Study of the $\pi\pi$ scatterings with a combination of all-to-all propagators and the HAL QCD method

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Introduction

Unconventional hadronic resonances (X, Y, Z, $\sigma/f_0(500)$, etc.)

Attempts to interpret them by some models

Multiquark state

Meson-molecule state

- We need to understand them from QCD non-perturbatively
- Methods for studying hadronic resonances from lattice QCD

  … Luscher's method, HAL QCD method

Introduction

Ultimately, we want to understand every hadronic resonance containing exotic ones by using the HAL QCD method

As a first step, we are trying to investigate the ρ meson resonance which emerges in the simplest, $\pi\pi$ scattering

Methods

HAL QCD method: construct an interaction potential from lattice QCD

Basic quantity: The Nambu-Bethe-Salpeter(NBS) wave function

$$\psi_{\mathbf{k}}(\mathbf{r}) = \sum_{\mathbf{x}} \langle 0|\pi(\mathbf{r}+\mathbf{x},0)\pi(\mathbf{x},0)|\pi\pi;\mathbf{k}\rangle \quad |\pi\pi;\mathbf{k}\rangle :$$ $\pi\pi$ scattering state with a relative momentum **k**

$$\left(\nabla^2 + k^2\right)\psi_{\mathbf{k}}(\mathbf{r}) = 0 \ (r > R)$$

$$\psi_l(k,r) \sim \frac{\sin\left(kr - \frac{l\pi}{2} + \delta_l\right)}{kr} \ (r > R)$$

$\left(\nabla^2 + k^2\right.$ ... $)$

Local operator based on the quark model

$$\pi^+(\mathbf{x},t) = \bar{d}(\mathbf{x},t)\gamma_5 u(\mathbf{x},t)$$

$U(\mathbf{r},\mathbf{r}')$: ~~energy independent but non-local potential~~

- faithful to the S-matrix
- depends on a choice of the operator π

- Derivative expansion $\quad U(\mathbf{r},\mathbf{r}') = \left(V_0(r) + V_1(r)\nabla^2 + \ldots\right)\delta(\mathbf{r}-\mathbf{r}')$
- In lattice QCD $\quad F(\mathbf{r},t) \equiv \sum_{\mathbf{x}} \langle \pi(\mathbf{r}+\mathbf{x},t)\pi(\mathbf{x},t)\mathcal{J}_{\pi\pi}(t_0)\rangle = \sum_n A_n \psi_n(\mathbf{r}) e^{-W_n(t-t_0)}$

Methods

Time-dependent HAL QCD method (N.Ishii et al.(2012))

- All of the elastic scattering states share the same potential

$$\left(\nabla^2 + k_n^2\right)\psi_n(\mathbf{r}) = 2\mu \int d^3\mathbf{r}' U(\mathbf{r},\mathbf{r}')\psi_n(\mathbf{r}'),\ \ (n = 0, 1, ..., n_{\rm th})$$

- They are unified into one equation through the "**R-correlator**"

R-correlator $R(\mathbf{r},t) \equiv \dfrac{F(\mathbf{r},t)}{e^{-2m_\pi t}} = \sum_n A_n \psi_n(\mathbf{r}) e^{-\Delta W_n t}$

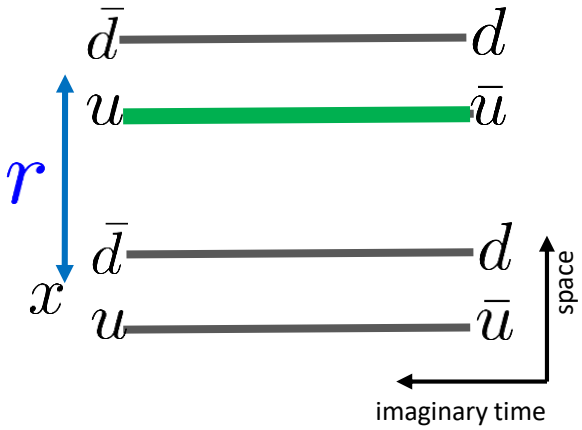
If we can neglect inelastic contributions, it satisfies

$$\left[\frac{\nabla^2}{m_\pi} - \frac{\partial}{\partial t} + \frac{1}{4m_\pi}\frac{\partial^2}{\partial t^2}\right] R(\mathbf{r},t) = \int d^3\mathbf{r}' U(\mathbf{r},\mathbf{r}') R(\mathbf{r}',t)$$

- ✓ We can obtain a reliable potential **at an early time** ($\mathcal{O}(e^{-\Delta W_1 t}) \gg \mathcal{O}(e^{-\Delta W_{\rm inela} t})$)
- ✓ We can use **all of the elastic states** to construct the potential

Methods

Difficulty in the calculation of the $I = 1$ $\pi\pi$ scattering



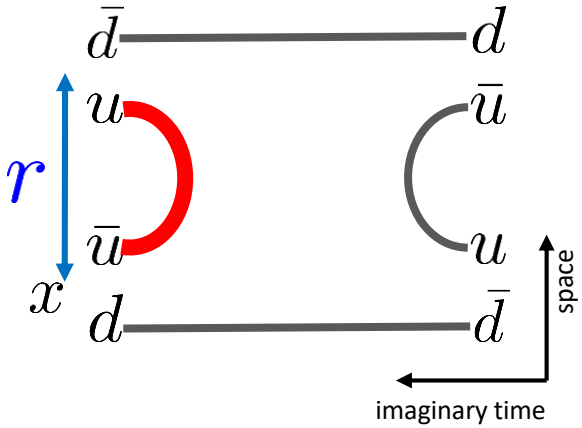
➢ Typical calculation (point-to-all propagators)

Solve the equation below for fixed x_0

$$D(x;y)\psi(y) = \delta_{x,x_0}$$

Then ψ is a propagator from fixed x_0 to every x

$$\psi(x) = D^{-1}(x;x_0)$$



➢ All-to-all propagators

A propagator from every point to every point

Naively, we need to calculate the point-to-all propagator N_{vol} times

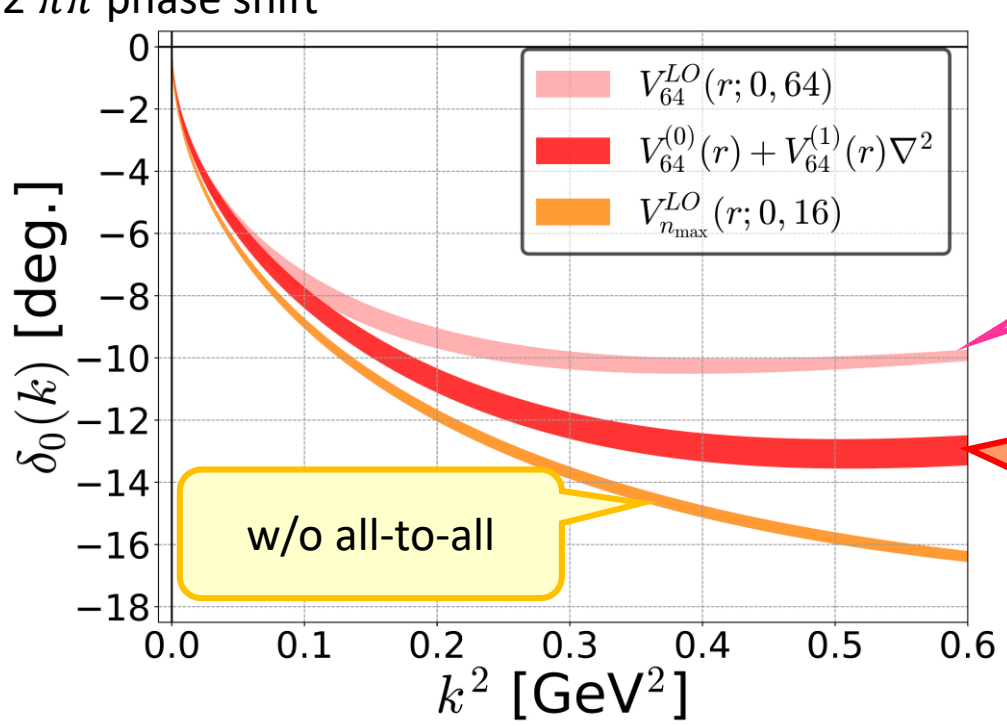
$\mathcal{O}(10^6) \sim \mathcal{O}(10^8)$

Naïve calculation is not realistic
Need for some approximations

Methods

Previous study: HAL QCD+LapH (D.Kawai et al. (2018))

I=2 $\pi\pi$ phase shift



Large deviation from a yellow line

w/o all-to-all

Approaching to a yellow line thanks to the 2nd derivative term

- All operators become non-local automatically due to the LapH method
  => **contributions from higher derivative terms are enhanced**

$$U(\mathbf{r},\mathbf{r}') = \left(V_0(r) + V_1(r)\nabla^2 + \ldots\right)\delta(\mathbf{r}-\mathbf{r}')$$

Methods

All-to-all method keeping the locality of operators:

The hybrid method (J.Foley et al.(2005))

Calculate a propagator approximately with eigenmodes of $H = \gamma_5 D$ and noisy estimators

➢Spectral decomposition of the propagator with eigenmodes of H

$$D^{-1} = \sum_{i=0}^{N_{\max}-1} \frac{1}{\lambda_i} v^{(i)} \otimes v^{\dagger(i)} \gamma_5$$

Practically, It is impossible to calculate all of the eigenmodes

Calculate a part of the propagator by N_{eig} low-lying eigenmodes

$$D_0^{-1} = \sum_{i=0}^{N_{\mathrm{eig}}-1} \frac{1}{\lambda_i} v^{(i)} \otimes v^{\dagger(i)} \gamma_5$$

Methods

- Remaining parts are estimated by noisy estimators

Noise vector η:

$$\begin{aligned} \langle \eta_n \otimes \eta_m^\dagger \rangle &= \delta_{nm} \\ |\eta_n|^2 &= 1 \end{aligned}$$

Solve an equation,

$$H\psi = \mathcal{P}_1 \eta$$

P_1: a projection operator for remaining parts

$$\langle \psi \otimes \eta^\dagger \rangle \gamma_5 = H^{-1} \mathcal{P}_1 \langle \eta \otimes \eta^\dagger \rangle \gamma_5 = H^{-1} \mathcal{P}_1 \gamma_5 = D^{-1} - D_0^{-1}$$

The expectation value is estimated by an average over independent noise vectors

Additional errors are introduced from the noisy estimator

Methods

- Noise reduction technique: dilution

Decompose a noise vector $\eta_{[r]}$ into linearly independent vectors $\eta_{[r]} = \sum_{j=0}^{N_{\rm dil}-1} \eta_{[r]}^{(j)}$

Example: color dilution in a calculation of $M^{-1}_{a\alpha,b\beta}(x,y)$ $\quad \eta^{(i)}_{a\alpha}(x) = 0$ If $i \neq a$

w/o dilution

$$\frac{1}{N_{\rm r}} \sum_r \left(M^{-1} \eta_{[r]} \otimes \eta^{\dagger}_{[r]} \right)_{a\alpha,b\beta} (x,y) = M^{-1}_{a\alpha,b\beta}(x,y) + \sum_{(c,\gamma,z)\neq(b,\beta,y)} C_{c,\gamma,z} M^{-1}_{a\alpha,c\gamma}(x,z)$$

Noise contamination from (c, γ, z)

w/ dilution

$$\frac{1}{N_{\rm r}} \sum_r \left(M^{-1} \eta^{(i)}_{[r]} \otimes \eta^{(i)\,\dagger}_{[r]} \right)_{a\alpha,b\beta} (x,y) = \frac{1}{N_{\rm r}} \sum_r \left(M^{-1} \eta^{(b)}_{[r]} \otimes \eta^{(b)\,\dagger}_{[r]} \right)_{a\alpha,b\beta} (x,y)$$

$$= M^{-1}_{a\alpha,b\beta}(x,y) + \sum_{(\gamma,z)\neq(\beta,y)} C'_{\gamma,z} M^{-1}_{a\alpha,b\gamma}(x,z)$$

Noise contamination is reduced thanks to color dilution (color index is fixed to b)

In our study

Color: full Time: full or J-interlace

Spin: full Space: none, even/odd, etc.

