Some theoretical issues in HAL QCD method

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0. Introduction

Some Issues in the HAL QCD method

HAL QCD method

A powerful method to investigate hadron interactions

Strategy

NBS wave function

$$\varphi^{\vec{k}}(\vec{x})e^{-W_{\vec{k}}t} = \langle 0|N(\vec{r},t)N(\vec{r}+\vec{x},t)|NN,W_{\vec{k}}\rangle \qquad W_{\vec{k}} = 2\sqrt{\vec{k}^2+m_N^2}$$

$$\to \sum_{lm} C_{lm}\frac{\sin(kx+\delta_l(k))}{kx}Y_{lm}(\Omega_{\vec{x}})$$

↓

energy-independent non-local potential

$$(E_{\vec{k}}-H_0)\,\varphi^{\vec{k}}(\vec{x}) = \int U(\vec{x},\vec{y})\varphi^{\vec{k}}(\vec{y})d^3y, \quad E_{\vec{k}} = \frac{\vec{k}^2}{m_N}, \ H_0 = \frac{-\nabla^2}{m_N}, \qquad W_{\vec{k}} \le W_{\rm th} = 2m_N + m_\pi$$

↓

Derivative expansion

$$U(\vec{x},\vec{y}) = V(\vec{x},\vec{\nabla})\delta^{(3)}(\vec{x}-\vec{y})$$

$$V(\vec{x},\vec{\nabla}) = V_0(x) + V_\sigma(x)(\vec{\sigma}_1\cdot\vec{\sigma}_2) + V_T(x)S_{12} + V_{\rm LS}(x)\vec{L}\cdot\vec{S} + O(\vec{\nabla}^2)$$

Some issues

Q1. The HAL QCD potential in the moving system ?

→ I. The HAL QCD potential from the moving system

Q2. Validity of the derivative expansion ? small parameter ?

→ II. Definition of the HAL QCD potential with the derivative expansion

Q3. Is the HAL QCD potential Hermite ?

→ III. Hermite potential from non-Hermite potential

Q4. Partial wave mixings in the cubic box ?

→ Sinya Gongyo's taik on 4/24.

Q5. Quark annihilation processes and resonances ?

→ Yutaro Akahoshi's talk on 4/16.

I. The HAL QCD potential from the moving system

Our Motivation

σ resonance from the $I = 0$ $\pi\pi$ scattering in the HAL QCD method

“vacuum” has the same quantum numbers →

“vacuum” state appears in the NBS wave function in center-of-mass system

↓

the potential describes the vacuum as the “deeply bound state” of two pions ?

Moving system no “vacuum” state in the NBS

But $$\varphi^{\vec{k}}(\vec{x})e^{-W_{\vec{k}}t} = \langle 0|N(\vec{r},t)N(\vec{r}+\vec{x},t)|NN, W_{\vec{k}}\rangle \to \sum_{lm} C_{lm}\frac{\sin(kx+\delta_l(k))}{kx}Y_{lm}(\Omega_{\vec{x}})$$

true only in the CM system

→ no definition of the potential directly from the boosted NBS

Generalized definition of the potential in the CM system

non-equal time NBS wave function in CM system

$$\varphi^{\vec{k}}(\vec{x}, x_4) = \left\langle 0 \left| N\left(\frac{\vec{x}}{2}, \frac{x_4}{2}\right) N\left(-\frac{\vec{x}}{2}, -\frac{x_4}{2}\right) \right| NN, W_{\vec{k}} \right\rangle$$

$$\simeq \sum_{lm} A_{lm} \frac{\sin(kx + \delta_l(k) + \pi l/2)}{kx} Y_{lm}(\Omega_{\vec{x}}) \qquad k = |\vec{k}|$$

➡

$$\boxed{(E_k - H_0)\, \varphi^{\vec{k}}(\vec{x}, x_4) = V_{x_4}(\vec{x}, \nabla) \varphi^{\vec{k}}(\vec{x}, x_4)}$$

also in Akahoshi's talk

the HAL QCD potential in the x_4 scheme

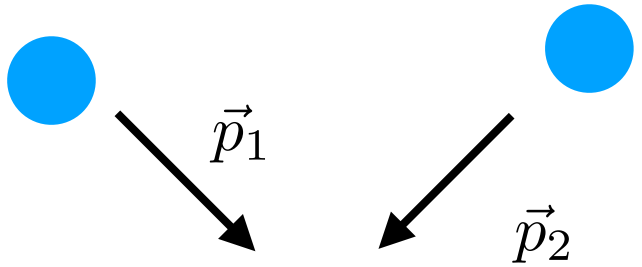
$x_4 = 0$: equal time scheme

Can we extract the generalized potential from the boosted NBS function ?

For simplicity, we consider this problem for scalar field.

CM and moving systems

Moving p, P



$\vec{V}$

boost velocity

CM p^*, P^*

$\vec{P} = \vec{p}_1 + \vec{p}_2 \qquad \vec{p} = (\vec{p}_1 - \vec{p}_2)/2$

$\vec{P}^* = 0$

$P_0^* \;:=\; (\Lambda P)_0 = \gamma(P_0 - \vec{V}\vec{P}), \quad \underline{P_k^* := (\Lambda P)_k = \gamma(P_k - V_k P_0) = 0,} \qquad \gamma \;=\; \frac{1}{\sqrt{1-\vec{V}^2}}$ boost factor

$$\boxed{\vec{V} \;=\; \vec{P}/P_0.}$$

boost velocity depends on energy

$$\underline{(\vec{p}^*)^2 \;:=\; (\vec{p}^*_\perp)^2 + (\vec{p}^*_\parallel)^2 = \vec{p}^2 - \frac{(\vec{P}\cdot\vec{p})^2}{P_0^2} = \frac{P_0^2 - \vec{P}^2}{4} - m^2.}$$

important relation

relation of NBS wave functions between 2 systems

NBS $\psi_{p,P}(x,X) \quad := \quad \langle 0|T\{\phi(x_1)\phi(x_2)\}|p_1,p_2\rangle$

$x := x_1 - x_2, X = (x_1+x_2)/2$

$p_i^2 - m^2 = 0$

$P_0 = \sqrt{\vec{p}_1^2 + m^2} + \sqrt{\vec{p}_2^2 + m^2}.$

➡

$$\begin{aligned} \psi_{p,P}(x,X) &= \langle 0|e^{i\hat{P}\cdot X} T\{\phi(x/2)\phi(-x/2)\} e^{-i\hat{P}\cdot X}|p_1,p_2\rangle \\ &= e^{-iP\cdot X}\varphi_{p,P}(x), \end{aligned}$$

moving CM

Scalar field $e^{-iP\cdot X}\varphi_{p,P}(x) = e^{-iP^*\cdot X^*}\varphi_{p^*,P^*}(x^*).$

$P\cdot X = P^*\cdot X^*$

➡

$$\boxed{\varphi_{p,P}(x) \quad = \quad \varphi_{p^*,P^*}(x^*),}$$

$$\boxed{p^* = (0,\vec{p}^*), \quad P^* = (P_0^*, \vec{0}) \quad x^* = \left(\gamma(x_0 - \vec{V}\vec{x}),\ \gamma(\vec{x}_{\|} - \vec{V}x_0),\ \vec{x}_{\perp}\right)}$$

Euclidean space-time (Wick rotation) $x_4 = ix_0$

$$
\begin{aligned}
X_0^* &= \gamma(X_0 - \vec{V}\vec{X}), \\
\vec{X}_\perp^* &= \vec{X}_\perp, \quad \vec{X}_\parallel^* = \gamma(\vec{X}_\parallel - \vec{V}X_0), \\
x_0^* &= \gamma(x_0 - \vec{V}\vec{x}), \\
\vec{x}_\perp^* &= \vec{x}_\perp, \quad \vec{x}_\parallel^* = \gamma(\vec{x}_\parallel - \vec{V}x_0),
\end{aligned}
$$

Minkowski

$$
\begin{aligned}
X_4^* &= \gamma(X_4 - i\vec{V}\vec{X}), \quad X_\parallel^* = \gamma(X_\parallel + i\vec{V}X_4), \quad X_\perp^* = X_\perp, \\
x_4^* &= \gamma(x_4 - i\vec{V}\vec{x}), \quad x_\parallel^* = \gamma(x_\parallel + i\vec{V}x_4), \quad x_\perp^* = x_\perp,
\end{aligned}
$$

Euclid

moving **CM**

$$\varphi_{p,P}(x) \quad = \quad \varphi_{p^*,P^*}(x^*),$$

$$p^* = (0, \vec{p}^*), \quad P^* = \left(P_0^*, \vec{0}\right) \quad x^* = \left(\gamma(x_4 - \underline{i\vec{V}\vec{x}}),\ \gamma(\vec{x}_\parallel + \underline{i\vec{V}x_4}),\ \vec{x}_\perp\right).$$

complex in CM !

$$(\vec{p}^*)^2 = \frac{P_0^2 - \vec{P}^2}{4} - m^2, \quad P_0^* = \frac{P_0}{\gamma}, \quad \vec{V} = \frac{\vec{P}}{P_0}, \qquad -iP\cdot X \quad = \quad -P_0X_4 + i\vec{P}\vec{X} = -P_0^*X_4^*.$$

HAL QCD potential from boosted NBS wave function

$$(E_{p^*} - H_0)\varphi_{p^*,P^*}(x^*) = V_{x_4^*}(\vec{x}, \nabla)\varphi_{p^*,P^*}(x^*)$$

generalized definition

using Boosted NBS

$$\varphi_{p,P}(x) \quad = \quad \varphi_{p^*,P^*}(x^*),$$

$$V_{x_4^*}(\vec{x*}, \nabla)\varphi_{p,P}(x) = \frac{(\vec{p*})^2 + \nabla^2_{x^*}}{m}\varphi_{p,P}(x)$$

CM Moving CM Moving

$$\nabla^2_{x^*} \quad = \quad \gamma^2\left(\nabla_{x_\parallel} + iV\partial_{x_4}\right)^2 + \nabla^2_{x_\perp}.$$

CM Moving

$$\boxed{V_{\gamma(x_4 - iVx_\parallel)}(\gamma(x_\parallel + \underline{iVx_4}), x_\perp, \nabla_{x^*})\varphi_{p,P}(x) \quad = \quad \frac{(\vec{p^*})^2 + \gamma^2\left(\nabla_{x_\parallel} + iV\partial_{x_4}\right)^2 + \nabla^2_{x_\perp}}{m}\varphi_{p,P}(x)}$$

$x_4 = 0$ (equal-time boosted NBS) is required.

