[C_k^\dagger A_k^\dagger\cdots C_1^\dagger A_1^\dagger],$$

$$\boxed{\gamma_{C_1,\dots,C_k} \propto M^{\dagger}{}_{C_1,\dots,C_k}^{A_1,\dots,A_k}\cdot\alpha_{A_1,\dots,A_k}}$$

Effective “design” of an ensemble

- We need to know OTO values for Haar random (or k-design) in advance.

  This is possible by using some heavy math machineries.

- 4m-point OTO correlation functions that are related to shockwave geometries.

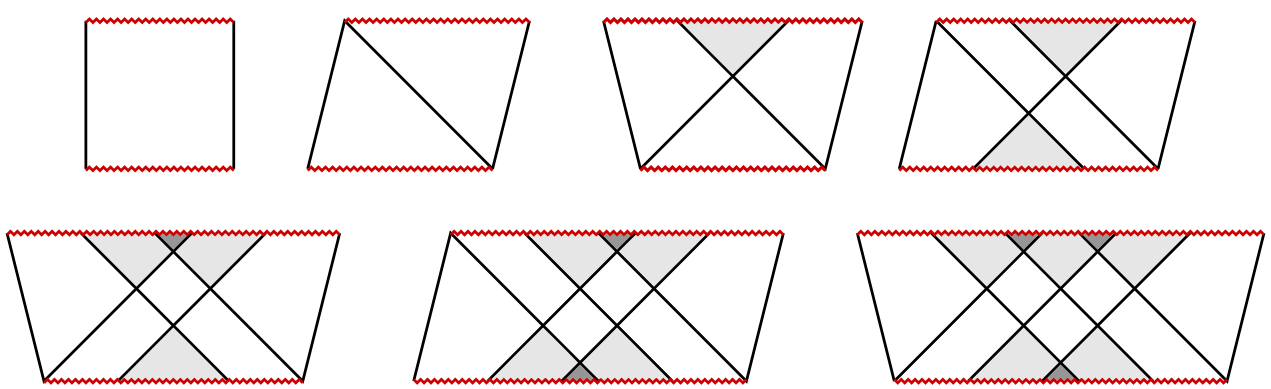
$$\boxed{|\text{4m-point}|_{\text{Haar}} \sim \frac{1}{d^{2m}}}$$

$$|\text{4m-point}|_{\text{Clifford}} \sim \frac{1}{d^2}$$

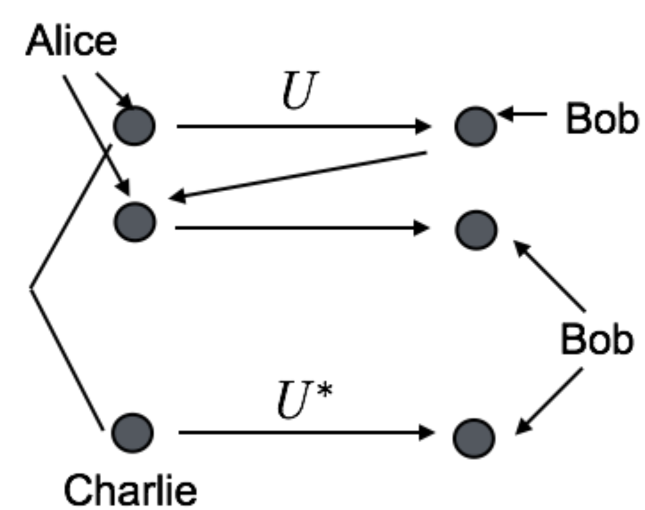
2-design

$$|\text{4m-point}| \sim \frac{1}{d^{\boxed{\mathcal{D}}}}$$

“effective design” of the ensemble



shockwave geometries



catch ball setup

Growth of design in an ER bridge ?

How do we define “design” in an ER bridge ?

- Unitary t-design considers an ensemble of unitary operators.
- Time-evolution of an ER bridge is given by a single Hamiltonian H.

Maybe, we can consider an ensemble of Hamiltonians ?