Results

Simulation details

- 2+1 flavor QCD configurations (CP-PACS+JLQCD, $a = 0.1214$[fm], $16^3 \times 32$)
- $m_\pi \approx 870$ MeV, $m_\rho \approx 1230$ MeV ($\boldsymbol{E_0 = -510}$ **MeV**)
- Calculations are held on Cray XC40 (YITP) and HOKUSAI Big-Waterfall (RIKEN)

Results

- I=2 $\pi\pi$ S-wave scattering

  Investigation into effectiveness of the hybrid method with the HAL QCD method

  We can compare our results with ones obtained without all-to-all propagators

- I=1 $\pi\pi$ P-wave scattering (preparatory calculation)

  Test calculation for the system containing quark annihilation diagrams with the hybrid method

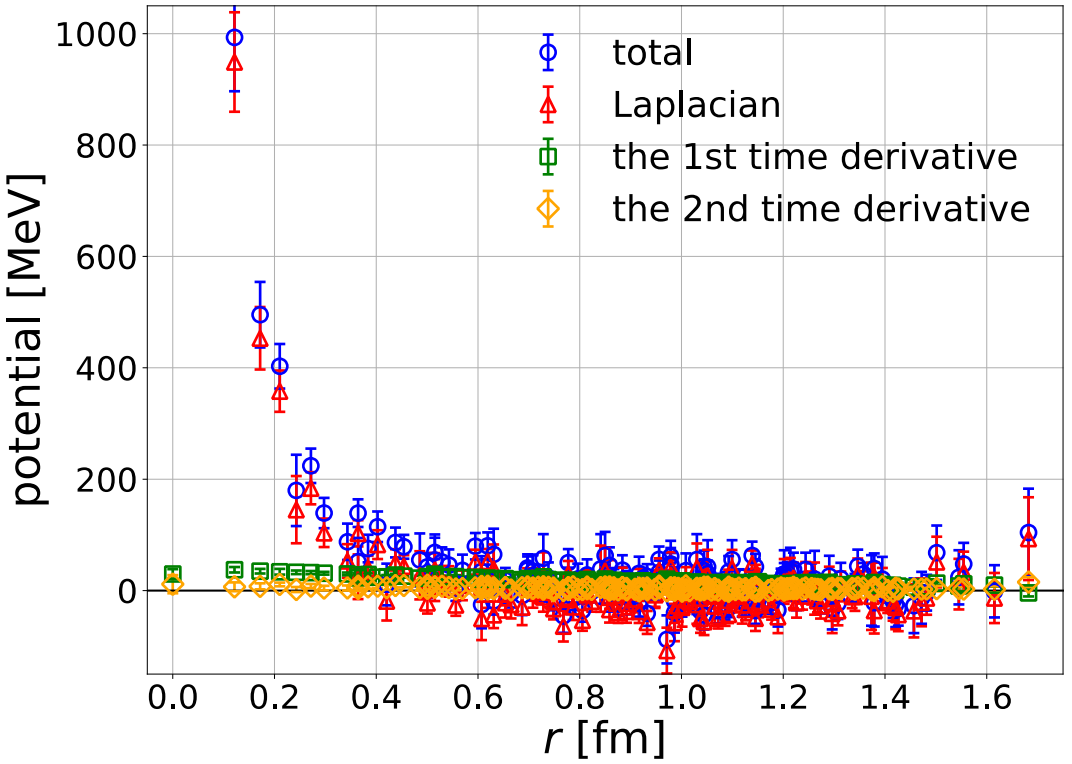
  We use a ρ shape source operator $\mathcal{J}_{\pi\pi}(t_0) = \bar{\rho}_i(t_0) = \frac{1}{\sqrt{2}}\left(\bar{u}\gamma_i u - \bar{d}\gamma_i d\right)(t_0)$

Result 1: I=2 $\pi\pi$ S-wave scattering

Result 1: I=2 $\pi\pi$ S-wave scattering

Behavior of the potential

- Bulk behavior of the potential are consistent with one without all-to-all propagators
- Statistical errors are enhanced due to the additional noise contamination
- The contamination mainly comes from the Laplacian part
  -> noise reductions in spatial directions are important

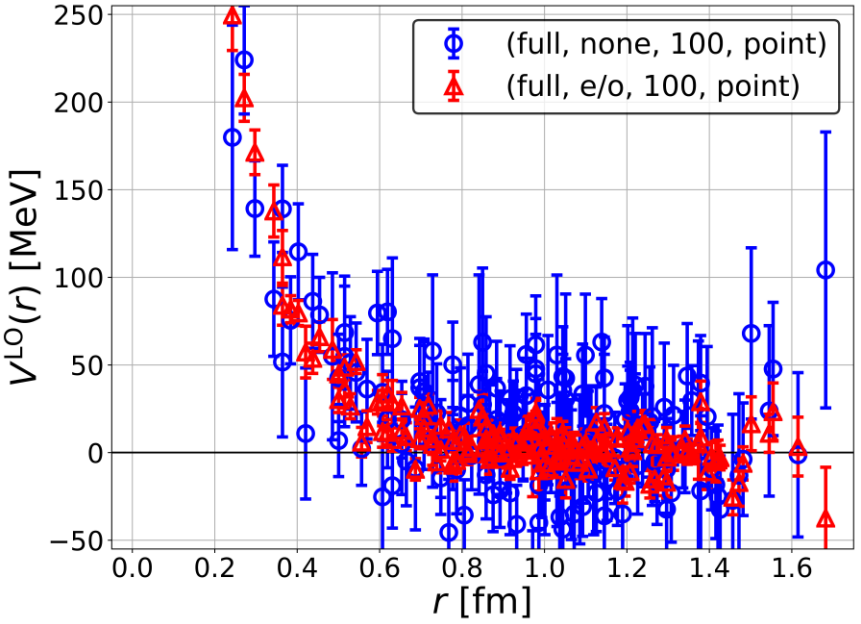


$$V^{\mathrm{LO}}(r) = \frac{\left[\frac{\nabla^2}{m_\pi} - \frac{\partial}{\partial t} + \frac{1}{4m_\pi}\frac{\partial^2}{\partial t^2}\right] R(\mathbf{r},t)}{R(\mathbf{r},t)}$$

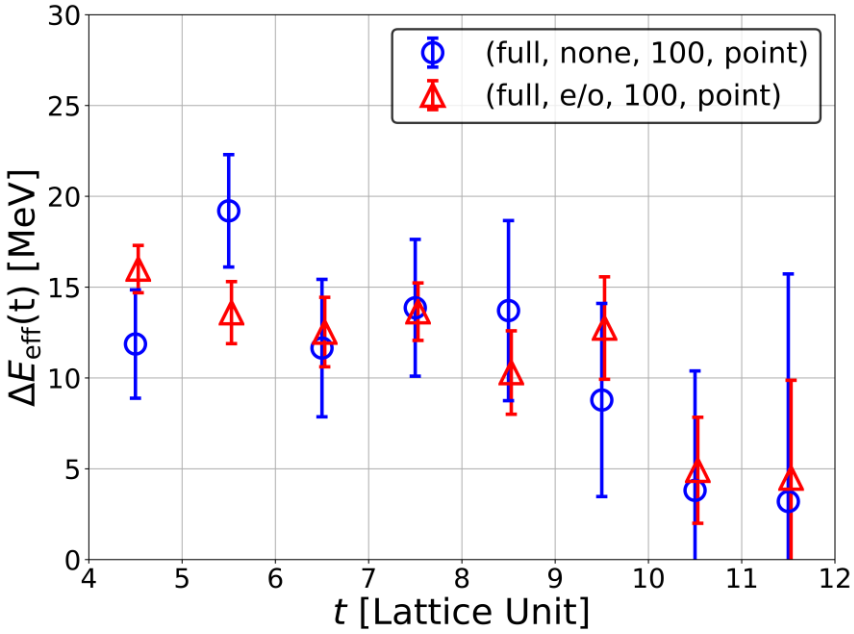
Result 1: I=2 $\pi\pi$ S-wave scattering

Importance of spatial dilutions

Potential



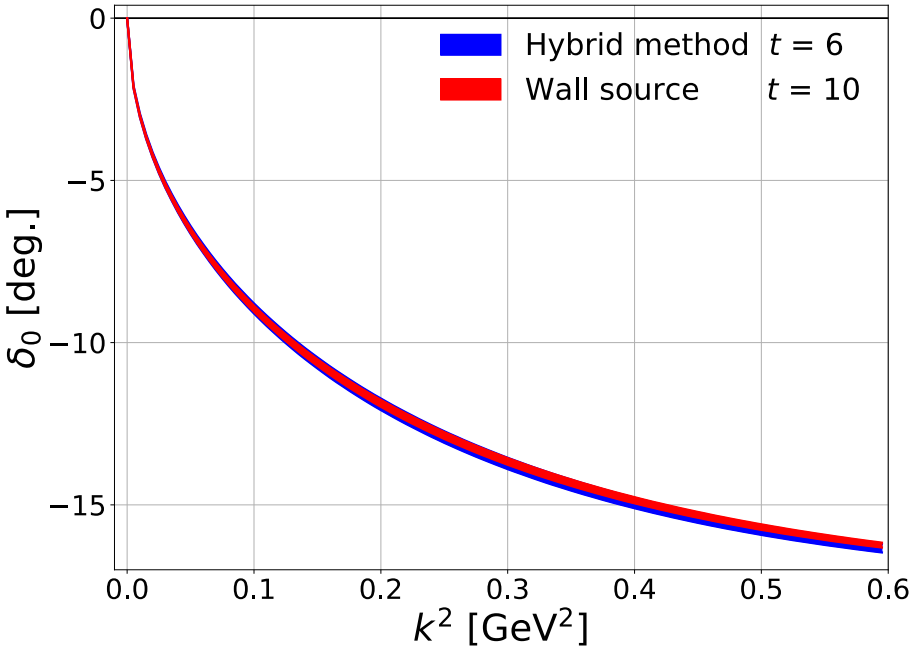
Energy shift $\Delta E_{\pi\pi} = E_{\pi\pi} - 2m_{\pi}$

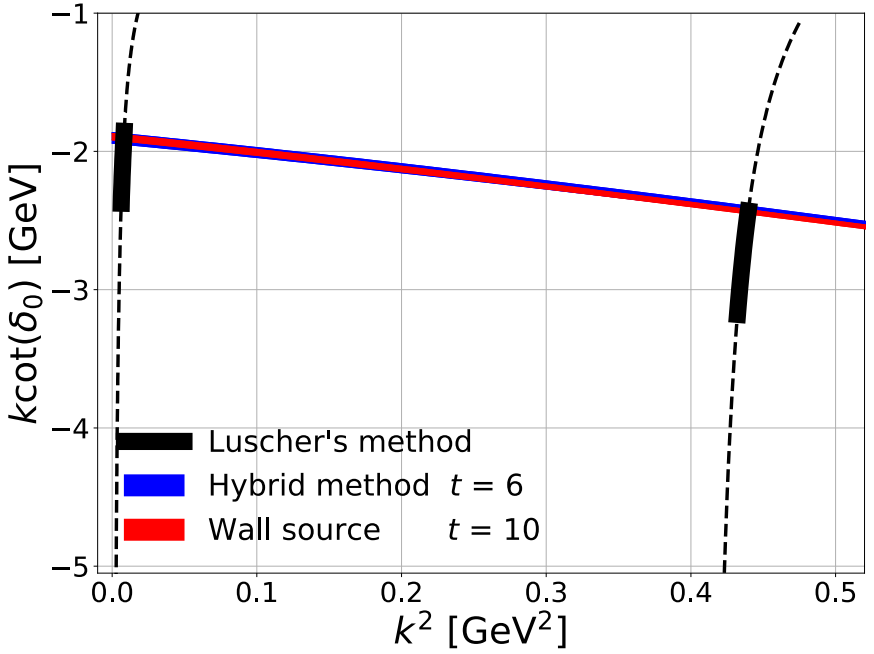


- Cancellation among different spatial points occurs in energy shift calculation
- **Fine spatial dilution is crucial, especially for the HAL QCD method**