$x_4 = 0$ →

$$V_{\underline{-i\gamma V x_{\parallel}}}(\gamma x_{\parallel}, x_{\perp}, \nabla_{x^*})\varphi_{p,P}(x) \quad = \quad \frac{(\vec{p}^*)^2 + \gamma^2 \left(\nabla_{x_{\parallel}} + iV\partial_{x_4}\right)^2 + \nabla_{x_{\perp}}^2}{m}\varphi_{p,P}(x).$$

Each $x_{\parallel}$ → potential in the $x_0^* = -\gamma x_{\parallel}$ scheme

Minkowski time from Euclidean correlates !

$$(\vec{p}^*)^2 = \frac{P_0^2 - \vec{P}^2}{4} - m^2, \quad P_0^* = \frac{P_0}{\gamma}, \quad \vec{V} = \frac{\vec{P}}{P_0},$$

Time-dependent HAL QCD method

R-correlator

$$R_{\vec{P}}(x,X)e^{i\vec{P}\cdot\vec{X}} \;:=\; \langle T\{\phi(x_1)\phi(x_2)\}J^{\dagger}_{\phi\phi}(0,\vec{P})\rangle \;=\; e^{i\vec{P}\cdot\vec{X}}\sum_{P_0^n\le W_{\rm th}} A_n e^{-P_0^n X_4}\varphi_{p^n,P^n}(x)+\cdots,$$

sum of NBS

$$V_{-i\gamma V x_\parallel}(\gamma x_\parallel, x_\perp, \nabla_{x^*})\varphi_{p,P}(x) \;=\; \frac{(\vec{p}^*)^2+\gamma^2\left(\nabla_{x_\parallel}+iV\partial_{x_4}\right)^2+\nabla^2_{x_\perp}}{m}\varphi_{p,P}(x).$$

$$\gamma_n^2 = \frac{(P_0^n)^2}{(P_0^n)^2-\vec{P}^2}$$

$$(\vec{p}^*)^2 = \frac{P_0^2-\vec{P}^2}{4}-m^2,\quad P_0^* = \frac{P_0}{\gamma},\quad \vec{V} = \frac{\vec{P}}{P_0},$$

$$\begin{aligned}
A_1(x,X) &:= \left(\partial^2_{X_4}-\vec{P}^2\right)R_{\vec{P}}(x,X) = \sum_n A_n e^{-P_0^n X_4}\underline{((P_0^n)^2-\vec{P}^2)}\varphi_{p^n,P^n}(x),\\
A_2(x,X) &:= \left(\partial^2_{X_4}-\vec{P}^2\right)^2 R_{\vec{P}}(x,X) = \sum_n A_n e^{-P_0^n X_4}\underline{((P_0^n)^2-\vec{P}^2)^2}\varphi_{p^n,P^n}(x),\\
B(x,X) &:= \nabla^2_{x_\perp}A_1(x,X),\\
C(x,X) &= \left(-\partial_{X_4}\nabla_{x_\parallel}+i|\vec{P}|\partial^2_{x_4}\right)^2 R_{\vec{P}}(x,X)\\
&= \sum_n A_n e^{-P_0^n X_4}\underline{(P_0^n)^2}\left(\nabla_{x_\parallel}+i\underline{V_n\partial_{x_4}}\right)^2\varphi_{p^n,P^n}(x), \qquad (41)
\end{aligned}$$

$$V_n = \frac{|\vec{P}|}{P_0^n}$$

$$\frac{A_2(x,X)}{4} - m^2 A_1(x,X) + B(x,X) + C(x,X)$$

$$= \sum_n A_n e^{-P_0^n X_4} \left[\left\{ \underbrace{\frac{(P_0^n)^2 - \vec{P}^2}{4} - m^2}_{= (\vec{p}_n^*)^2} + \nabla_{x_\perp}^2 \right\} + \underbrace{\frac{(P_0^n)^2}{(P_0^n)^2 - \vec{P}^2}}_{= \gamma_n^2} (\nabla_{x_\parallel} + iV_n \partial_{x_4})^2 \right] \left((P_0^n)^2 - \vec{P}^2 \right) \varphi_{p^n,P^n}(x)$$

Set $x_4 = 0$ $\Big\downarrow$

$$V_{-i\gamma V x_\parallel}(\gamma x_\parallel, x_\perp, \nabla_{x^*})\varphi_{p,P}(x) = \frac{(\vec{p}^*)^2 + \gamma^2 \left(\nabla_{x_\parallel} + iV\partial_{x_4}\right)^2 + \nabla_{x_\perp}^2}{m}\varphi_{p,P}(x).$$

$$= m \sum_n A_n e^{-P_0^n X_4} V_{-i\gamma_n V_n x_\parallel}(\gamma_n x_\parallel, x_\perp, \nabla_{x^*})((P_0^n)^2 - \vec{P}^2)\varphi_{p^n,P^n}(x)$$

$$= m V_{x_4^*=0}(x_\parallel = 0, x_\perp, \nabla_{x^*}) \sum_n A_n e^{-P_0^n X_4}((P_0^n)^2 - \vec{P}^2)\varphi_{p^n,P^n}(x)$$

Set $x_\parallel = 0$

$$= m V_{x_4^*=0}(x_\parallel = 0, x_\perp, \nabla) A_1(x,X)$$

LO potentail

$$\boxed{V^{\rm LO}_{x_4^*=0}(0, x_\perp) = \frac{A_2(x,X)/4 - m^2 A_1(x,X) + B(x,X) + C(x,X)}{m A_1(x,X)}.}$$

$x_4 = x_\parallel = 0$

Future investigations

1. Check the formula for he simple system

$$I = 2\ \pi\pi \text{ moving system}$$

2. resonance in the HAL QCD potential

$$\sigma \text{ resonance in } I = 0\ \pi\pi \text{ moving system}$$

3. extension to fermions (Baryons)

lower components mix

relativistic formulation for the “potential” ?

II. Definition of the HAL QCD potential with the derivative expansion

$$(E_{\vec{k}} - H_0)\varphi^{\vec{k}}(\vec{x}) = \int U(\vec{x},\vec{y})\varphi^{\vec{k}}(\vec{y})d^3y, \quad \boxed{W_{\vec{k}} \le W_{\text{th}}}$$

This equation does not fix the non-local potential due to the restriction of energies. Therefore this potential is ambiguous. We have to fix the definition of the potential (scheme) explicitly.

We here propose a scheme to fix the potential completely using the derivative expansion.

For simplicity, let us consider the scalar particles and ignore the angular momentum dependent part of the potential.

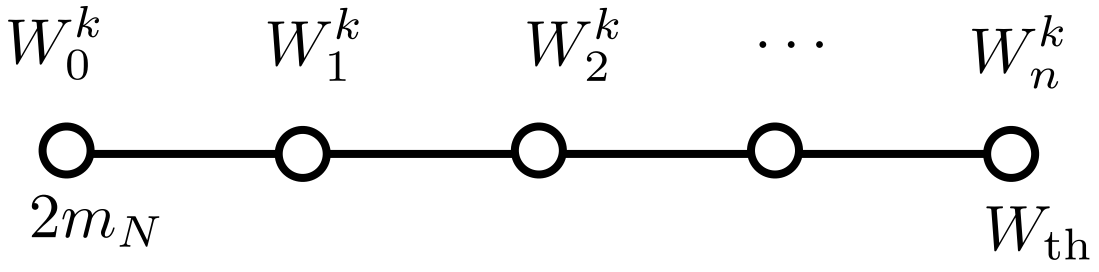
- We consider the expansion in terms of ∇^2 (but not $\mathbf{L}^2$).
- Terms with odd number of ∇ are not included. This is our scheme.
- The potential must be non-Hermitian. We can make it Hermitian as seen later.

Of course, the scheme is not unique. One may use a different one.

Definition of the potential

choice of energy

$$W_k^n := 2m_N + \frac{k}{n}(W_{\rm th} - 2m_N), \quad k = 0, 1, \cdots, n$$



approximated potential at the n-th order

$$V^{(n)}(\mathbf{x}, \nabla) := \sum_{k=0}^{n} V_k^{(n)}(\mathbf{x})(\nabla^2)^k$$

$(\mathbf{L}^2)^k$ terms are ignored for simplicity.

Coefficient functions $V_k^{(n)}(\mathbf{x})$ can be determined from

faithful to phase shifts

$$\sum_{k=0}^{n} V_k^{(n)}(\mathbf{x})(\nabla^2)^k \varphi_{p_k}(\mathbf{x}) = (\epsilon_{p_k} - H_0)\varphi_{p_k}(\mathbf{x})$$

$$W_n^k = 2\sqrt{p_k^2 + m_N^2}$$

$$\begin{pmatrix} \varphi_{p_0}(\mathbf{x}) & \nabla^2\varphi_{p_0}(\mathbf{x}) & \cdots & (\nabla^2)^n\varphi_{p_0}(\mathbf{x}) \\ \varphi_{p_1}(\mathbf{x}) & \nabla^2\varphi_{p_1}(\mathbf{x}) & \cdots & (\nabla^2)^n\varphi_{p_1}(\mathbf{x}) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_{p_n}(\mathbf{x}) & \nabla^2\varphi_{p_n}(\mathbf{x}) & \cdots & (\nabla^2)^n\varphi_{p_n}(\mathbf{x}) \end{pmatrix} \begin{pmatrix} V_0^{(n)}(\mathbf{x}) \\ V_1^{(n)}(\mathbf{x}) \\ . \\ . \\ V_n^{(n)}(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} (\epsilon_{p_0} - H_0)\varphi_{p_0}(\mathbf{x}) \\ (\epsilon_{p_1} - H_0)\varphi_{p_1}(\mathbf{x}) \\ . \\ . \\ (\epsilon_{p_n} - H_0)\varphi_{p_n}(\mathbf{x}) \end{pmatrix}$$

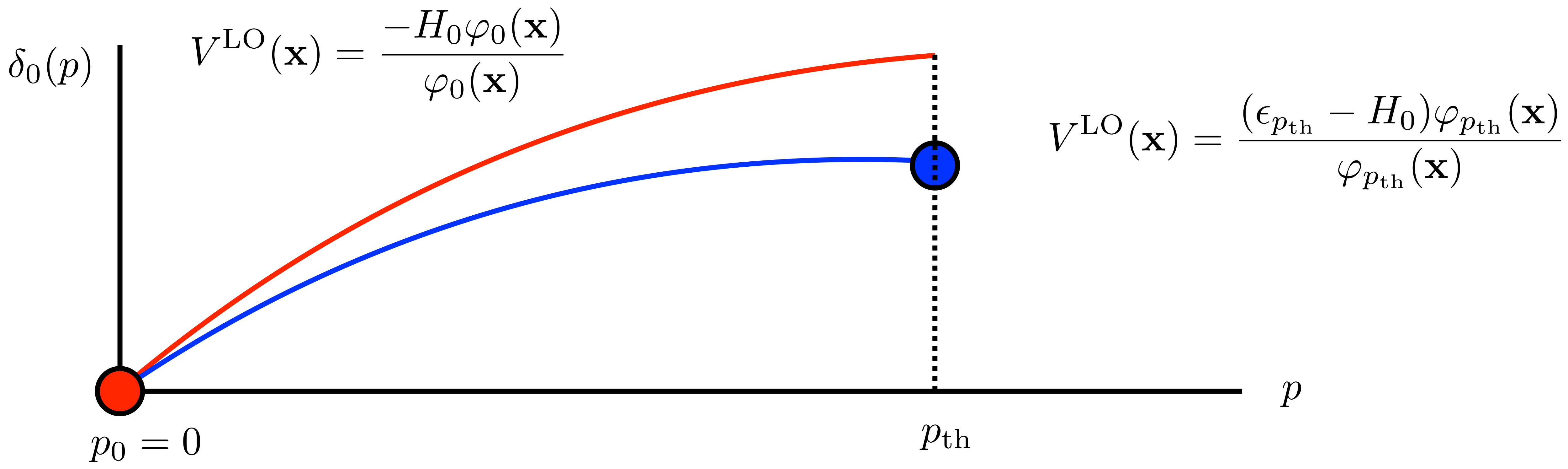
a number of unknowns = a number of equations ➡ $V_k^{(n)}(\mathbf{x})$

➡ **Def of potential**

$$\boxed{V(\mathbf{x}, \nabla) := \lim_{n\to\infty} V^{(n)}(\mathbf{x}, \nabla) = \lim_{n\to\infty} \sum_{k=0}^{n} V_k^{(n)}(\mathbf{x})(\nabla^2)^k}$$

In practice

Take $n = 1$ with $W_0^1 \simeq 2m_N$ and $W_1^1 \simeq W_{\rm th}$.