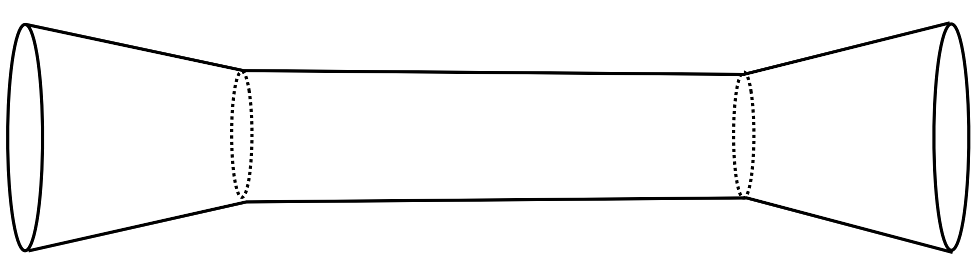
eg) Sachdev-Ye-Kitaev model

random variables

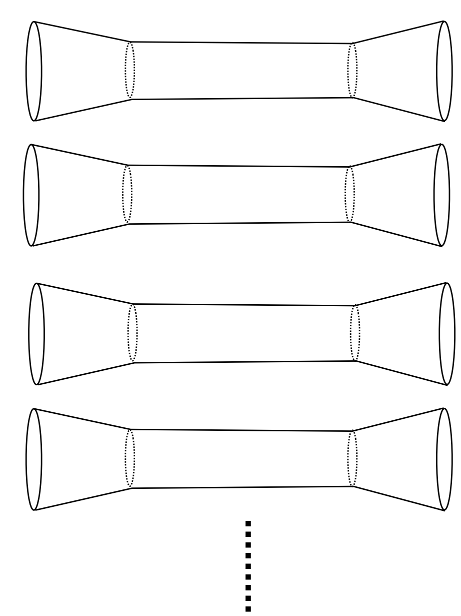
$$H = \frac{1}{4!}\sum_{i,j,k,l=1}^{N} \boxed{J_{ijkl}}\, \chi_i \chi_j \chi_k \chi_l$$

we can compute disorder average analytically

Or, we can imagine very high-energy DOF, which can be integrated out.



H



H_1

H_2

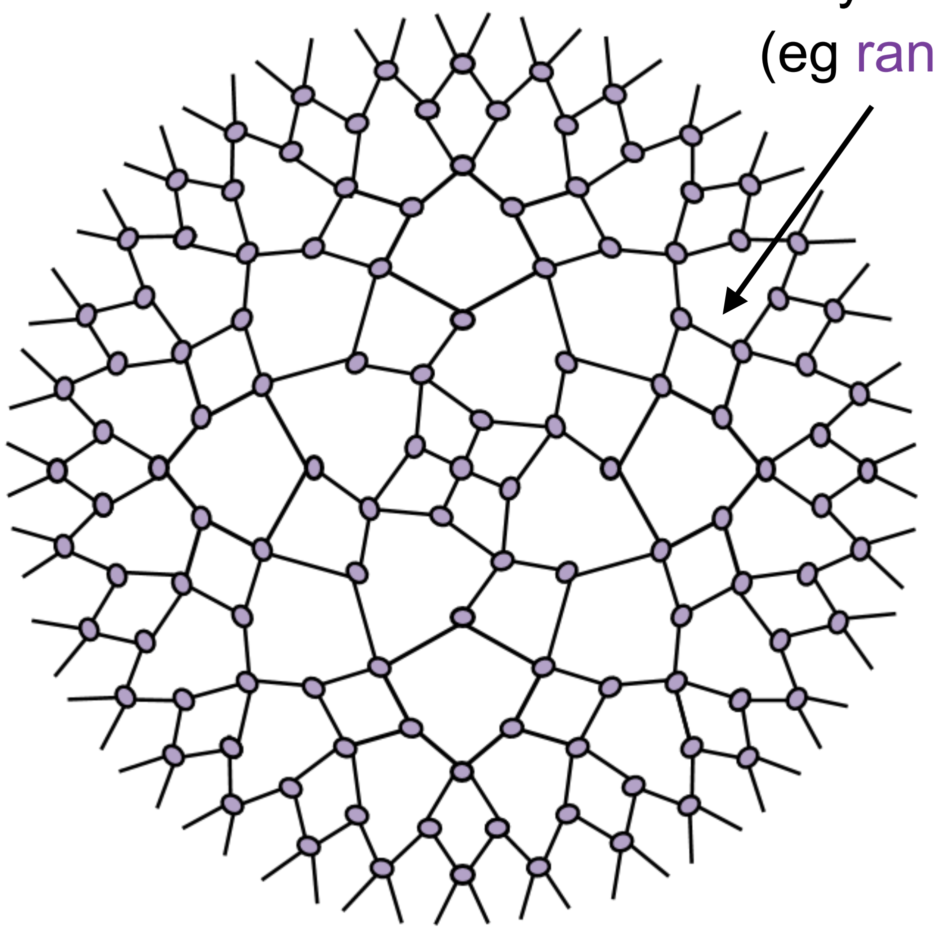
H_3

H_4

Conclusion / Speculation

toy model of AdS/CFT

very entangled tensor
(eg random tensor)



out-of-time ordered correlator

probe of space-time

$$\mathrm{OTO} = \langle A(0)B(t)C(0)D(t)\rangle$$

scrambling / chaos

complexity / design