Result 1: I=2 $\pi\pi$ S-wave scattering

Consistency check with results without all-to-all propagators





Results with the hybrid method are reasonable

Result 2: I=1 P-wave $\pi\pi$ scattering

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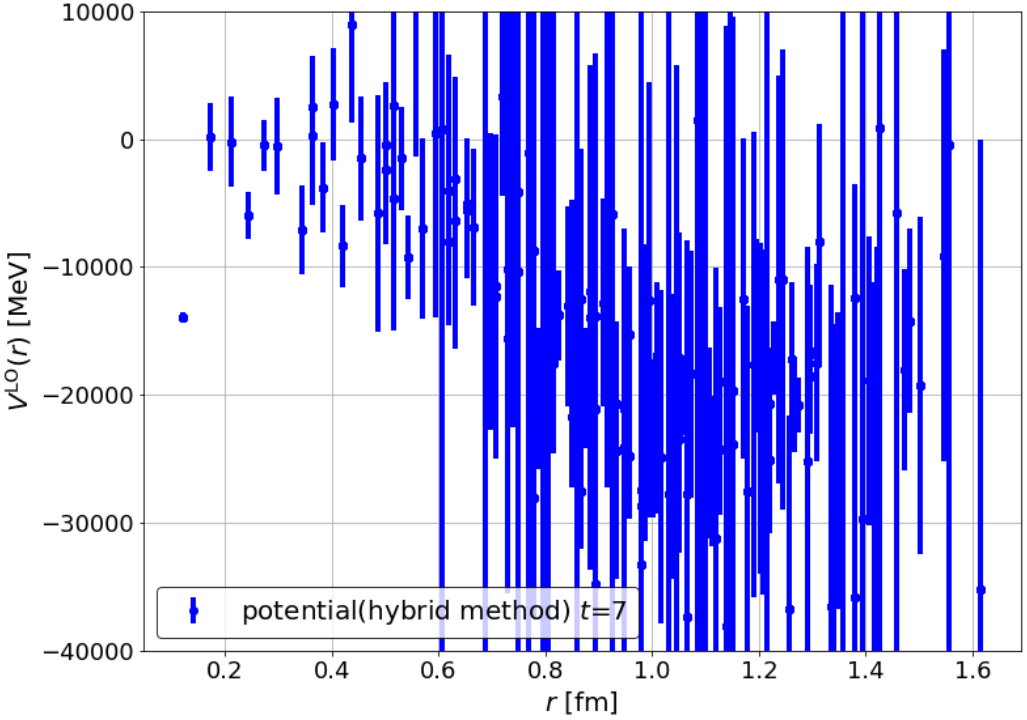
Potential with the same setup as the $I = 2$ calculation



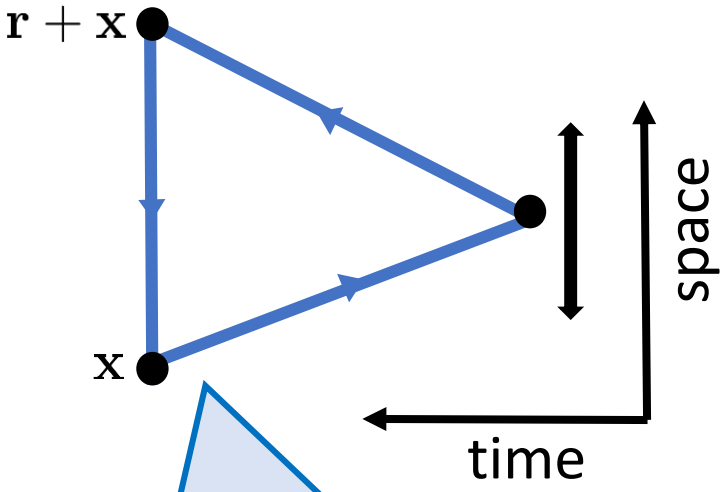
Extremely large statistical errors
(due to the quark annihilation diagrams)

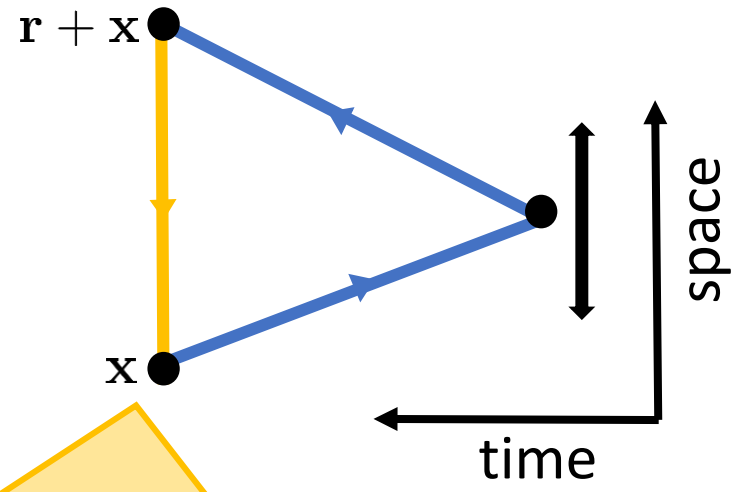
More noise reductions are needed

Result 2: I=1 P-wave $\pi\pi$ scattering

Efforts of noise reductions

1. Changing dilution setups for each propagators

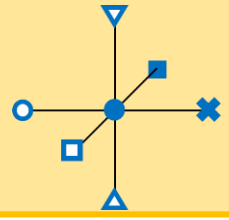




All of the propagators share the same dilution setup

Finer spatial dilution for this part

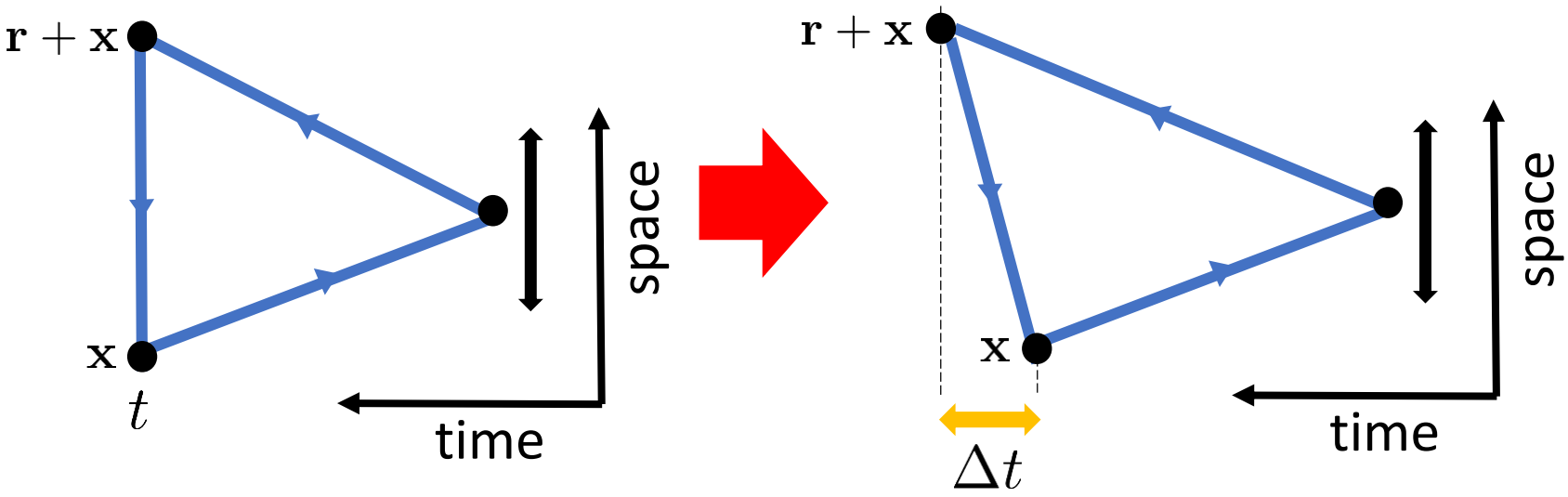
To enable us to use as independent diluted vectors as possible in Laplacian calculation



Result 2: I=1 P-wave $\pi\pi$ scattering

Efforts of noise reductions

2. Taking the different-time scheme for the NBS wave function



- Motivated by the fact that there is no equal-time propagation in the $I = 2$ case
- Note: potentials depend on the scheme we choose, but **physical quantities are independent of it**

3. Taking an average over the noise vectors

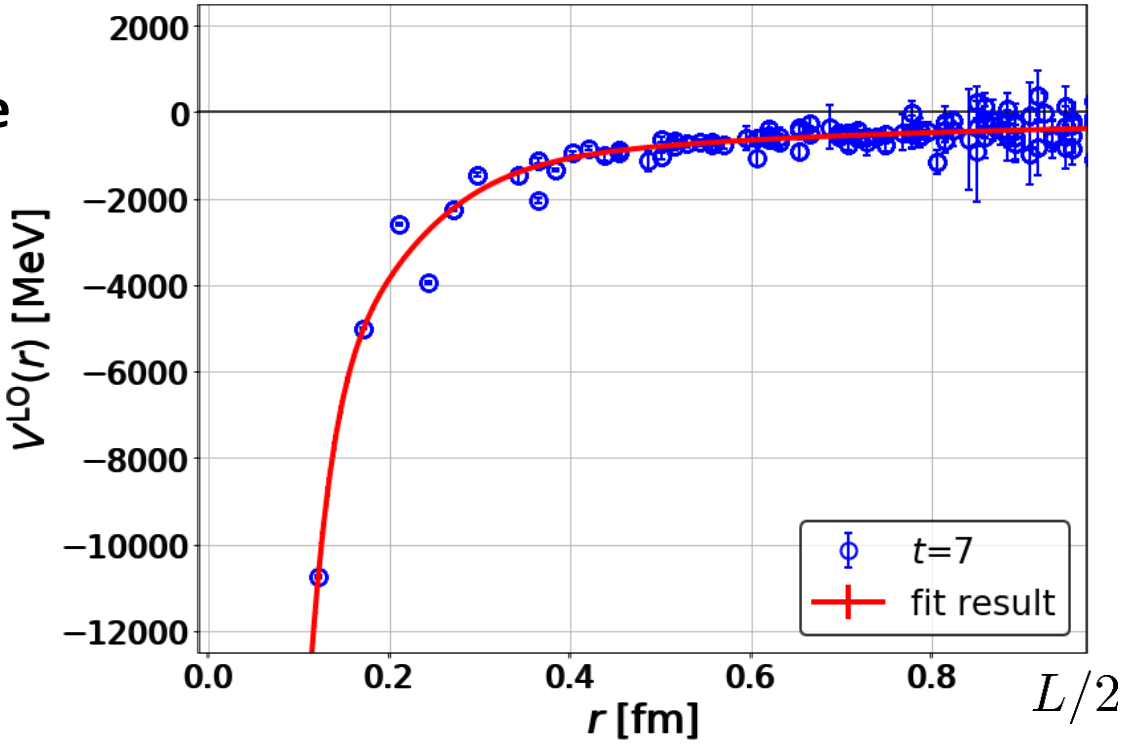
Result 2: I=1 P-wave $\pi\pi$ scattering

Resultant potential and binding energy

Very strong attractive force

Ground state energy

$E_0 = -453 \pm 9$ [MeV]



- A bound state **exists** (related to ρ meson)
- Long-tail structure … need for considering finite volume effects in fitting