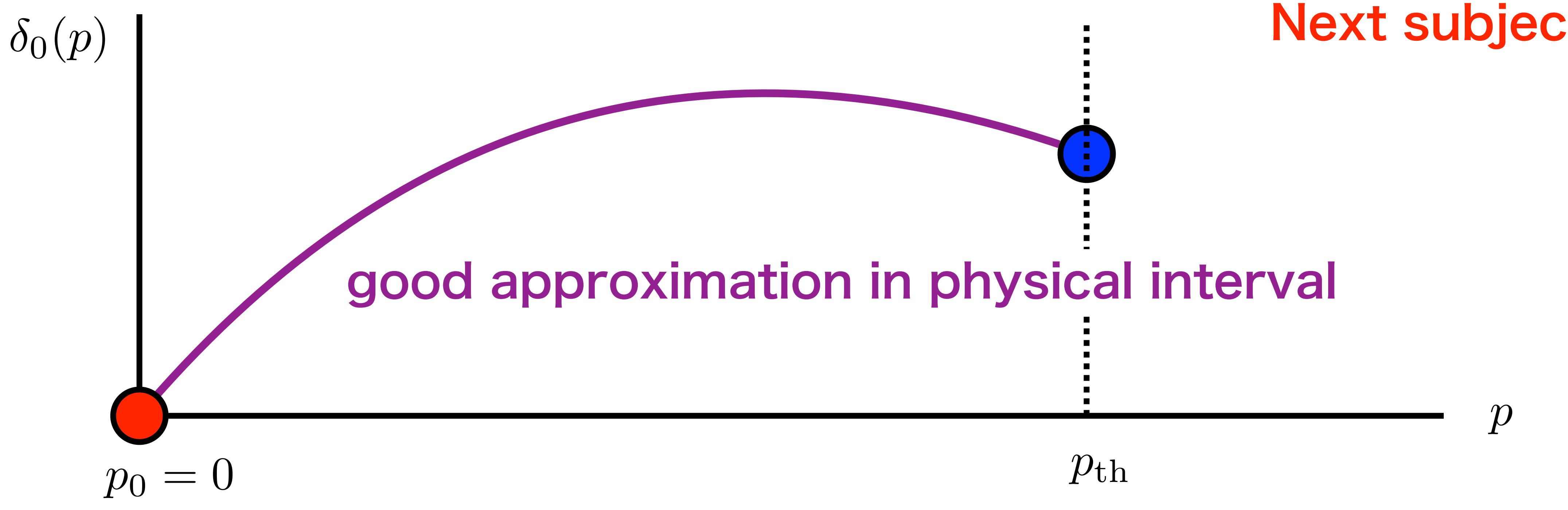


$$V^{\rm LO}(\mathbf{x}) = \frac{-H_0\varphi_0(\mathbf{x})}{\varphi_0(\mathbf{x})}$$

$$V^{\rm LO}(\mathbf{x}) = \frac{(\epsilon_{p_{\rm th}} - H_0)\varphi_{p_{\rm th}}(\mathbf{x})}{\varphi_{p_{\rm th}}(\mathbf{x})}$$

➡ $H = H_0 + V_0^{(1)}(r) + V_1^{(1)}(r)\nabla^2$ ➡ $\tilde{H} = H_0 + \tilde{V}_0^{(1)}(r) + \nabla^i V_1^{(1)}(r)\nabla_i$

Next subject



good approximation in physical interval

Demonstration

Separable potential $U(\vec{x},\vec{y}) = w v(\vec{x}) v(\vec{y})$ $v(\vec{x}) = e^{-\mu x}, \quad x := |\vec{x}|$

L=0 wave function R: IR cut-off

$$\psi_k^0(x) = \frac{e^{i\delta(k)}}{kx}\left[\sin(kx+\delta(k)) - \sin\delta(k) e^{-\mu x}\left(1 + x\frac{\mu^2+k^2)}{2\mu}\right)\right] \qquad x \le R$$

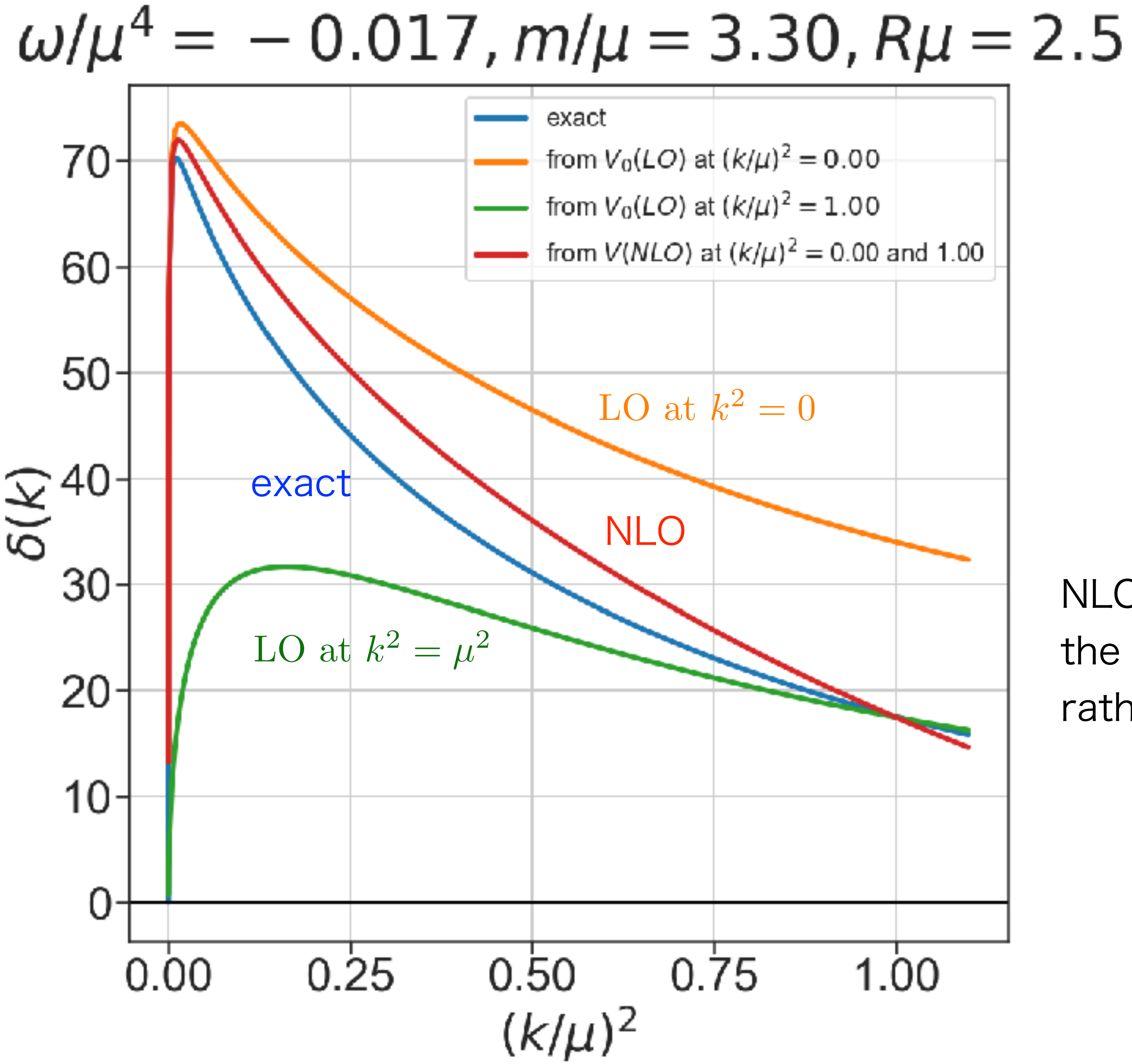
$$= C\frac{e^{i\delta(k)}}{kx}\sin(kx+\delta_R(k)) \qquad x > R$$

phase shift $\delta_R(k)$ is exactly calculable.

separable potential $U(\vec{x},\vec{y})$

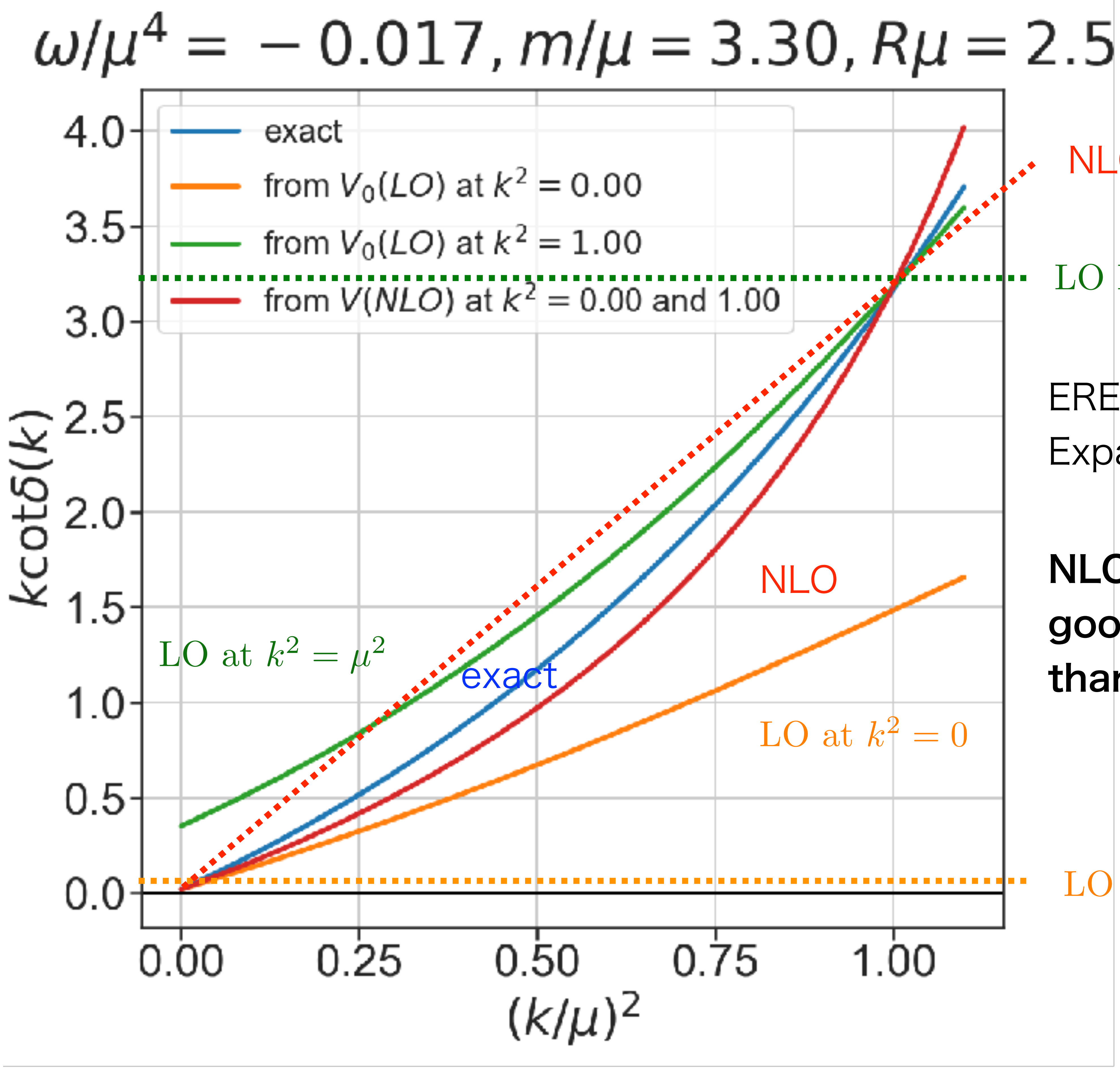
LO potential $V_0^{\rm LO}(r)$ from $k^2 = 0$ or $k^2 = \mu^2$

NLO potential $V_0^{\rm NLO}(r) + V_1^{\rm NLO}(r)\nabla^2$



NLO potential reproduces the exact phase shift rather well.

$k\cot(\delta_0(k))$



NLO ERE

LO ERE ar $k^2 = \mu^2$

NLO

LO at $k^2 = \mu^2$

exact

LO at $k^2 = 0$

LO ERE ar $k^2 = 0$

ERE =Effective Range Expansion

NLO potential is equally good to or even better than NLO ERE.

III. Hermitian potential from non-Hermitian potential

The HAL QCD potential is non-Hermitian in general, since NBS wave functions are not orthogonal to each other. (If Hermite, eigenfunctions must be orthogonal.)

Can we make non-Hermitan potential Hermite ?

$$\boxed{H\psi = E\psi, E \in \mathcal{R}} \qquad H = H_0 + U \qquad H_0 = -\frac{1}{m_N}\nabla^2,$$

real eigenvalues

U : non-Hermitian

?

➡

$$\tilde{H} = H_0 + V,$$

V : Hermitian

$$\tilde{H}\phi = E\phi, \quad \boxed{\tilde{H} = R^{-1}HR,} \; \psi = R\phi$$

1. LO in the derivative expansion (n=1)

$$V_2^{ji} = V_2^{ij}$$

$$\boxed{U = V_0 + V_1\nabla_i + \frac{1}{2}V_2^{ij}\nabla_i\nabla_j = V_0 + \bar{V}_1^i\nabla_i + \frac{1}{2}\nabla_j V_2^{ij}\nabla_j,}$$

$$\tilde{V}_1^i := V_1^i - (\nabla_j V_2^{ij})/2$$

Hermite

$$\tilde{H} = H_0 + \tilde{V}_0 + \frac{1}{2}\nabla_i V_2^{ij}\nabla_j + \boxed{\left\{\tilde{V}_1^i - \frac{2}{m_N}R^{-1}\nabla^i R + V_2^{ij}R^{-1}V_j R\right\}\nabla_i} = 0$$

$$\tilde{V}_0 := V_0 - \frac{1}{m_N}R^{-1}\nabla^2 R + V_1^i R^{-1}\nabla_i R + \frac{1}{2}V_2^{ij}R^{-1}(\nabla_i\nabla_j R).$$

Rotationally symmetric case:

$$\tilde{V}_1^i := \hat{r}^i V_1(r), \quad V_2^{ij} := \delta^{ij} V_{2a} + \hat{r}^i \hat{r}^j V_{2b}, \qquad \hat{r}^i := \frac{r^i}{r}$$

➡

$$\frac{dR(r)}{dr} = \frac{m_N}{2} \frac{\bar{V}_1(r)}{1 - \frac{m_N}{2} V_2(r)} R(r), \quad V_2 := V_{2,a} + V_{2,b},$$

➡

$$R(r) = \exp\left[\frac{m_N}{2} \int_{r_\infty}^{r} ds \, \frac{\bar{V}_1(s)}{1 - \frac{m_N}{2} V_2(s)} \right]$$

At this order, we can make the potential Hermitian without approximation.

$$\tilde{V}_0 = V_0 - \frac{\tilde{V}_1}{r} - \frac{\tilde{V}_1'}{2} + \frac{m}{4} \frac{\tilde{V}_1^2}{1 - \frac{m}{2} V_2} \qquad \tilde{V}_1 := V_1 - V_{2a} - \frac{V_{2a}' + V_{2b}'}{2}$$

2. NLO (n=2)

$$H_1 = H^0 + U_1, \;\; H^0 = H_0 + U_0 \qquad O(\nabla^2)$$

$$O(\nabla^4) \quad U_1 = V_3 + V_4 = \underbrace{U_{1,4}}_{\textbf{Hermitian}} + U_{1,3} + \underbrace{U_{1,2}}_{\textbf{Hermitian}} + U_{1,1} \qquad U_{1,n} : n\text{-th derivative term}$$

$$\tilde{H}_1 := R^{-1} H_1 R, \quad R := R_0(1 + R_1) \qquad R^{-1} \simeq (1 - R_1)R_0^{-1}$$

$$(1 - R_1)R_0^{-1}(H^0 + U_1)R_0(1 + R_1)$$

$$\tilde{H}^0 := R_0^{-1} H^0 R_0 = \tilde{V}_0 + \frac{1}{2}\nabla_i H_0^{ij} \nabla_j \;\; \textbf{Hermitian (n=1)}$$

$$\tilde{H}_1 \simeq \tilde{H}^0 + R_0^{-1} U_1 R_0 + [\tilde{H}^0, R_1] + \cancel{[R_0^{-1} U_1 R_0, R_1]}$$

neglected as higher order (approximation)

$$\tilde{U}_1 := R_0^{-1} U_1 R_0 = \underbrace{U_{1,4}}_{\textbf{H}} + \tilde{U}_{1,3} + \underbrace{\tilde{U}_{1,2}}_{\textbf{H}} + \tilde{U}_{1,1} + \underbrace{\tilde{U}_{1,0}}_{\textbf{H}}$$

Take $R_1 = R_{1,0} + R_{1,2}, \quad R_{1,2} := \frac{1}{2} R^{ij}_{1,2} \nabla_i \nabla_j$

$[\tilde{H}^0, R_1] \quad \Rightarrow$

$$X_0 := [\tilde{H}^0, R_{10}] = X_{0,1} + X_{0,0}, \quad X_2 := [\tilde{H}^0, R_{12}] = X_{2,3} + X_{2,2} + X_{2,1} + X_{2,0}$$

$\Downarrow$

$$\tilde{H}_1 = \underline{\tilde{H}^0 + U_{1,4} + \tilde{U}_{1,2} + \tilde{U}_{1,0} + X_{2,2} + X_{2.0} + X_{0,0}}$$

Hermitian

$$\boxed{\tilde{U}_{1,3} + X_{2,3}} = 0 \longrightarrow R_{12} \qquad \boxed{\tilde{U}_{1,1} + X_{2,1} + X_{0,1}} = 0 \longrightarrow R_{10}$$

The potential can be made Hermitian within the derivative expansion.

Higher orders can be treated similarly. (detail skipped)

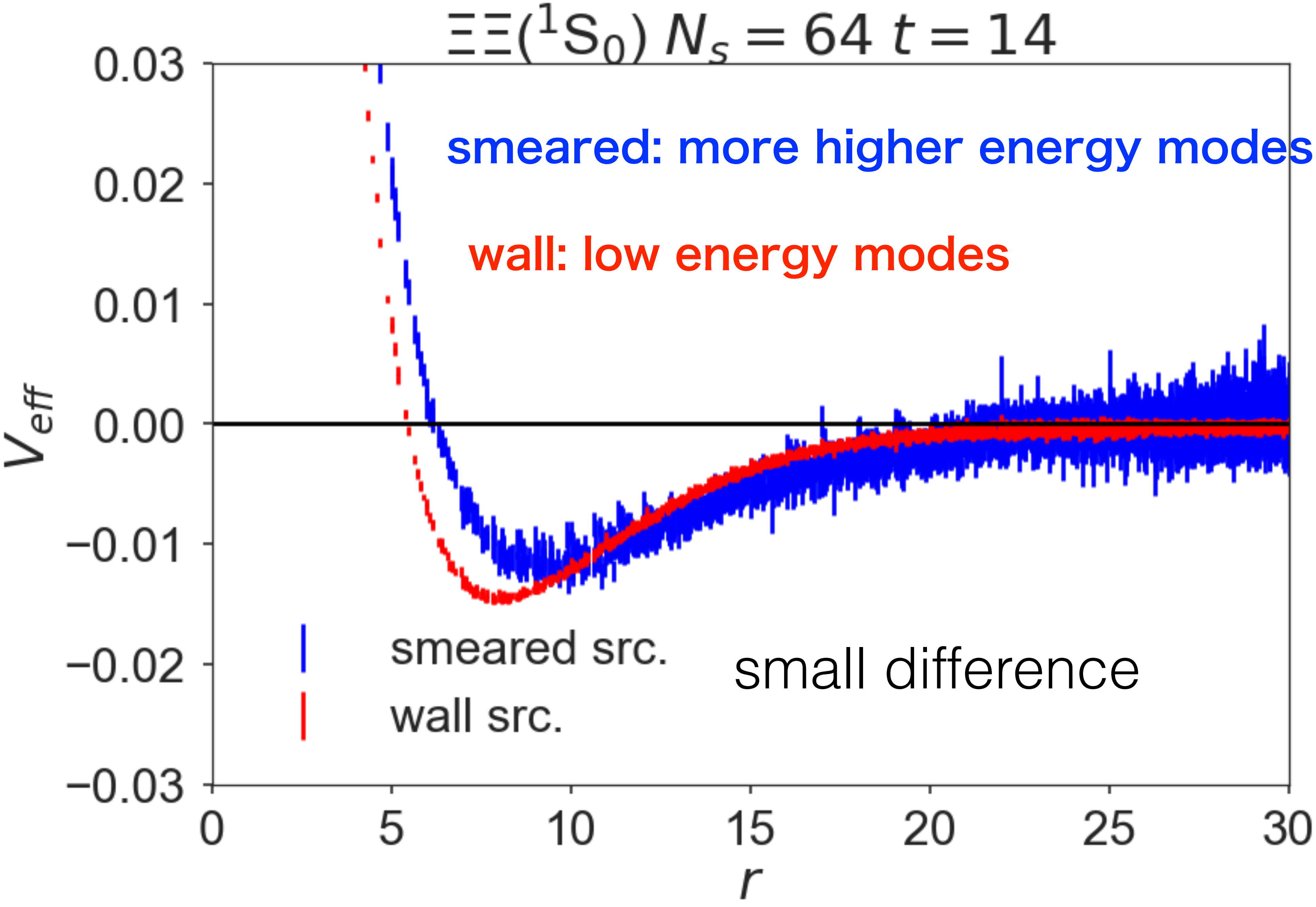
Example of II & III

2+1 flavor QCD

$a = 0.09\ \mathrm{fm}\ (a^{-1} = 2.2\ \mathrm{GeV})$

$m_\pi = 0.51\ \mathrm{GeV},\ m_N = 1.32\ \mathrm{GeV},\ m_K = 0.62\ \mathrm{GeV},\ m_\Xi = 1.46\ \mathrm{GeV}$