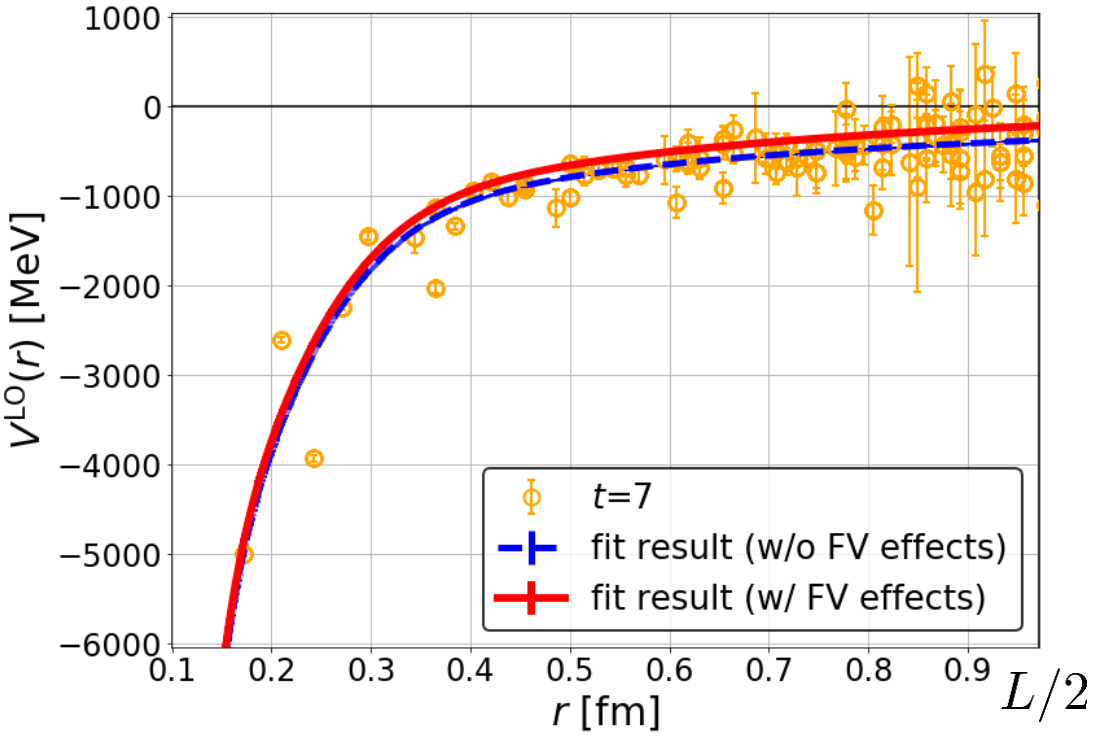
Result 2: I=1 P-wave $\pi\pi$ scattering

Finite volume effects in the potential fitting

$$V_{\mathrm{FV}}(\mathbf{r}) = V(\mathbf{r}) + \sum_{\mathbf{n}\in\mathcal{N}} V(\mathbf{r}+L\mathbf{n}),\ \mathcal{N} = \{\mathbf{n}|\mathbf{n} = (\pm1,0,0),(0,\pm1,0),(0,0,\pm1)\}$$

Ground state energy

$E_0 = -374 \pm 16$ [MeV]



- Smaller binding energy than that from the naïve fitting (previous slide)

Result 2: I=1 P-wave $\pi\pi$ scattering

Comparison with the expected g.s. energy

Our result (w/ FV effects)

$E_0 \approx -370\ \mathrm{MeV}$

$\neq$

Expected g.s. energy

$E_0 \approx -510\ \mathrm{MeV}$

Possible origins of this difference

- Interaction does not fit in a box (R > L/2) … reliable calculation is hard
- Leading-order potential is not a good approximation for this system
- Systematic errors from the fitting?

Summary

- As a first step for future resonance studies, we study the $\pi\pi$ scatterings with the HAL QCD method + the hybrid method
- From the I=2 calculation, It is confirmed that we can obtain meaningful results with the hybrid method
- In the I=1 calculation, we see that noise contamination becomes large due to the quark annihilation diagrams
- Thanks to the additional noise reductions, we get a precise potential enough to calculate the binding energy, and we obtain $E_0 \approx -370$ MeV

Future work

- ρ meson resonance study

  We have to improve our method to reduce numerical costs
- Further studies of hadronic resonances

  I=0 $\pi\pi$ scattering($\sigma/f_0(500)$), other meson-meson systems

Backup

What dilution really does

Consider a noise vector $\eta = (1,1,1,1,1,1)$

Without the dilution,

$$\eta \otimes \eta^{\dagger} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\langle \eta \otimes \eta^{\dagger} \rangle = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Diluted vectors : $\eta^{(1)} = (1,1,0,0,0,0), \eta^{(2)} = (0,0,1,1,0,0), \eta^{(3)} = (0,0,0,0,1,1)$

Remaining noise contamination

$$\sum_{j=1}^{3} \eta^{(j)} \otimes \eta^{(j)\dagger} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Block off-diagonal noise contamination becomes exactly 0

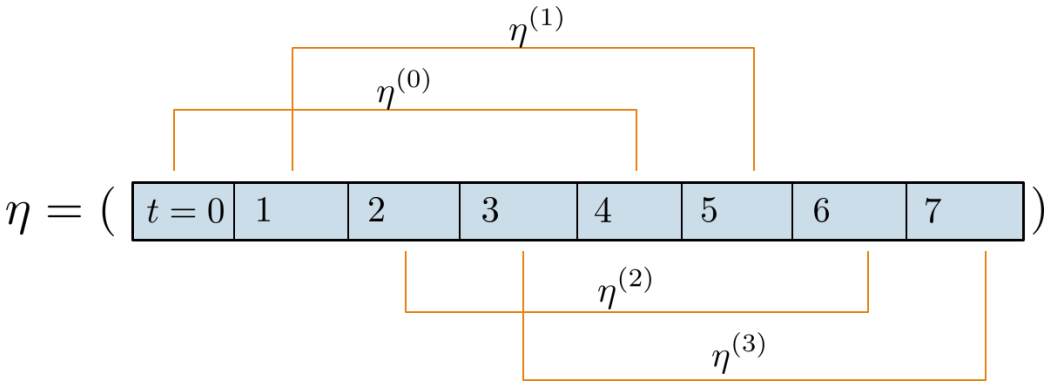
Details of dilutions

- J-interlace time dilution

$$\eta = \sum_{i=0}^{J-1} \eta^{(i)}$$

$$\eta^{(i)}(\mathbf{x}, t) = 0 \text{ if } i \neq t \bmod J$$

Schematically, $(L_t = 8$, 4-interlace$)$



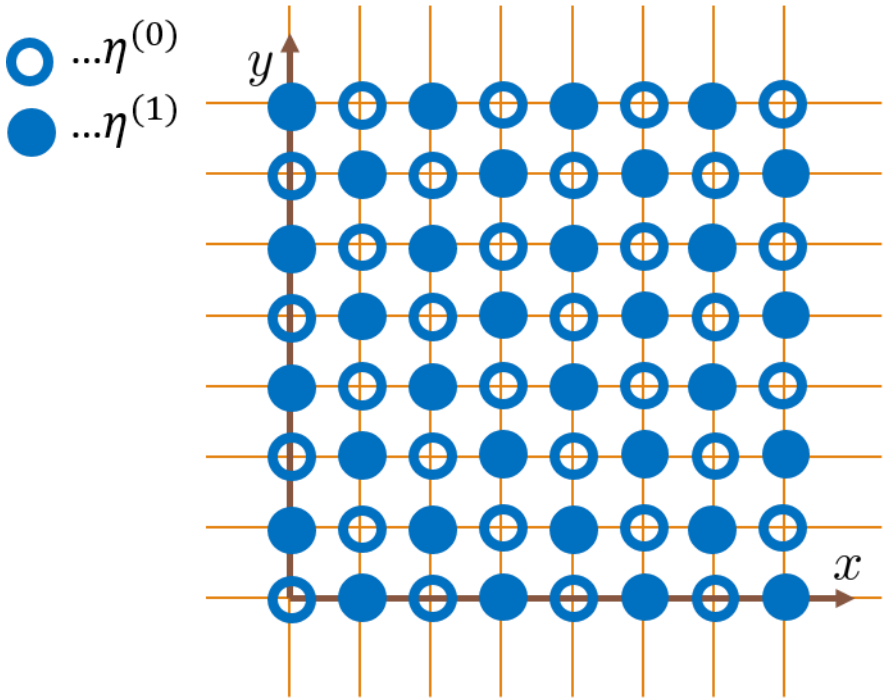
Details of dilutions

- Space-even/odd dilution

$$\eta = \sum_{i=0}^{1} \eta^{(i)}$$

Decompose into two vectors by an even/odd parity of $n_x + n_y + n_z$

On $z = 0$ surface,

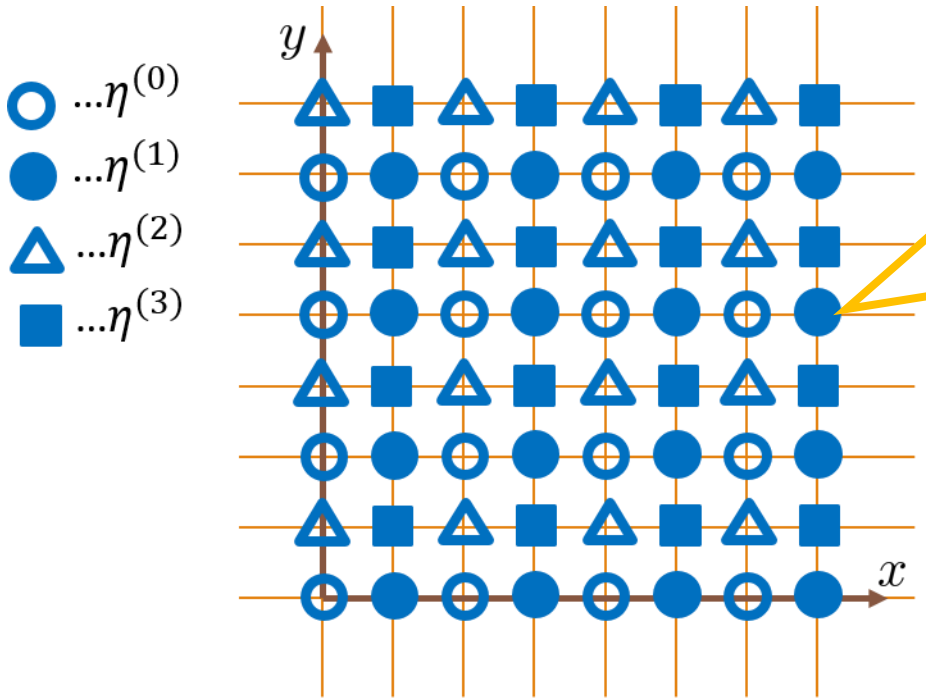


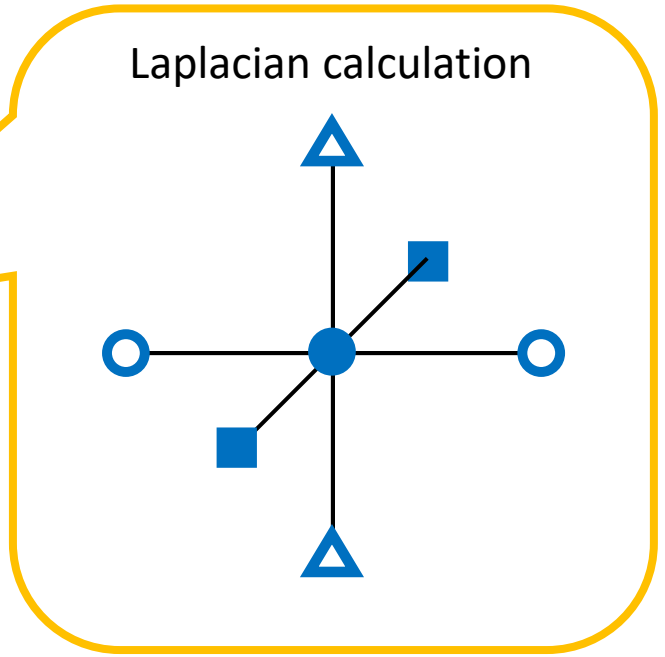
Details of dilutions

- Space-4 dilution

$$\eta = \sum_{i=0}^{3} \eta^{(i)}$$

On $z = 0$ surface,



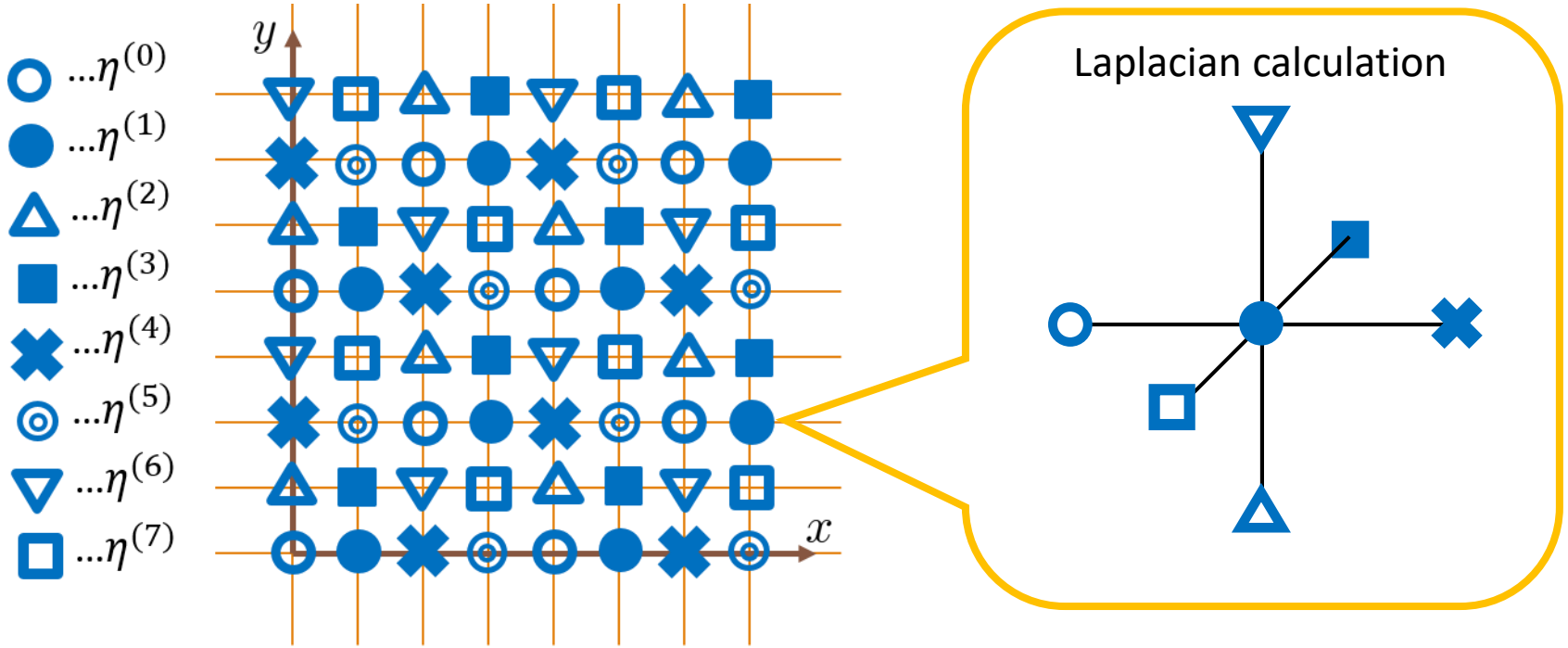


Details of dilutions

- Space-8 dilution

$$\eta = \sum_{i=0}^{7} \eta^{(i)}$$

On $z = 0$ surface,



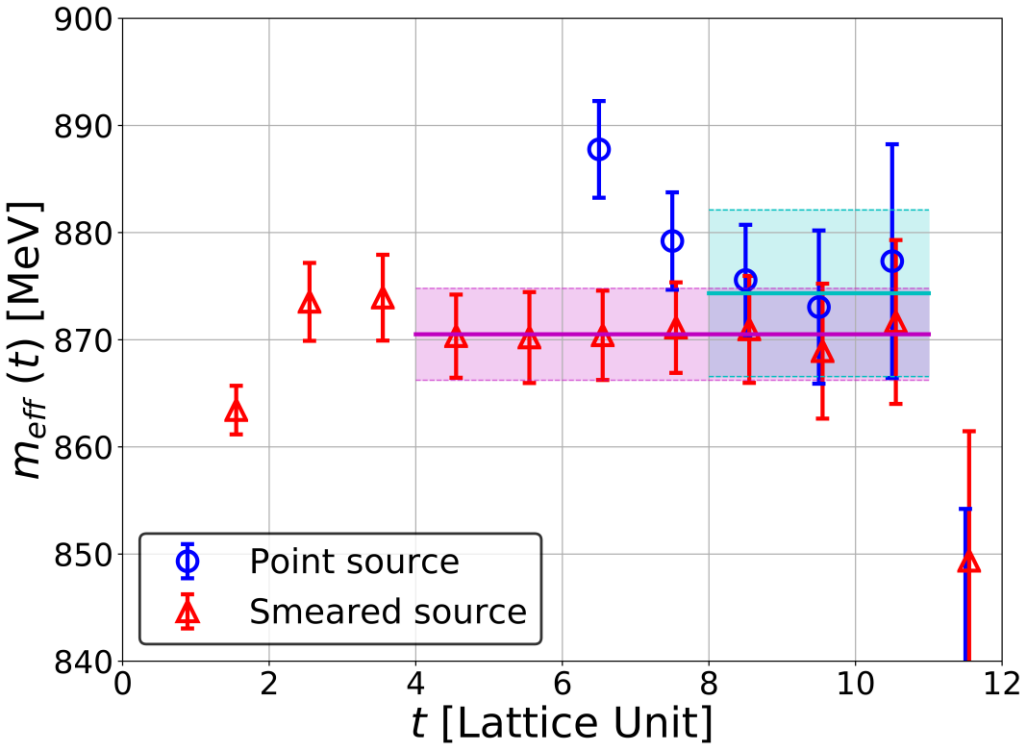
Details of calculations

- I=2 $\pi\pi$ calculation
  - 16-interlace time, full color, full spin, 4-space dilution
  - Neig = 100
  - Smearing: exponential smearing with the Coulomb gauge
  - #. of confs: 60 (60 x 32 time slices) for consistency check,
    20 (20 x 32 time slices) for studies of systematics
- I=1 $\pi\pi$ calculation
  - Using different-time scheme ($\Delta t = 1$ in Lattice Unit)
  - 16-interlace time, full color, full spin, space-4 (src to sink)
  - 4-interlace time, full color, full spin, space-8 * even/odd (sink to sink)
  - Neig = 100
  - Smearing: exponential smearing with the Coulomb gauge
  - #. of confs: 60 (statistics: 60 x 32 time slice)
  - #. of noise samples: 24

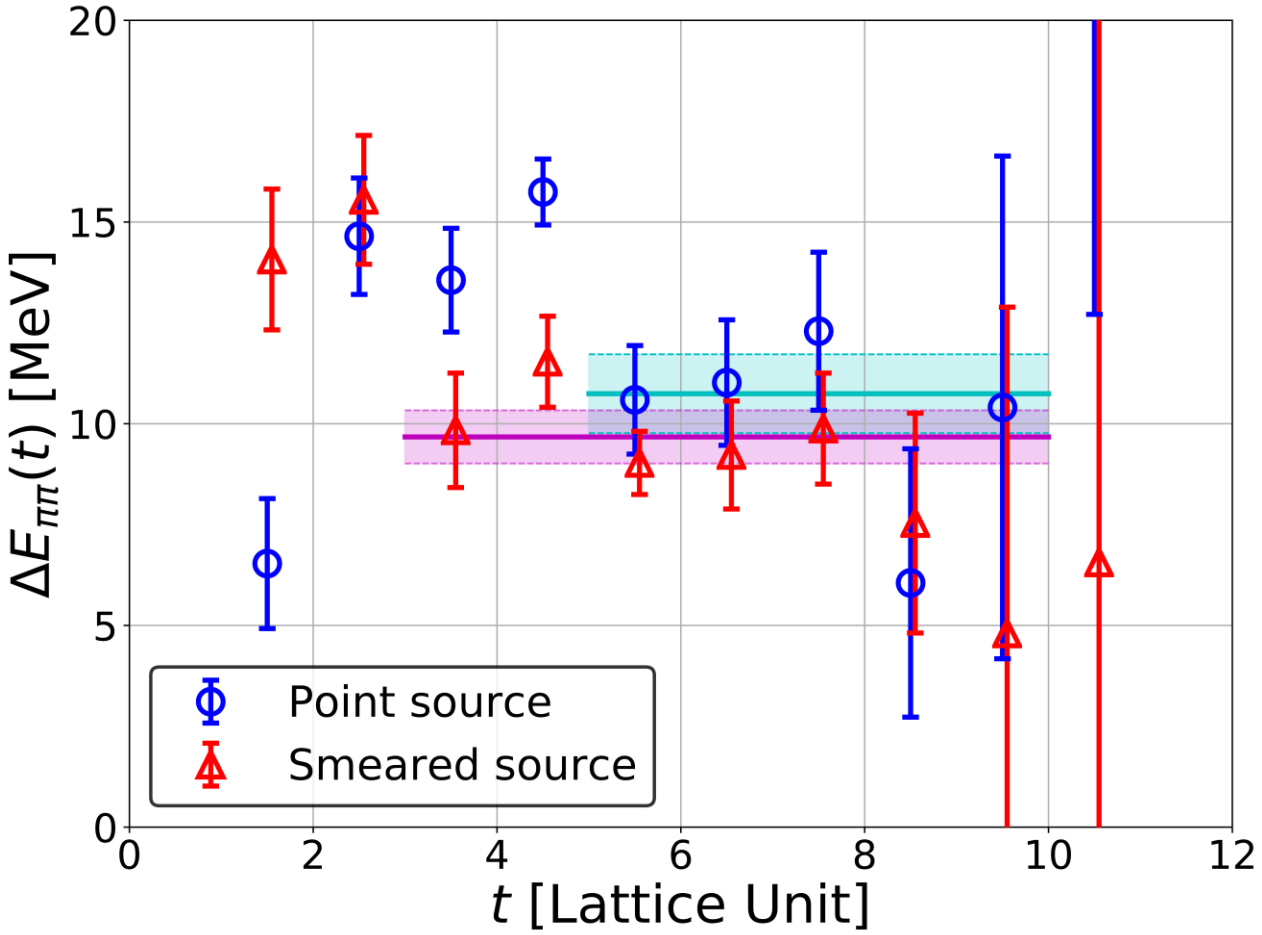
Exponential smearing

$$q_s(\mathbf{x},t) = \sum_{\mathbf{y}} f(\mathbf{x}-\mathbf{y})q(\mathbf{y},t), \qquad f(\mathbf{x}) = \begin{cases} ae^{-b|\mathbf{x}|} & (\ 0 < |\mathbf{x}| < (L-1)/2\ ) \\ 1 & (\ |\mathbf{x}| = 0\ ) \\ 0 & (\ |\mathbf{x}| \geq (L-1)/2\ ) \end{cases}$$

We take a=1.0, b=0.47 (lattice unit) to get a plateau of pion mass at an early time