$n = 0\ \Xi\Xi$ potential



smeared: more higher energy modes

wall: low energy modes

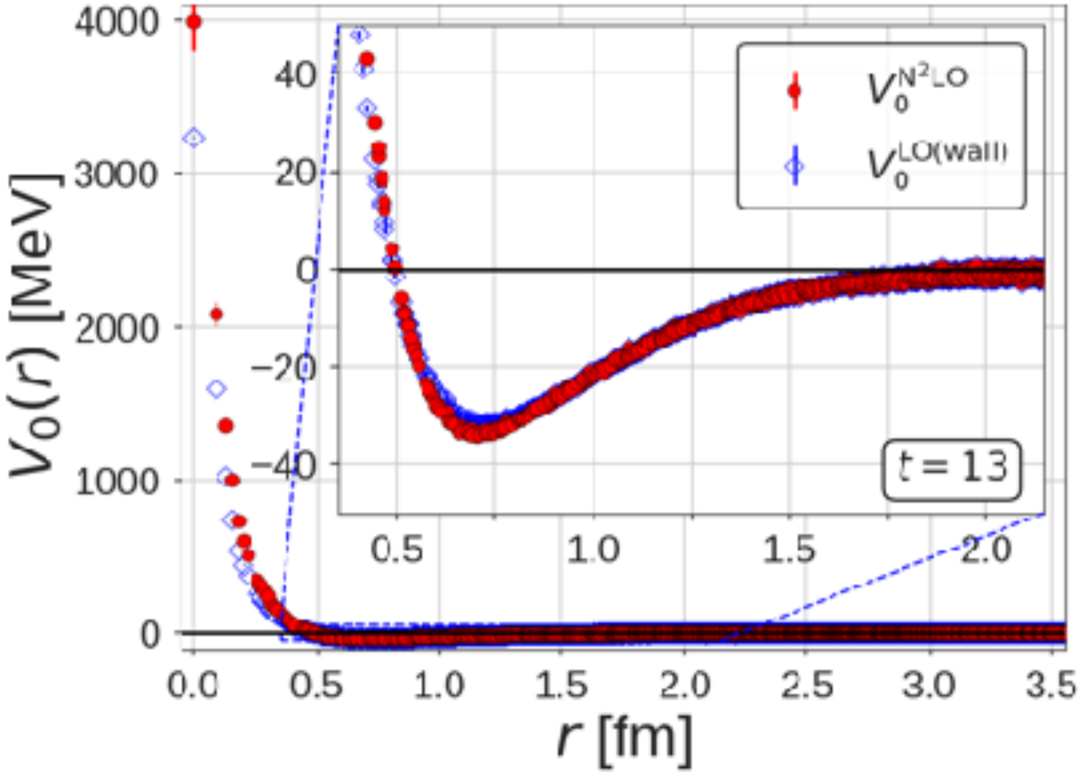
small difference

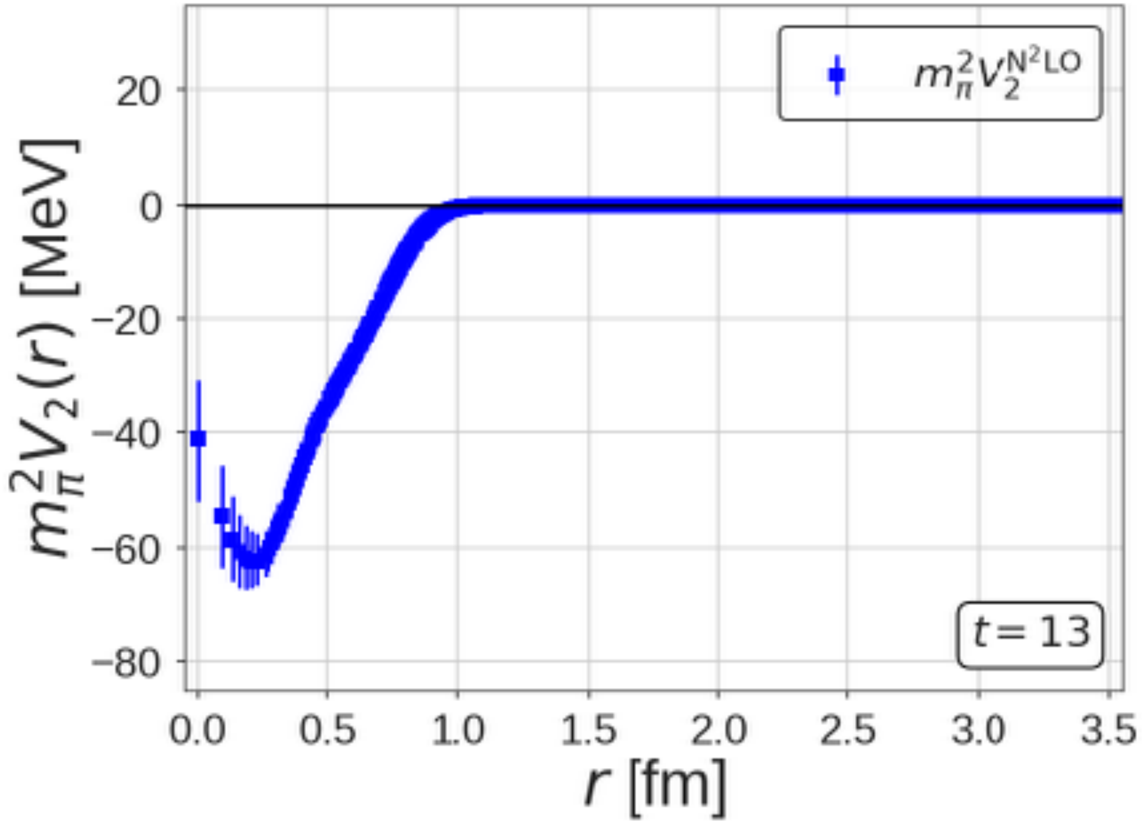
From the difference, we can determine two terms.

$V_0(r)$ **Hermite**

$V_0(r) + V_1(r)\nabla^2$

$m_\pi^2 V_1(r)$ **non-Hermite**

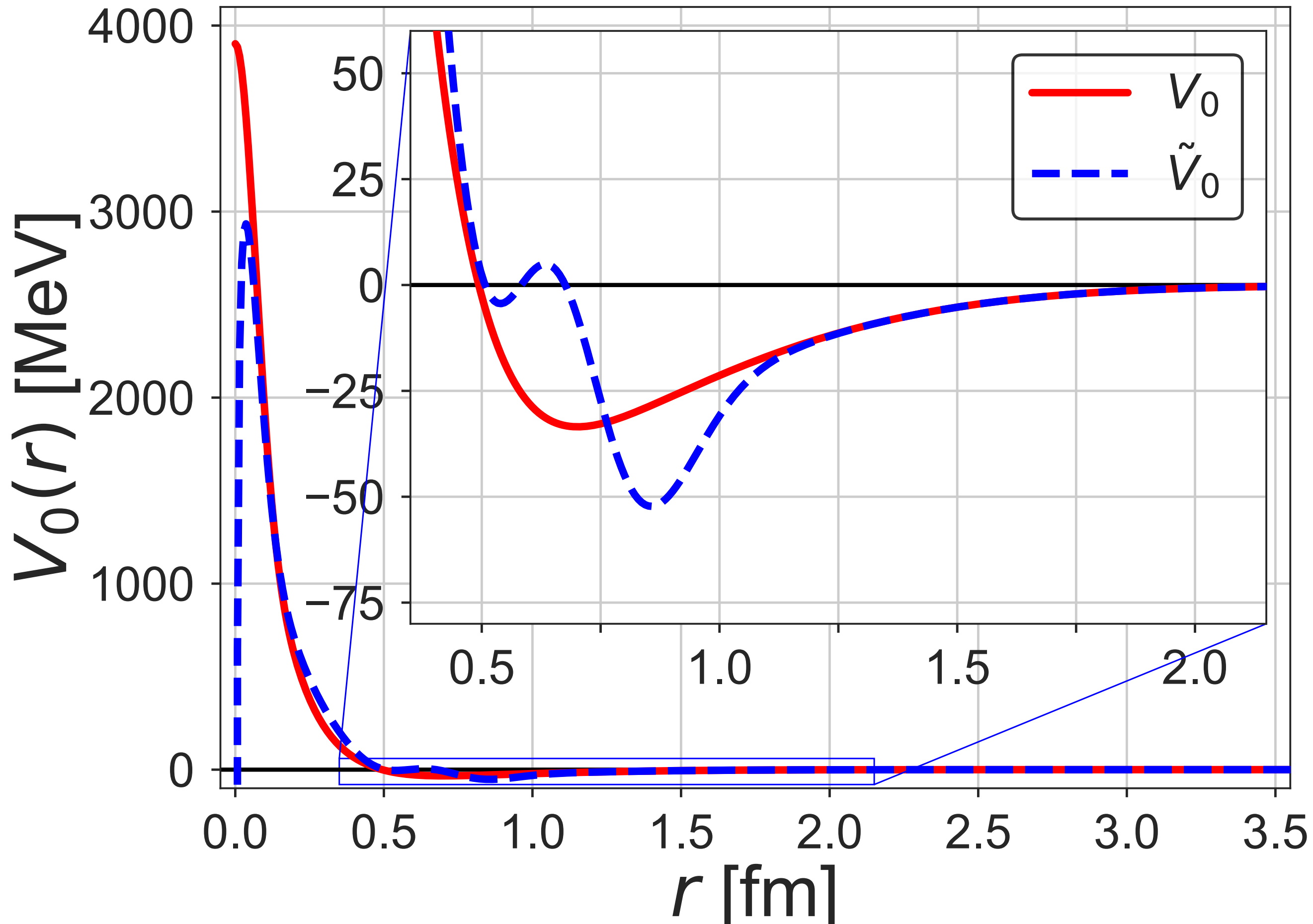




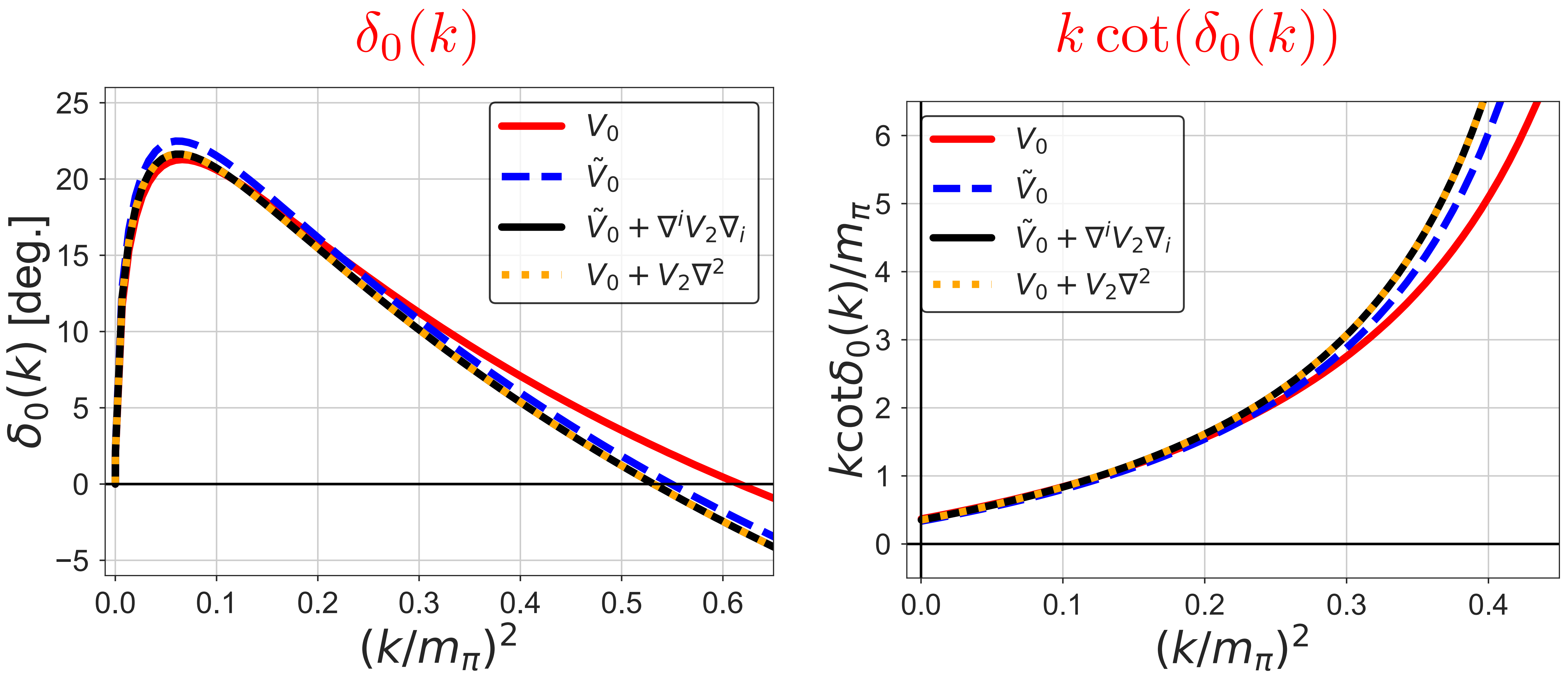
Hermitization

$\tilde{V}_0(r)$

$\tilde{V}_0(r) + \nabla^i V_1(r) \nabla_i$



Scattering phase shift



$\delta_0(k)$

$k\,\cot(\delta_0(k))$

Effects of NLO contributions gradually show up as energy increases.

LO local potential after Hermitization is better than non-Hermitian one.

Our answers

Q1. The HAL QCD potential in the moving system ?