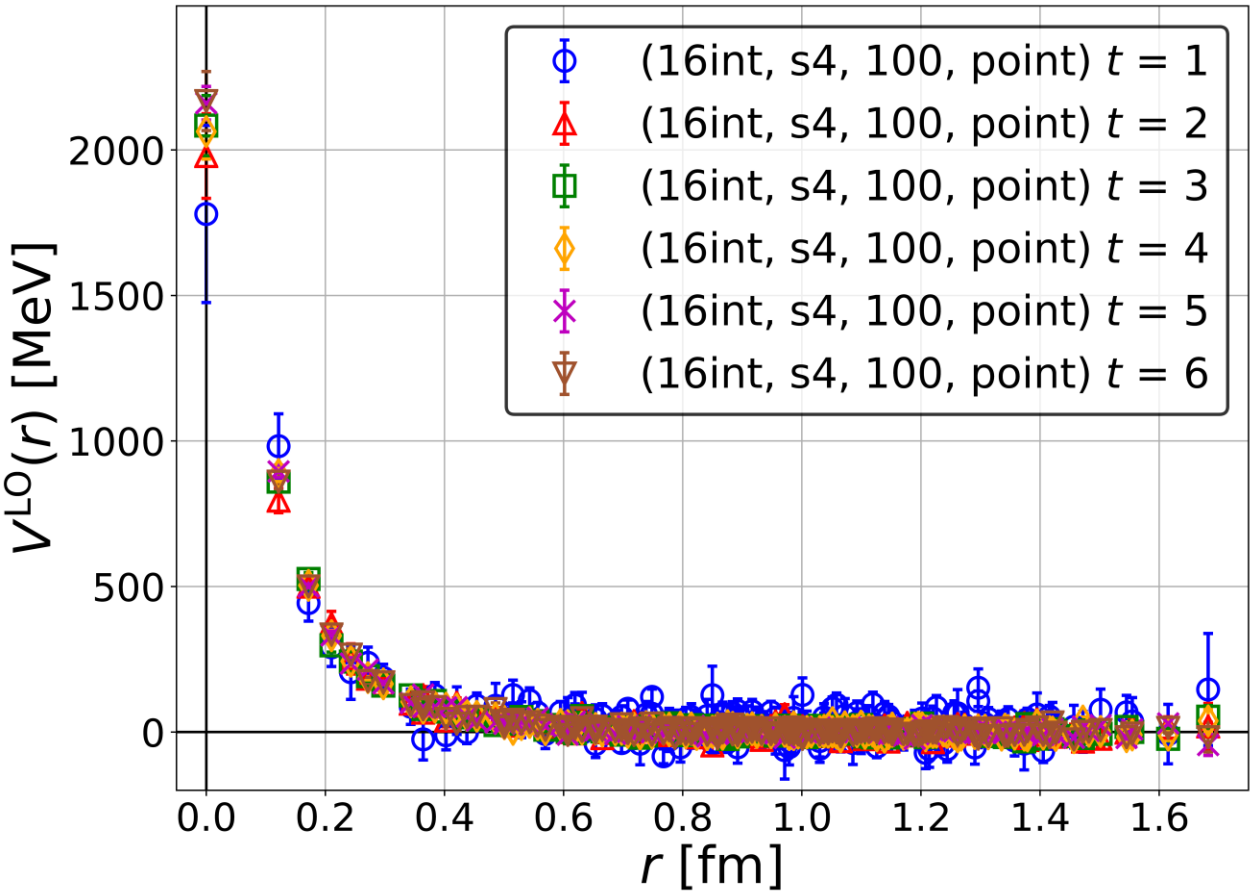


I=2 effective energy shift



ΔE is saturated around t = 5 -> potentials at t >= 5 can be reliable

Time dependence of potentials

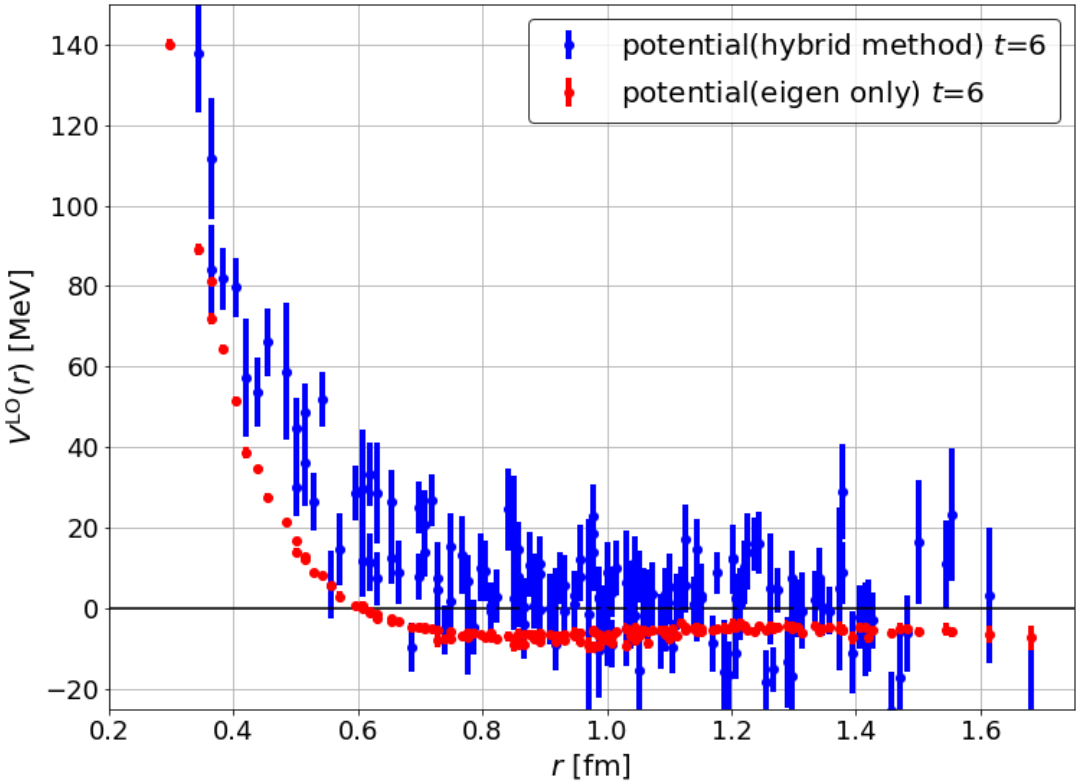


Potentials are saturated around t = 6

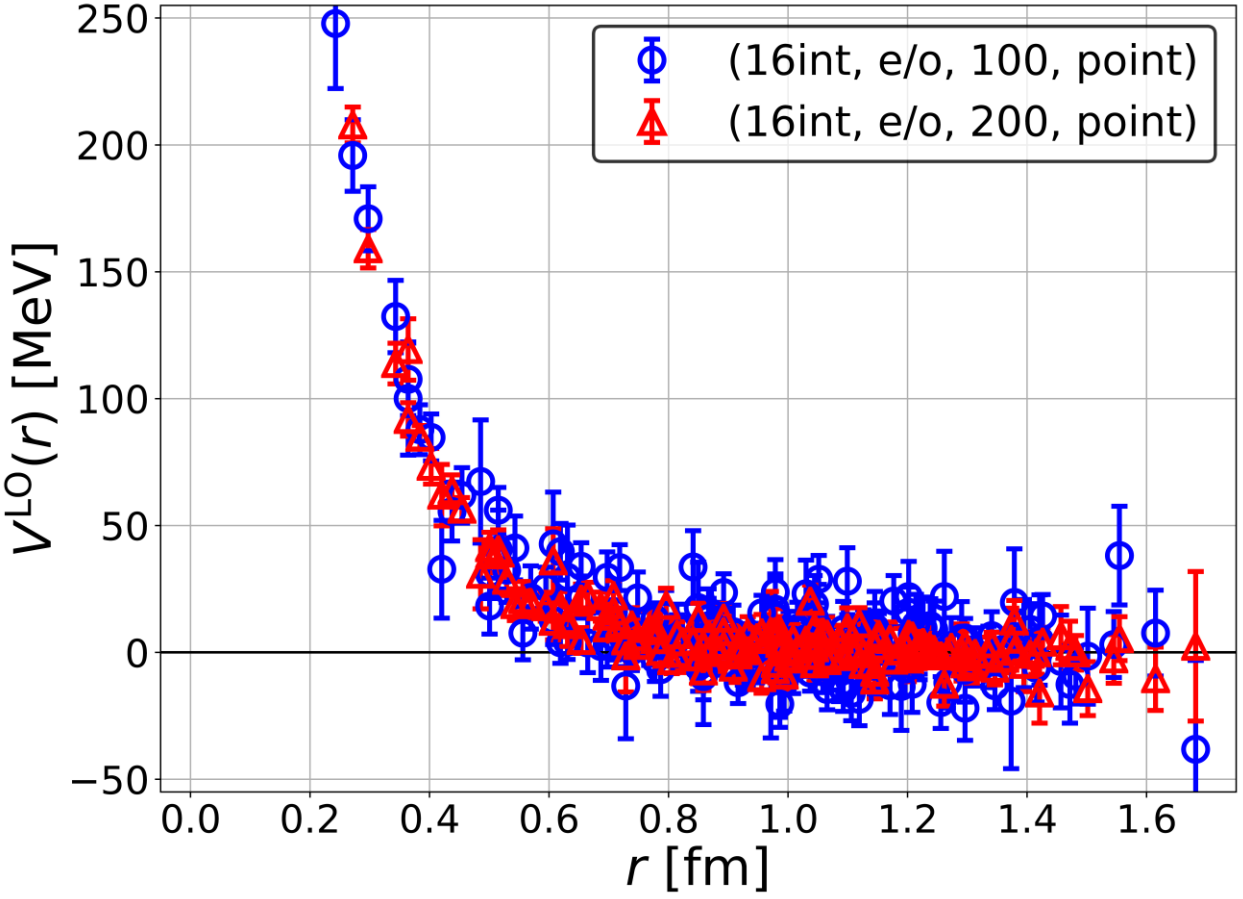
Importance of noisy estimators

There is significant difference between results w/ and w/o noisy estimators

Noisy estimators are important to get a correct potential



Dependence on Neig



- Errors are reduced if we use more eigenmodes
- Note: there is an optimal Neig which depends on lattice setups and numerical costs

Potential fitting

We use a 2-Gaussian fitting function for I=2 case

$$V(r) = a_0 e^{-\left(\frac{r}{a_1}\right)^2} + a_2 e^{-\left(\frac{r}{a_3}\right)^2}$$

- Result of the fitting

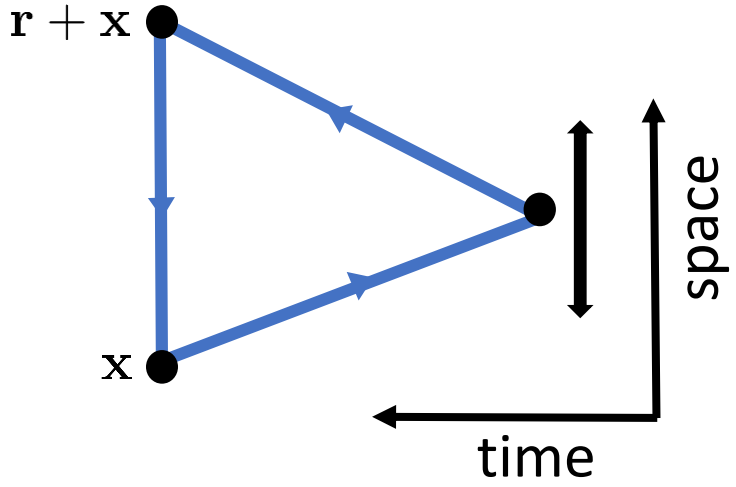
a_0 [MeV]	a_1 [fm]	a_2 [MeV]	a_3 [fm]	χ^2/dof
2047.7	0.11	377.9	0.32	1.27