→ I. The potential can be construed from the boosted NBS.

Q2. Validity of the derivative expansion ? small parameter ?

→ II. The derivative expansion is a part of the definition.

Q3. Is the HAL QCD potential Hermite ?

→ III. The HAL QCD potential is non-Hermite, but can be made Hermite.

Q4. Partial wave mixings in the cubic box ?

→ Sinya Gongyo's talk on 4/24.

Q5. Quark annihilation processes and resonances ?

→ Yutaro Akahoshi's talk on 4/16.

States and operators

Lorentz tr. of EM op.

$$\hat{P}'_\mu := U\hat{P}_\mu U^{-1} = (\Lambda^{-1})_\mu{}^\nu \hat{P}_\nu,$$

Lorentz tr. of states

$$U|p\rangle = |\Lambda p\rangle, \qquad (\Lambda p)_\mu = \Lambda_\mu{}^\nu p_\nu \qquad \hat{P}_\mu|p\rangle = p_\mu|p\rangle$$

⇐ $$\hat{P}_\mu U|p\rangle = UU^{-1}\hat{P}_\mu U|p\rangle = U\Lambda_\mu{}^\nu \hat{P}_\nu|p\rangle = \underline{\Lambda_\mu{}^\nu p_\nu} U|p\rangle,$$

⇒ $$U|p_1,p_2\rangle = |\Lambda p_1,\Lambda p_2\rangle, \qquad U|0\rangle = |0\rangle,$$

Scalar field op.

$$\begin{aligned} U\phi(x)U^{-1} &= e^{i\Lambda^{-1}\hat{P}\cdot x}U\phi(0)U^{-1}e^{-i\Lambda^{-1}\hat{P}\cdot x} = e^{i\hat{P}\cdot\Lambda x}\phi(0)e^{-i\hat{P}\cdot\Lambda x} \\ &= \phi(\Lambda x) \end{aligned}$$

⇐ $$\phi(x) = e^{i\hat{P}\cdot x}\phi(0)e^{-i\hat{P}\cdot x}, \quad U\phi(0)U^{-1} = \phi(0)$$

$$\Lambda^{-1}A\cdot B := g^{\mu\nu}(\Lambda^{-1})_\mu{}^\alpha A_\alpha B_\nu = g^{\alpha\beta}A_\alpha\Lambda_\beta{}^\nu B_\nu = A\cdot\Lambda B$$

Procedure

1. At non-zero boost $\vec{P}$, calculate $\varphi_{p,P}(x)$ at fixed energy or $R_{\vec{P}}(x, X)$.
2. Define projections as

$$(P_{\parallel})_{ij} = \frac{P_i P_j}{\vec{P}^2}, \quad P_{\perp} = 1 - P_{\parallel}.$$

3. If P_0 can be measured, calculate

$$\vec{V} = \frac{\vec{P}}{P_0}, \quad \gamma = \frac{1}{\sqrt{1 - \vec{V}^2}}, \quad (\vec{p}^*)^2 = \frac{P_0^2 - \vec{P}^2}{4} - m^2.$$

4. Calculate $V_{-i\gamma V x_{\parallel}}(\gamma x_{\parallel}, x_{\perp}, \nabla)$ from $\varphi_{p,P}$ or $V_{x_4^*=0}(0, x_{\perp}, \nabla)$ from $R_{\vec{P}}$.

IV. Partial wave decomposition in the HAL QCD method

T. Miyamoto, et al. (HAL QCD), in preparation

Lattice QCD in the finite box

Rotational symmetry $O(3,\mathbf{R})$

angular momentum is conserved

partial wave decomposition is possible

Cubic symmetry $O(3,\mathbf{Z})$

a finite number of irreducible representation

different partial waves are mixed

l	rep.	basis polynomials	independent elements
0	$\boxed{A_1^+}$	1	
1	T_1^-	r_i	$i=1,2,3$
2	E^+	$r_i^2-r_j^2$	$(i,j)=(1,2),(2,3)$
2	T_2^+	r_ir_j	$i\neq j$
3	A_2^-	$r_1r_2r_3$	
3	T_1^-	$5r_i^3-3r^2r_j$	$i=1,2,3$
3	T_2^-	$r_i(r_j^2-r_k^2)$	$(i,j,k)=(1,2,3),(2,3,1),(3,1,2)$
4	$\boxed{A_1^+}$	$5(r_1^4+r_2^4+r_3^4)-3r^4$	
4	E^+	$7(r_i^4-r_j^4)-6r^2(r_i^2-r_j^2)$	$(i,j)=(1,2),(2,3)$
4	T_1^+	$r_ir_j^3-r_jr_i^3$	$i\neq j$
4	T_2^+	$7(r_ir_j^3+r_jr_i^3)-6r^2r_ir_j$	$i\neq j$

$$\mathbf{0}=A_1^+,\ \mathbf{1}=T_1^-,\ \mathbf{2}=E^+\oplus T_2^+,$$

$$\mathbf{3}=A_2^-\oplus T_1^-\oplus T_2^-,\ \mathbf{4}=A_1^+\oplus E^+\oplus T_1^+\oplus T_2^+,$$

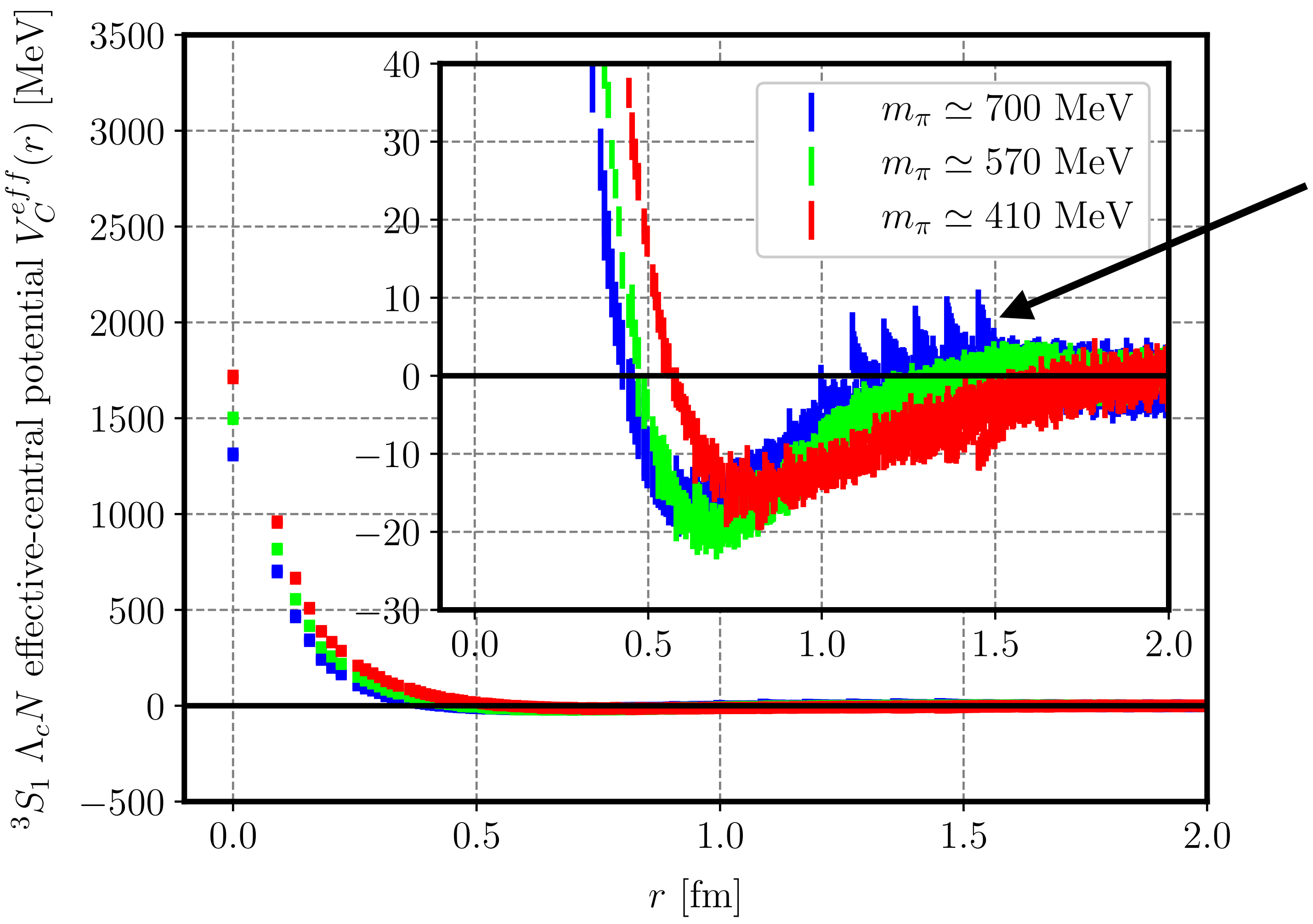
Motivation

Ex. $\varphi_{\mathrm{NBS}}(\vec{x}) \xrightarrow{\text{S-wave projection}} \varphi^{L=0}_{\mathrm{NBS}}(r = |\vec{x}|)$

continuous space $\varphi^{L=0}(r) = \int_s d\Omega\, Y^*_{00}(\theta,\phi)\, \varphi(\vec{x}, r = |\vec{x}|)$ spherical surface integral

discrete space $\varphi^{A_1}(\vec{x}) = \frac{1}{48} \sum_{g \in O(3,\mathbf{Z})} \varphi(g^{-1}\vec{x})$ average over cubic group

$$A_1 = \mathbf{0} \oplus \mathbf{4} \oplus \mathbf{6} \oplus \cdots$$



manifestation of higher (L =4,6, ..) partial waves

Can we remove these higher partial waves ?

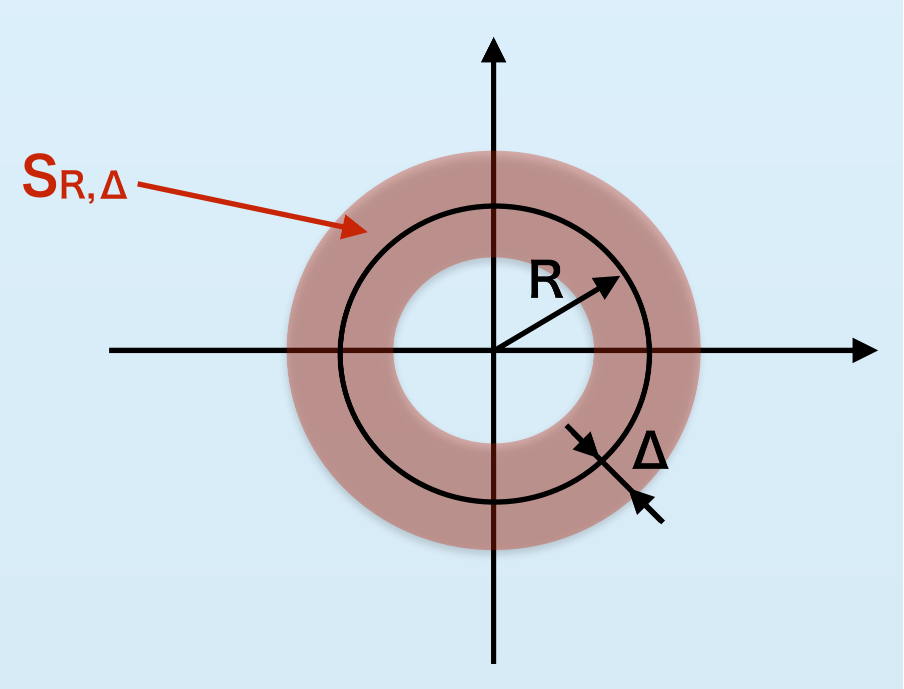
Misner's method

C. W. Misner, Class. Quantum Grav. **21** (2004) S243 - S247

setup

$$\psi(r,\theta,\phi)=\sum_{lm} g_{lm}(r)Y_{lm}(\theta,\phi) \quad \Rightarrow \quad g_{lm}(r)\ ?$$

A complete set in the shell $S_{R,\Delta}=\{\vec{x}|R-\Delta\le|\vec{x}|\le R+\Delta\}$



$$\mathcal{Y}^{R,\Delta}_{nlm}(r,\theta,\phi):=G^{R,\Delta}_n(r)Y_{lm}(\theta,\phi)$$

$$\int_{R-\Delta}^{R+\Delta} r^2dr\,G^{R,\Delta}_n(r)G^{R,\Delta}_m(r)=\delta_{n,m}$$

Ex.

$$G^{R,\Delta}_n(r)\equiv \underbrace{P_n\left(\frac{r-R}{\Delta}\right)}_{\text{Legendre polynomial}}\frac{1}{r}\sqrt{\frac{2n+1}{2\Delta}}$$

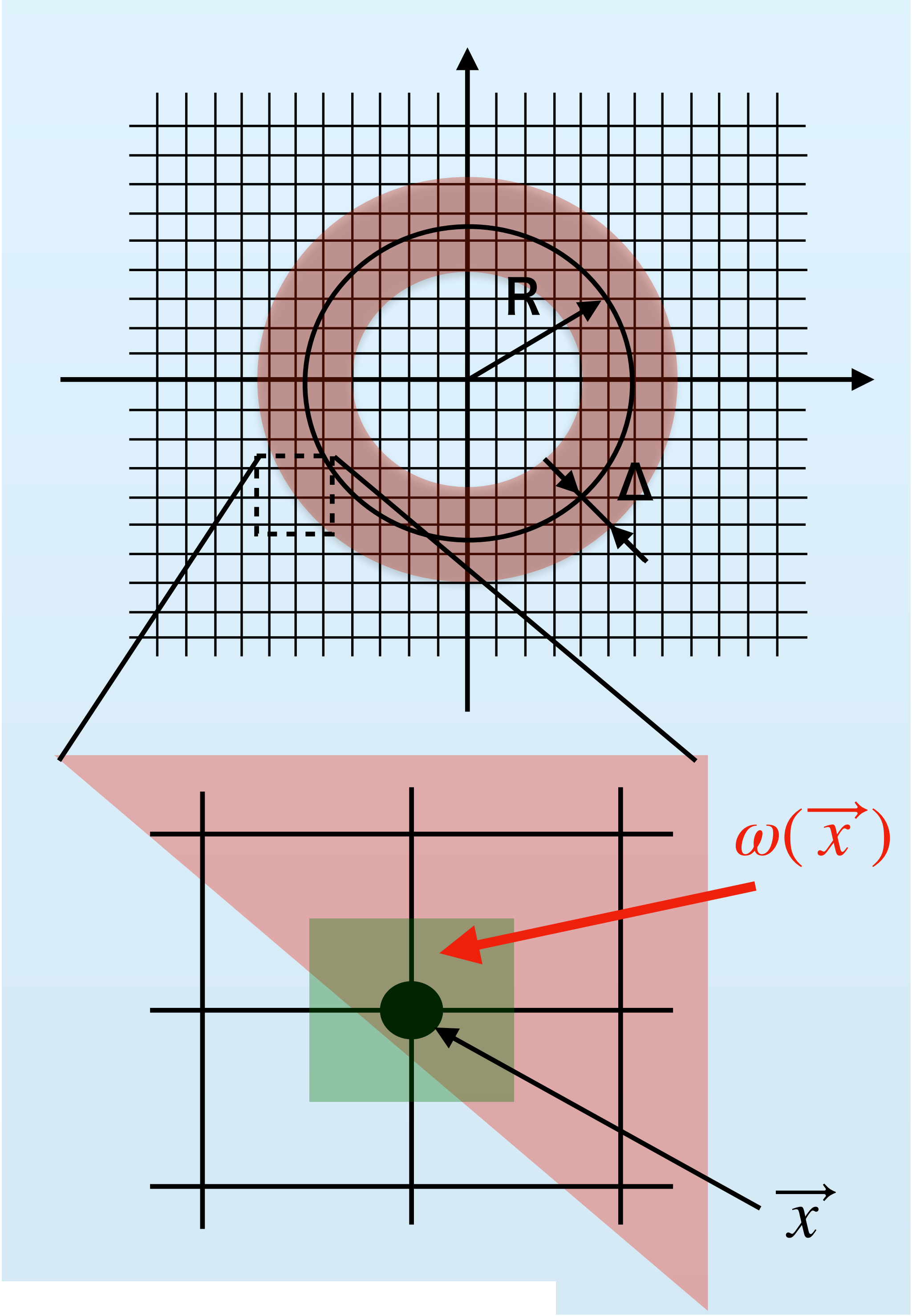
$$\psi(r,\theta,\phi)=\sum_{nlm} c_{nlm}\,\mathcal{Y}^{R,\Delta}_{nlm}(r,\theta,\phi) \quad \text{in } S_{R,\Delta} \quad \Rightarrow \quad c_{nlm}=\int_{S_{R,\Delta}} d^3r\,\overline{\mathcal{Y}^{R,\Delta}_{nlm}(r,\theta,\phi)}\,\psi(r,\theta,\phi)$$

$$\because \int_{S_{R,\Delta}} d^3r\,\overline{\mathcal{Y}^{R,\Delta}_{nlm}}(\vec{r})\mathcal{Y}^{R,\Delta}_{n'l'm'}(\vec{r})=\delta_{nn'}\delta_{ll'}\delta_{mm'}$$

$$\Rightarrow \quad g_{lm}(R)=\sum_n c_{nlm}G^{R,\Delta}_n(R)$$

exact in continuous space

discrete space



$$\langle f|g\rangle_c \equiv \int_{\vec{x}\in S_{R,\Delta}} d^3x\, \overline{f(\vec{x})}g(\vec{x})$$

$$\langle \mathcal{Y}_A^{R,\Delta}|\mathcal{Y}_B^{R,\Delta}\rangle_c = \delta_{AB}$$

(A,B = nlm)

$$\langle f|g\rangle_d \equiv \sum_{\vec{x}\in S_{R,\Delta}} \underline{\omega(\vec{x})}\,\overline{f(\vec{x})}g(\vec{x})$$

$$\langle \mathcal{Y}_A^{R,\Delta}|\mathcal{Y}_B^{R,\Delta}\rangle_d \neq \delta_{AB}$$

weight function

dual basis

$$\langle \mathcal{Y}_A^{R,\Delta}|\mathcal{Y}_B^{R,\Delta}\rangle_d = G_{AB} = G_{BA}^*$$

$$\mathcal{Y}_{adj,A}^{R,\Delta} \equiv \sum_B G_{AB}^{-1}\mathcal{Y}_B^{R,\Delta}$$

$$\langle \mathcal{Y}_{dual,A}^{R,\Delta}|\mathcal{Y}_B^{R,\Delta}\rangle_d = \delta_{AB}$$

Since a number of points in the shell is finite, we have to truncate $n \leq n_{\max}$ and $l \leq l_{\max}$ to have G_{AB}^{-1}.

This introduces an approximation !

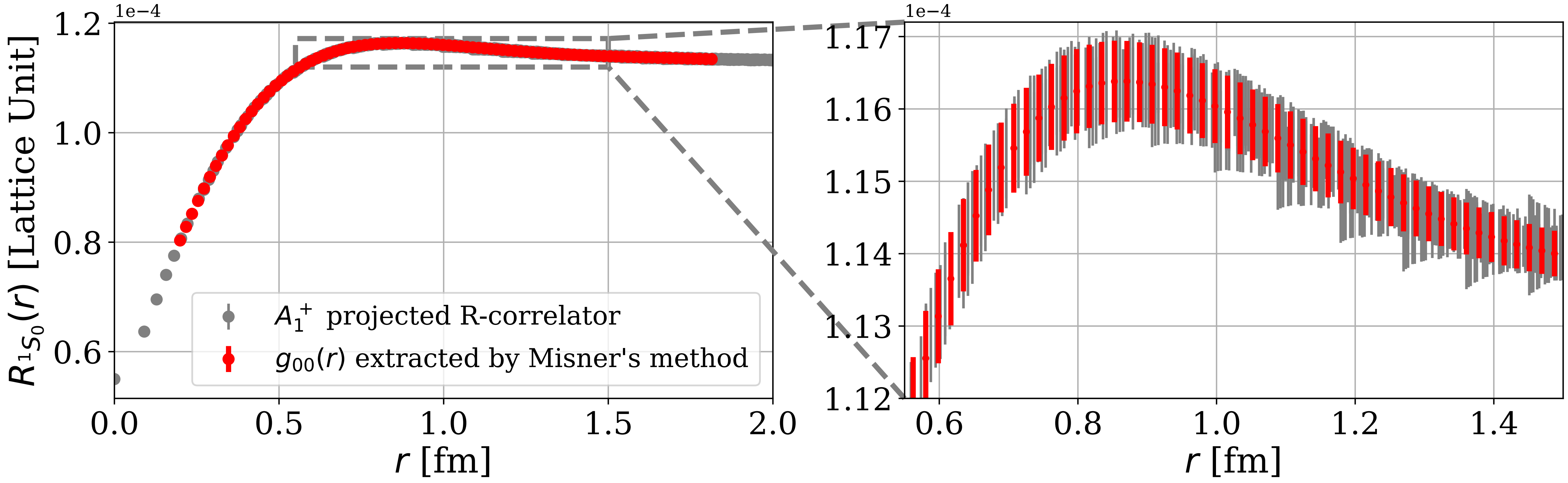
$$\left(\sum_B = \sum_{n=0}^{n_{\max}}\sum_{l=0}^{l_{\max}}\sum_{m=-l}^{l}\right)$$

A choice of Δ, $n_{\max}$ and $l_{\max}$

$G_n^{R,\Delta}$ has $O(\Delta^{n_{\max}+2})$ discretization errors → $\Delta \sim a, n_{\max} \geq 2$

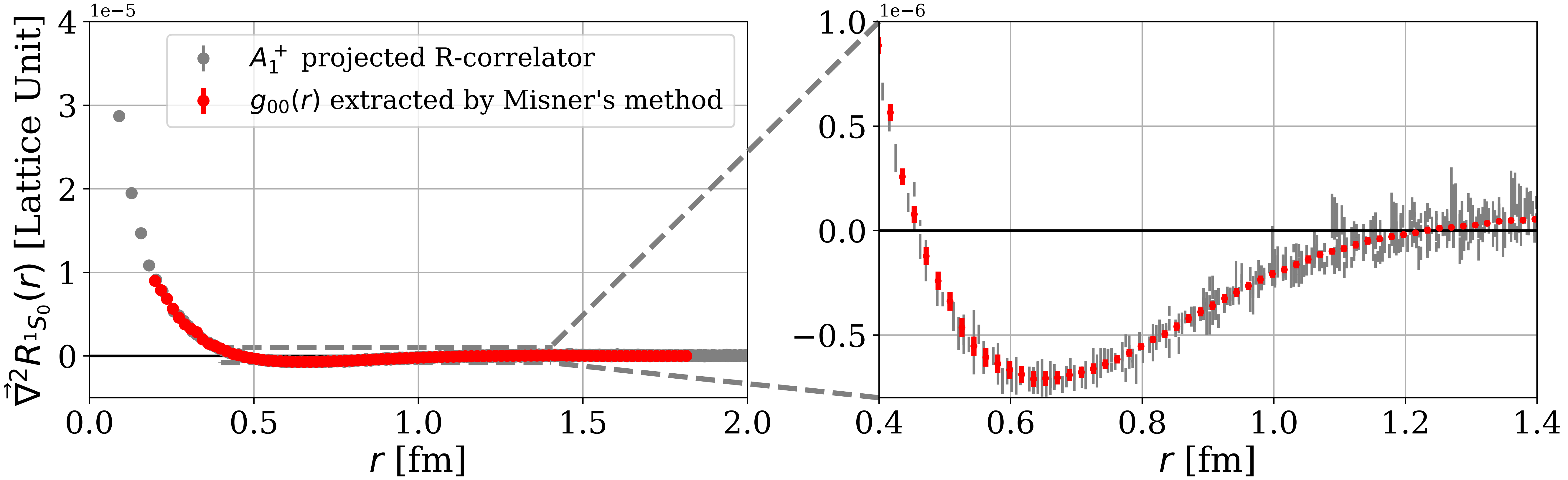
Our choice $\Delta = a, n_{\max} = 2, l_{\max} = 6$

NBS wave function



Higher L contributions seem to be removed by the Misner’s method !