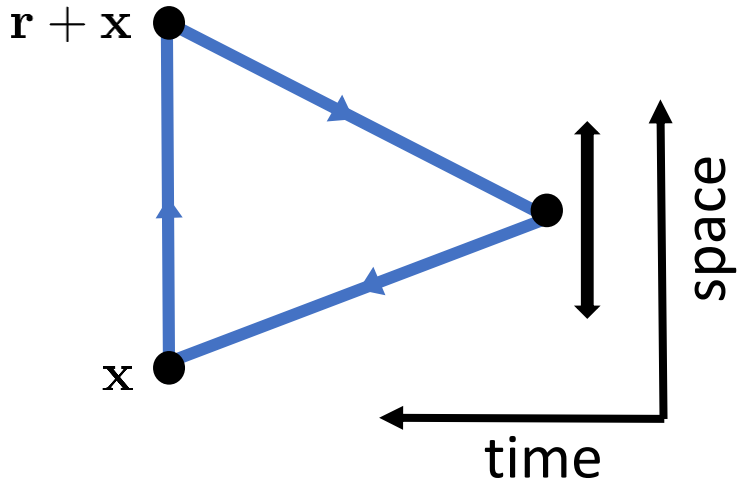
ρ source calculation

We use a ρ shape source operator

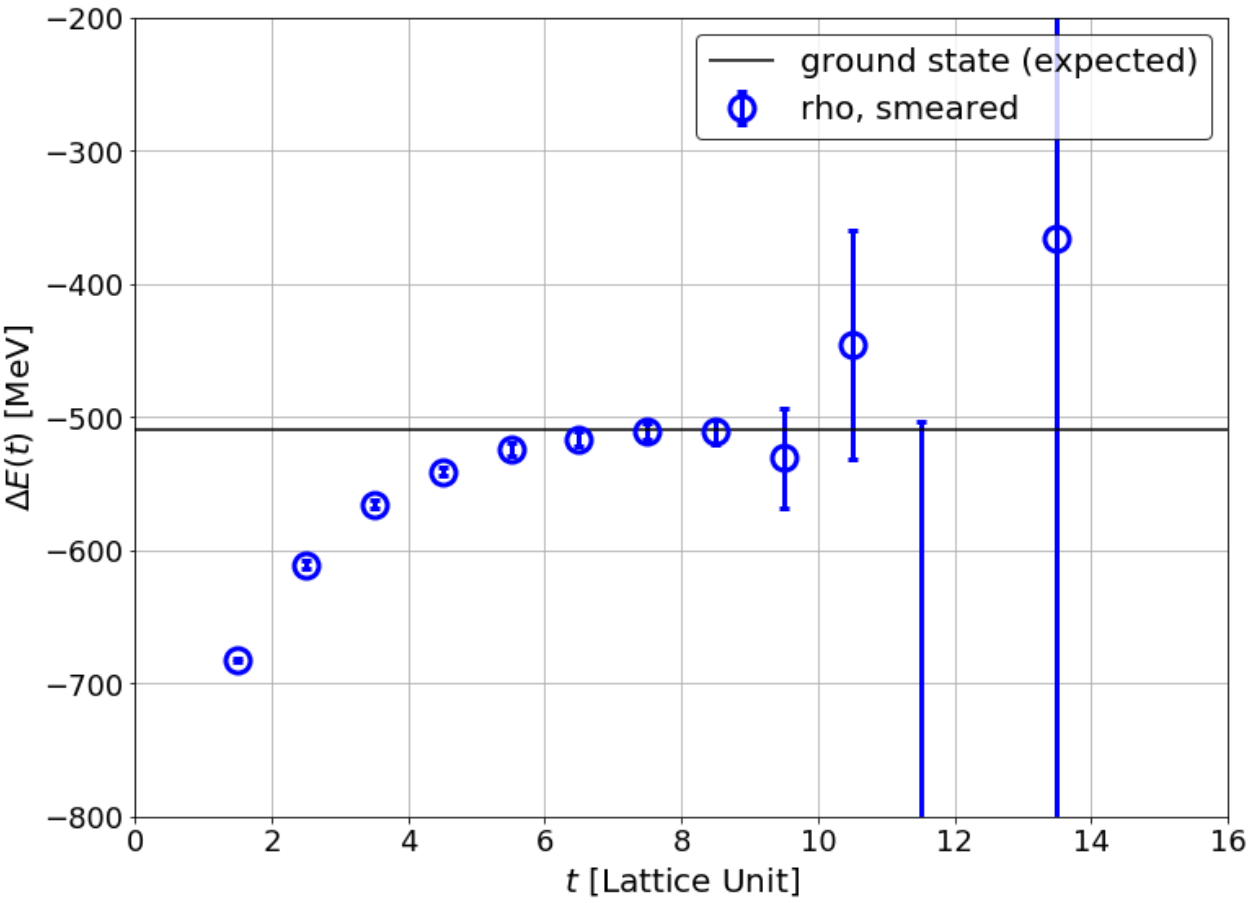
$$\mathcal{J}_{\pi\pi}(t) = \bar{\rho}^0(t) = \sum_{\mathbf{x}} \frac{1}{\sqrt{2}} \left(\bar{u}(\mathbf{x},t)\gamma_i u(\mathbf{x},t) - \bar{d}(\mathbf{x},t)\gamma_i d(\mathbf{x},t) \right)$$

Then, triangle diagrams contribute to the correlator





I=1 effective energy shift



t=7 is sufficient for the ground state saturation

Potential fitting

We use a fitting function defined below for I=1 case

$$V(r) = \frac{a_0}{r} e^{-\left(\frac{r}{a_1}\right)^2} + a_2 e^{-\left(\frac{r}{a_3}\right)^2} + a_4 e^{-\left(\frac{r}{a_5}\right)^2}$$

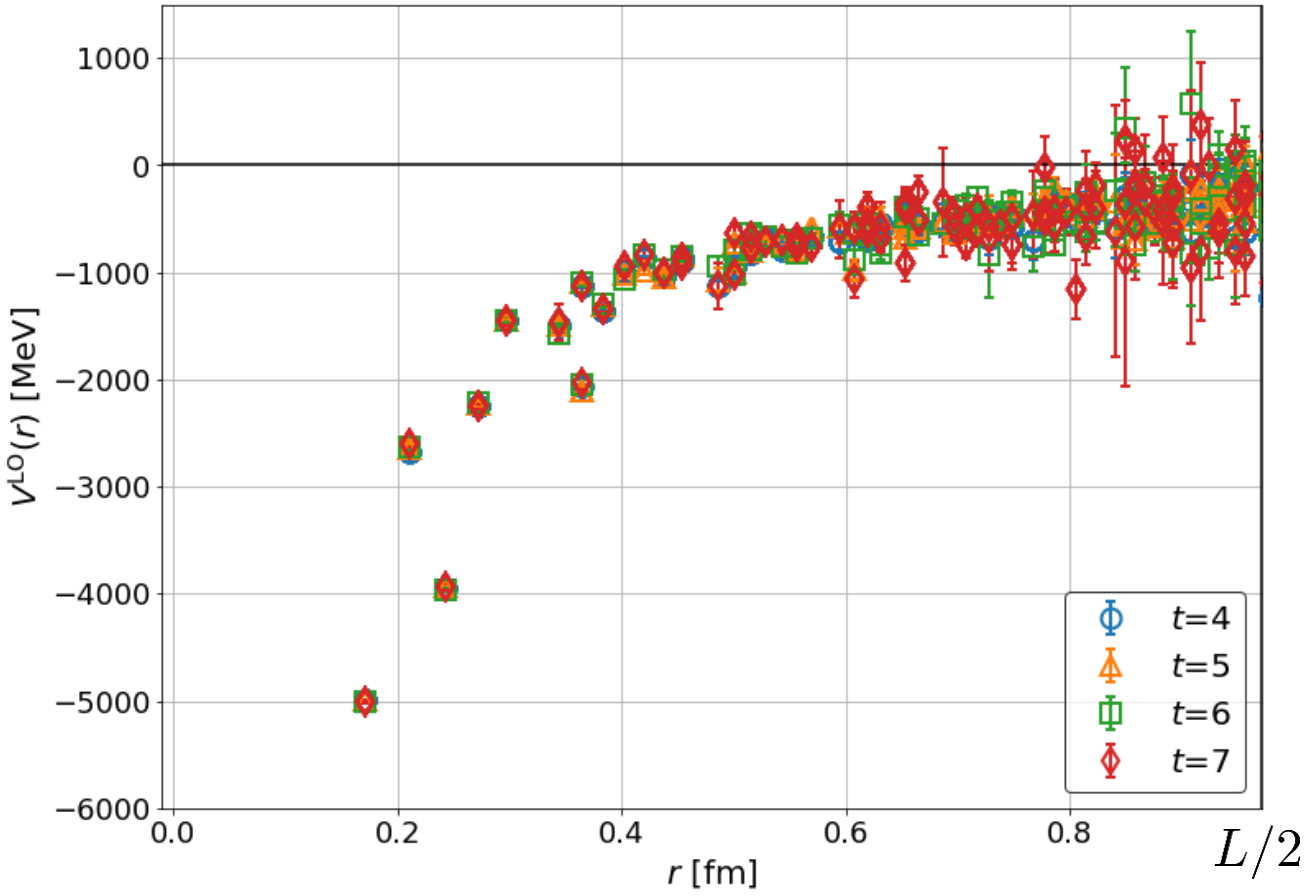
This function has an inter-quark potential behavior in short range ($r \sim 0$)

$$V(r) \approx \frac{a_0}{r} - \frac{a_0}{a_1^2} r + a_2 + a_4$$

➢ Results of the fitting (in lattice unit)

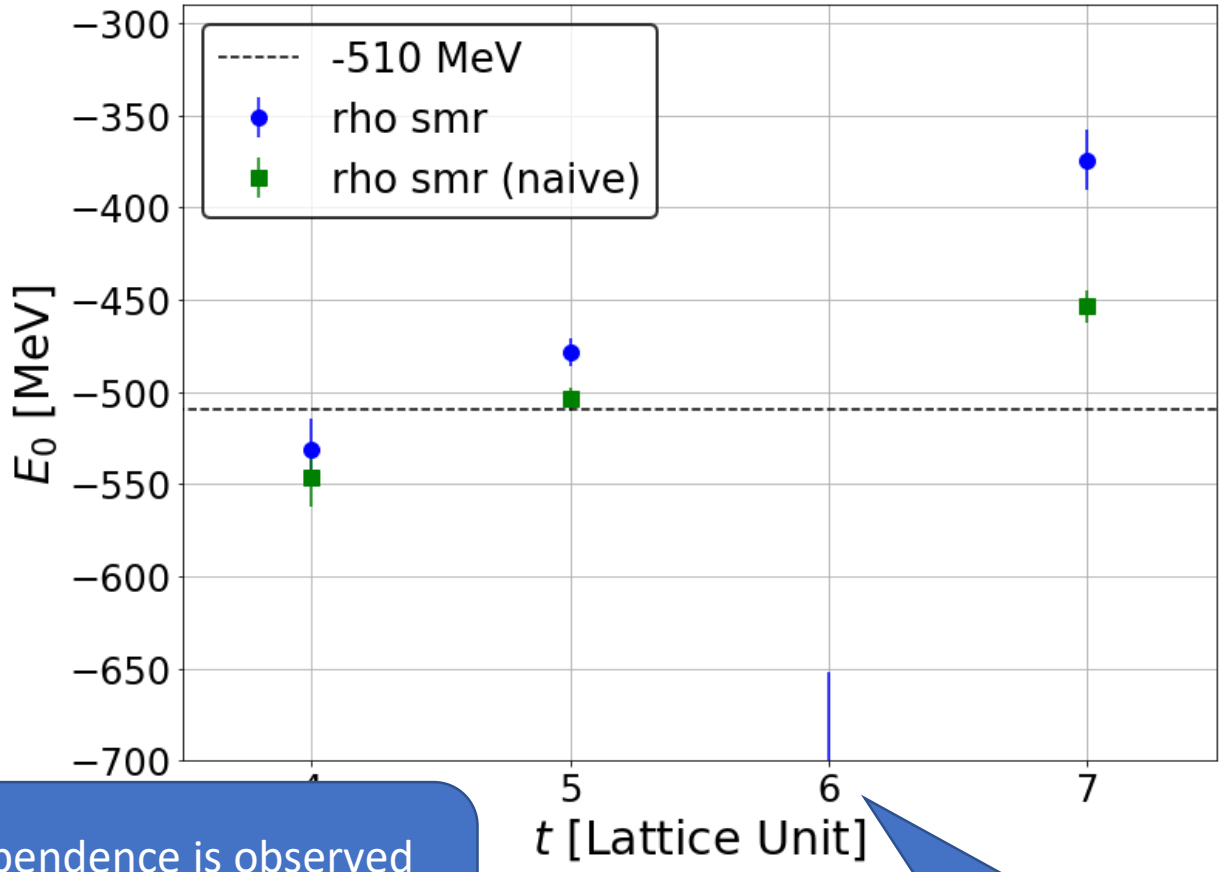
	a_0	a_1	a_2	a_3	a_4	a_5	χ^2/dof
w/ FV effects	-3.80	2.83	-0.43	7.70	-15.5	0.77	9.40
w/o FV effects	-3.81	2.89	-0.42	10.6	-15.5	0.77	9.40

Time dependence of the potential



Potentials are saturated around t = 4, 5, 6, 7

Time dependence of g.s. energies



Time dependence is observed
Although the potentials seem to be saturated in this region

Fitting does not work well at t = 6

NLO analysis of the potential

We can obtain the NLO potential by solving linear equations below:

$$\left[\frac{\nabla^2}{m_\pi} - \frac{\partial}{\partial t} + \frac{1}{4m_\pi}\frac{\partial^2}{\partial t^2}\right] R_0(\mathbf{r},t) = V_0(r)R_0(\mathbf{r},t) + V_1(r)\nabla^2 R_0(\mathbf{r},t)$$

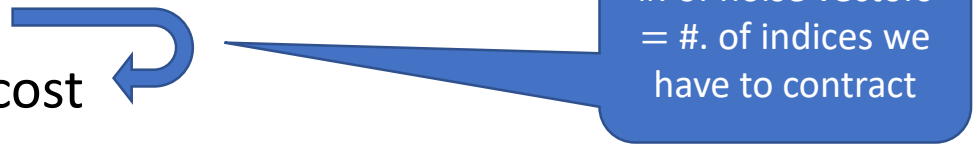
$$\left[\frac{\nabla^2}{m_\pi} - \frac{\partial}{\partial t} + \frac{1}{4m_\pi}\frac{\partial^2}{\partial t^2}\right] R_1(\mathbf{r},t) = V_0(r)R_1(\mathbf{r},t) + V_1(r)\nabla^2 R_1(\mathbf{r},t)$$

(R_0, R_1 are the R-correlators calculated with different source operators)

Need for a new strategy

The required properties of the new method

- Using less noise vectors
- Smaller computational cost



To satisfy those properties, we consider combining some propagator calculation techniques

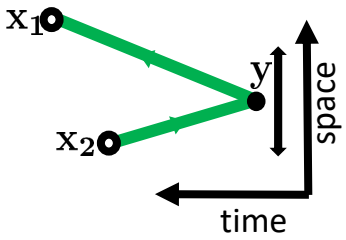
- hybrid method
- point-to-all propagator
- sequential propagator
- one-end trick (2 noise vectors -> 1 noise vector)

One-end trick

- Generate a noise vector $\eta_{[r]}(\mathbf{z})$ in each time slice (ex. Z4 noise)
- Then calculate ξ, χ

$$\xi_{t_0[r]}(x) \equiv \sum_{\mathbf{z}} D^{-1}(x;\mathbf{z},t_0)\eta_{[r]}(\mathbf{z})$$

$$\chi_{t_0[r]}(x) \equiv \sum_{\mathbf{z}} D^{-1}(x;\mathbf{z},t_0)\gamma_5\Gamma^\dagger\eta_{[r]}(\mathbf{z})$$



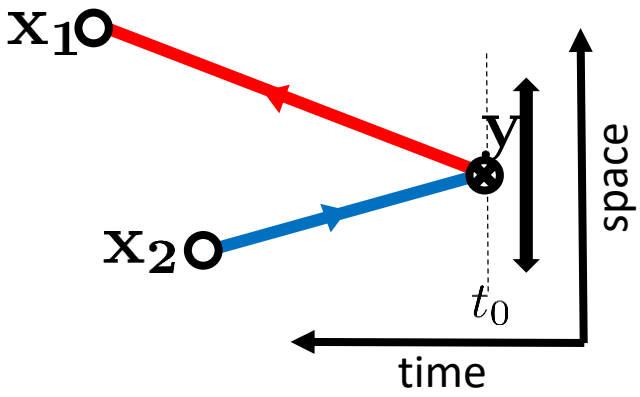
- Using ξ, χ, we obtain

$$\sum_{\mathbf{y}} D^{-1}(\mathbf{x}_1,t_1;\mathbf{y},t_0)\Gamma D^{-1}(\mathbf{y},t_0;\mathbf{x}_2,t_2) \approx \frac{1}{N_\mathrm{r}}\sum_{r=0}^{N_\mathrm{r}-1}\xi_{t_0[r]}(\mathbf{x}_1,t_1)\otimes\chi^\dagger_{t_0[r]}(\mathbf{x}_2,t_2)\gamma_5$$

- ➢ Dilution technique can be used in this method
- ➢ We can combine this method with smearing and momentum projection

Sequential propagator

- Consider a part of diagram below (same as the one-end trick):

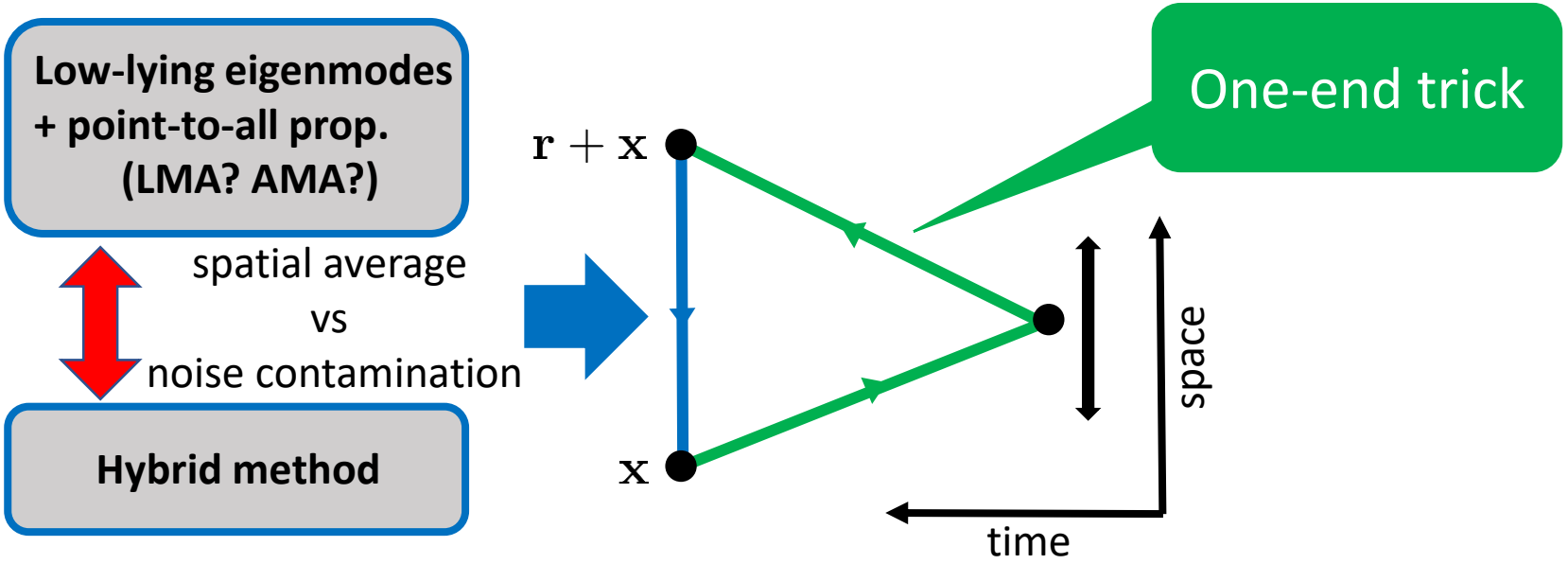


$$= \sum_{\mathbf{y}} D^{-1}(\mathbf{x}_1, t_1; \mathbf{y}, t_0) \Gamma D^{-1}(\mathbf{y}, t_0; \mathbf{x}_2, t_2)$$

- We can calculate the red part **exactly**
- Note: we have to calculate $D^{-1}(\mathbf{y}, t_0; \mathbf{x_2}, t_2)$ in advance

I=1 calculation (ρ source)

- Triangle diagram



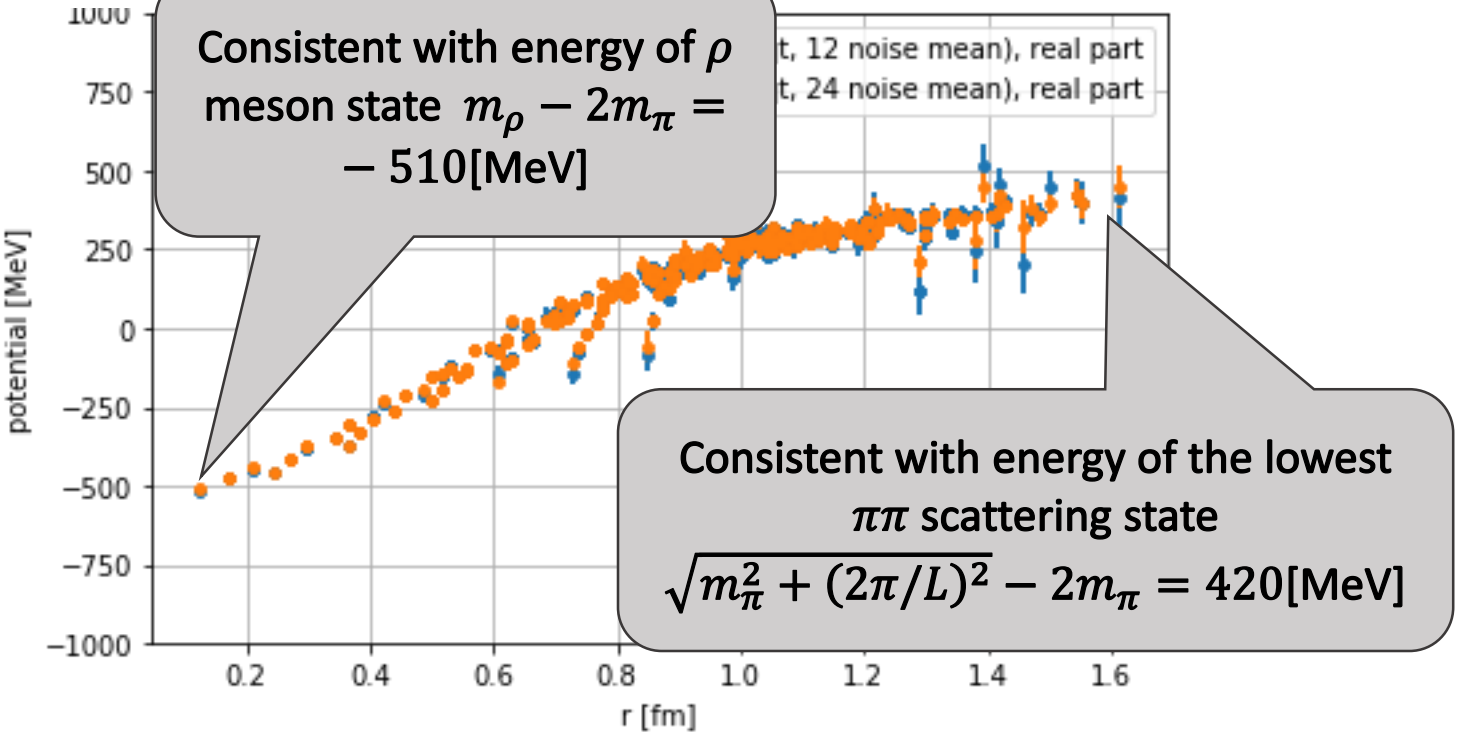
✓ There is ambiguity in a choice of a method to calculate a sink-to-sink propagator

✓ #. of noise vectors =< 2

Behavior of 1st time derivative term

$$-\frac{\partial R(\mathbf{r},t)}{\partial t}/R(\mathbf{r},t) \sim \Delta W$$

$$\mathcal{J}_{\pi\pi} = (\bar{\pi}\bar{\pi})_{I=1,I_z=0}$$



From this behavior, we can conclude that ρ meson state dominates in a short-range region, and on the other hand, $\pi\pi$ scattering state dominates in a long-range region