Laplacian term



Conventional HAL QCD

$$\vec{\nabla}^2\varphi^{A_1}(\vec{x}) \simeq \sum_{k=1}^{3} \frac{\varphi^{A_1}(\vec{x}+a\vec{k}) + \varphi^{A_1}(\vec{x}-a\vec{k}) - 2\varphi^{A_1}(\vec{x})}{a^2}$$

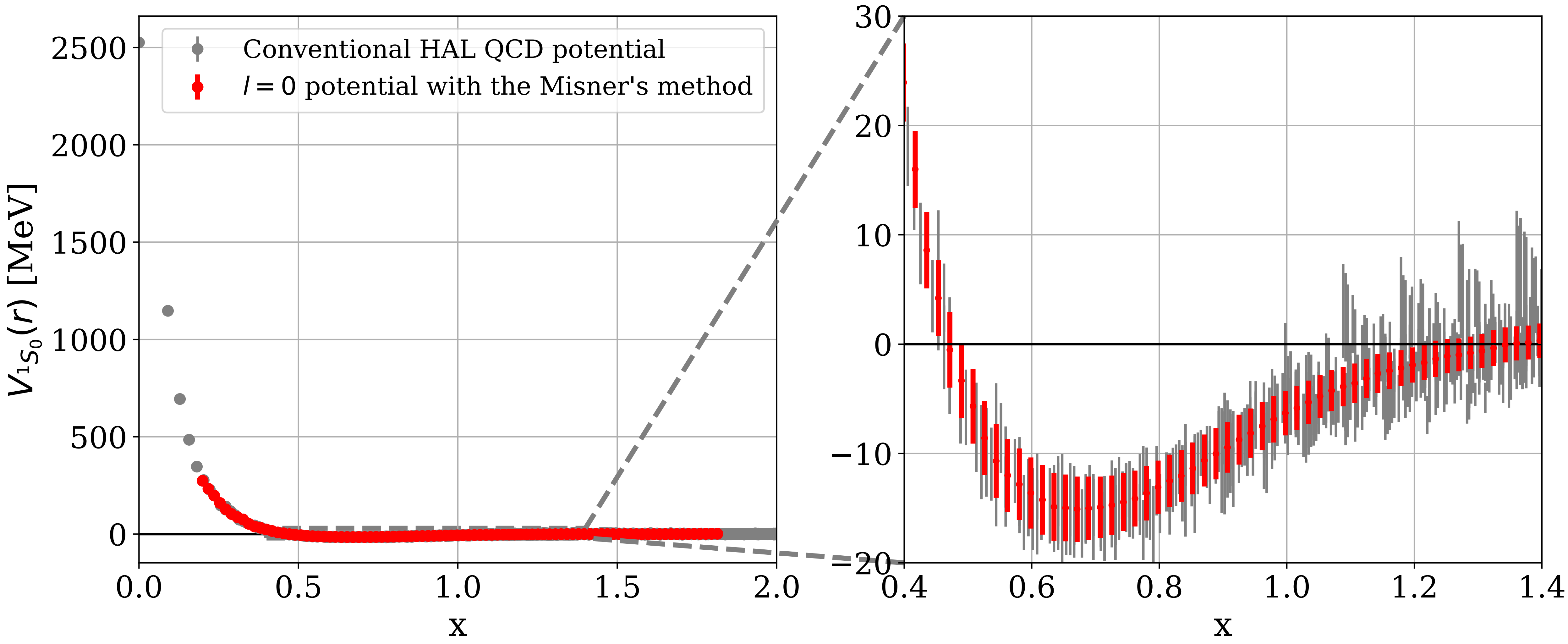
Misner's method

$$\vec{\nabla}^2 g_{lm}(r) = \sum_{n=0}^{n_{\max}} c_{nlm}^{R,\Delta} \frac{1}{r}\frac{\partial^2}{\partial r^2}\left[rG_n^{R,\Delta}(r)\right]$$

analytic derivative

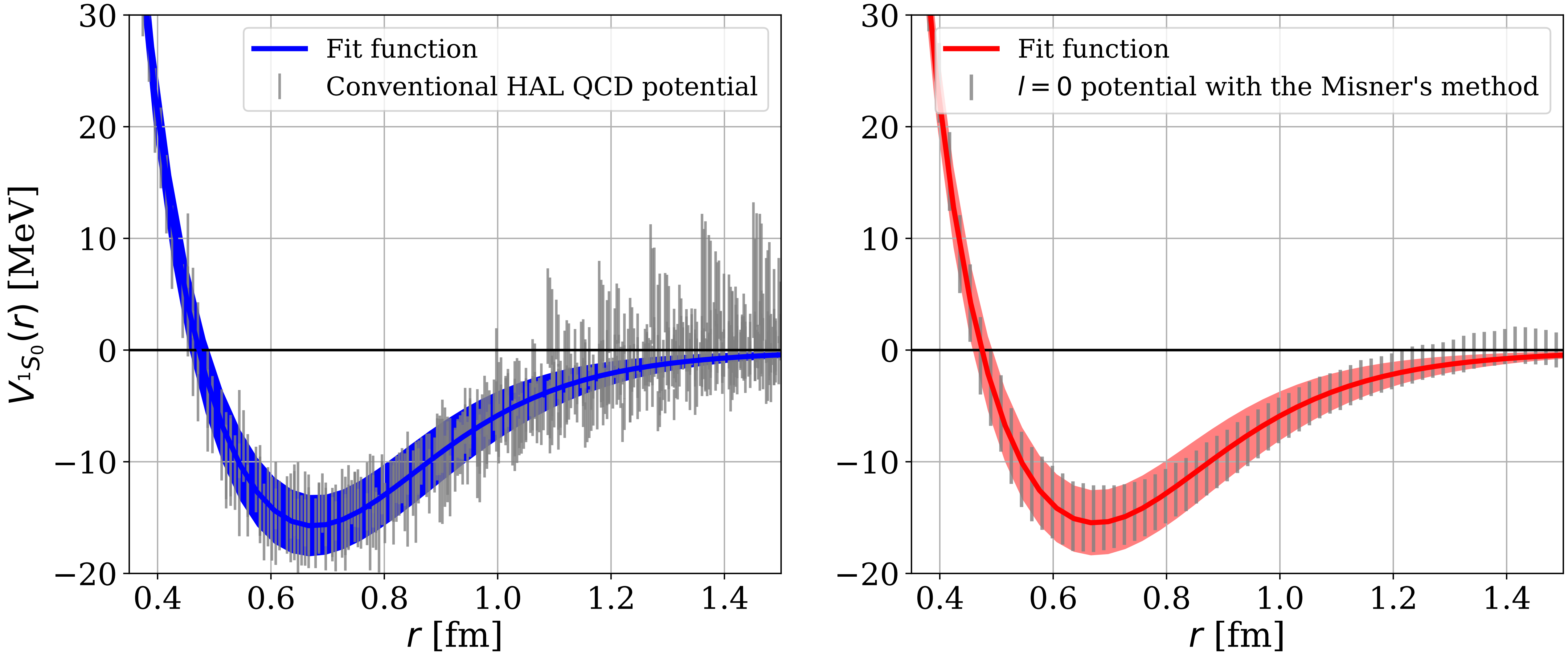
The finite difference approximation enhances higher partial wave contributions.

Potential



The Misner’s method can remove large fluctuations caused by the contamination from higher partial waves to the S=0 component.

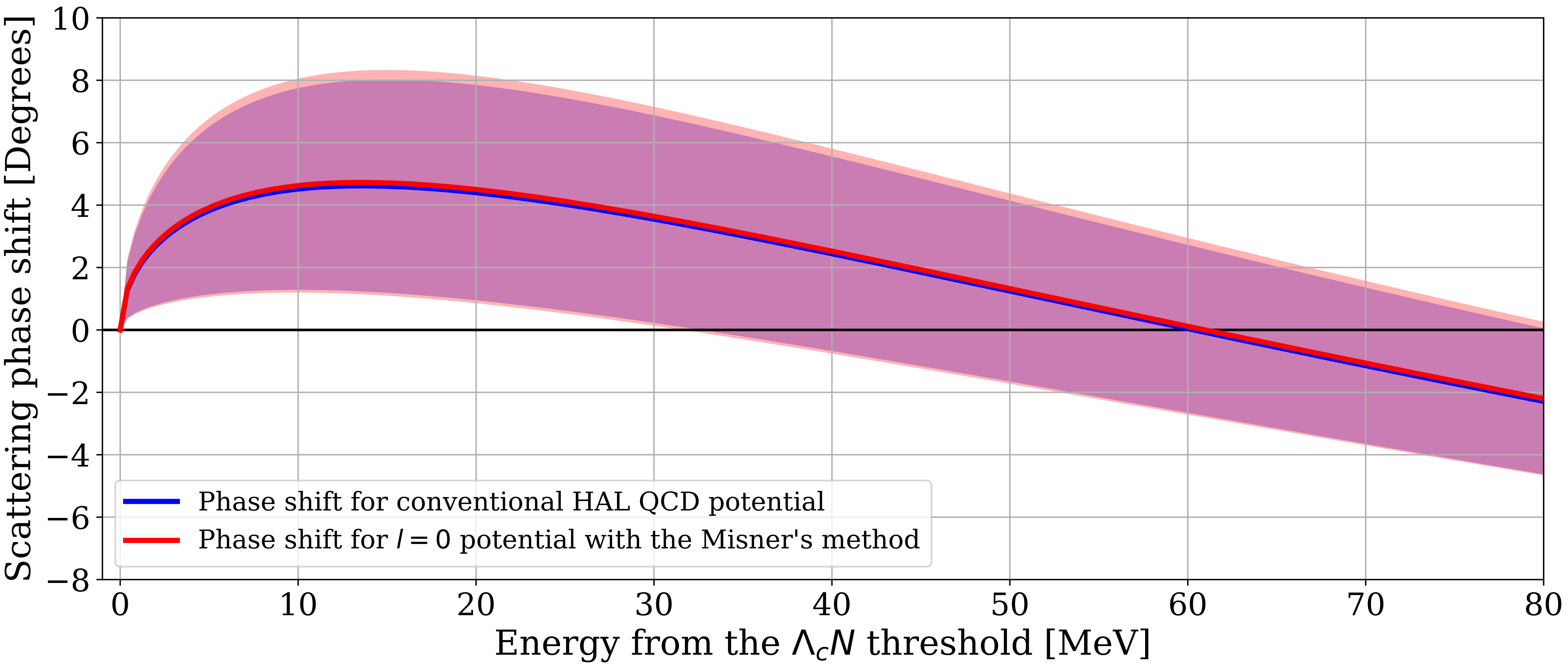
Fits



Fit to the conventional HAL QCD data $\simeq$ **The Misner's method**

Statistical errors of the fit to the conventional HAL QCD data are not affected by contaminations from higher partial waves.

Scattering phase shift



Almost identical between the conventional result and the Misner's method.

No improvement of statistical errors,
but we have more confidence on validity of our results !