Two-pion contributions to the $(g-2)_\mu$

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ALBERT EINSTEIN CENTER
FOR FUNDAMENTAL PHYSICS

Frontiers in Lattice QCD, Kyoto, 16.4.2019

Based on:

[HLbL]

JHEP09(14)091, JHEP09(15)074, JHEP04(17)161, PRL(17)
in collab. with M. Hoferichter, M. Procura and P. Stoffer and
PLB738(2014)6 .... +B. Kubis
and work in progress with F. Hagelstein and L. Laub

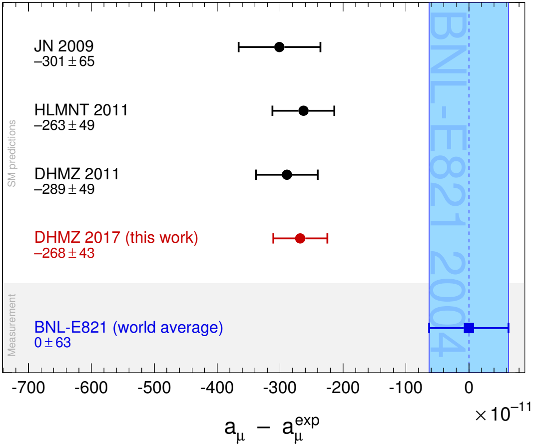
[HVP]

JHEP02(19)006, in collab. with M. Hoferichter and P. Stoffer

Outline

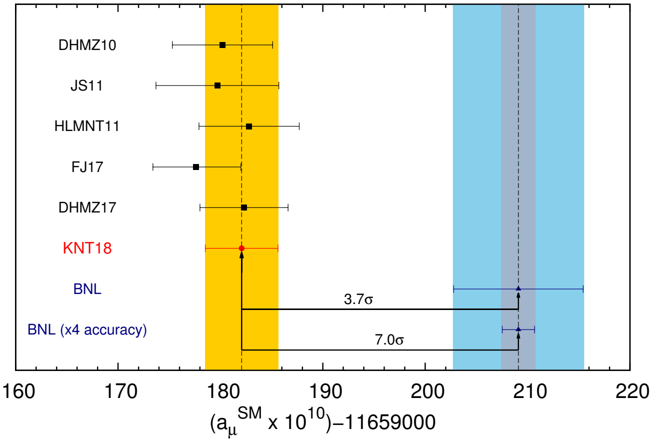
Status of $(g-2)_\mu$, experiment vs SM

Davier, Hoecker, Malaescu, Zhang 2017



Status of $(g-2)_\mu$, experiment vs SM

Keshavarzi, Nomura, Teubner, 2018 (KNT18)



Fermilab experiment's goal: error $\times 1/4$, should be matched by theory:

$\Rightarrow$ Muon "$(g-2)$ Theory Initiative" led by A. El-Khadra and C. Lehner

Status of $(g-2)_\mu$, experiment vs SM

KNT 18

	$a_\mu\ [10^{-11}]$	$\Delta a_\mu\ [10^{-11}]$
experiment	116 592 089.	63.
QED $\mathcal{O}(\alpha)$	116 140 973.21	0.03
QED $\mathcal{O}(\alpha^2)$	413 217.63	0.01
QED $\mathcal{O}(\alpha^3)$	30 141.90	0.00
QED $\mathcal{O}(\alpha^4)$	381.01	0.02
QED $\mathcal{O}(\alpha^5)$	5.09	0.01
QED total	116 584 718.97	0.07
electroweak, total	153.6	1.0
HVP (LO) [KNT 18]	6 932.7	24.6
HVP (NLO) [KNT 18]	−98.2	0.4
HLbL [update of Glasgow consensus–KNT 18]	98.0	26.0
HVP (NNLO) [Kurz, Liu, Marquard, Steinhauser 14]	12.4	0.1
HLbL (NLO) [GC, Hoferichter, Nyffeler, Passera, Stoffer 14]	3.0	2.0
theory	116 591 820.5	35.6

Status of $(g-2)_\mu$, experiment vs SM

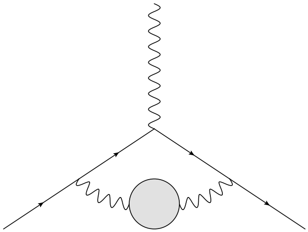
KNT 18

$$a_\mu^{\mathrm{exp}} - a_\mu^{\mathrm{SM}} = 268.5 \pm 72.4 \qquad [3.7\sigma]$$

Keshavarzi, Nomura, Teubner, 2018

Theory uncertainty comes from hadronic physics

- Hadronic contributions responsible for most of the theory uncertainty
- Hadronic vacuum polarization (HVP) can be systematically improved



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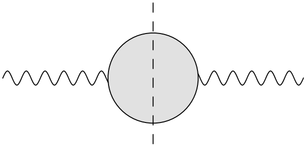
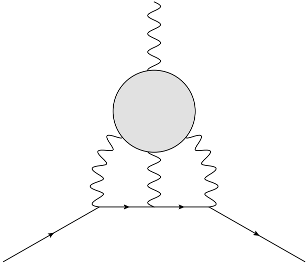


  - basic principles: unitarity and analyticity
  - direct relation to experiment: $\sigma_{\text{tot}}(e^+e^- \to \gamma^* \to \text{hadrons})$
  - dedicated e^+e^- program: BaBar, Belle, BESIII, CMD3, KLOE2, SND
  - alternative approach: lattice (ETMC, Mainz, HPQCD, BMW, RBC/UKQCD)

Theory uncertainty comes from hadronic physics

- Hadronic contributions responsible for most of the theory uncertainty
- Hadronic vacuum polarization (HVP) can be systematically improved
- Hadronic light-by-light (HLbL) is more problematic:



  - 4-point fct. of em currents in QCD
  - "*it cannot be expressed in terms of measurable quantities*"
  - until recently, only model calculations
  - lattice QCD is making fast progress

Muon $g-2$ Theory Initiative

Steering Committee:

GC
Michel Davier
Simon Eidelman
Aida El-Khadra (co-chair)
Christoph Lehner (co-chair)
Tsutomu Mibe (J-PARC E34 experiment)
Andreas Nyffeler
Lee Roberts (Fermilab E989 experiment)
Thomas Teubner

Workshops:

- First plenary meeting, Q-Center (Fermilab), 3-6 June 2017
- HVP WG workshop, KEK (Japan), 12-14 February 2018
- HLbL WG workshop, U. of Connecticut, 12-14 March 2018
- Second plenary meeting, Mainz, 18-22 June 2018
- Third plenary meeting, Seattle, 9-13 September 2019

Dispersive approach for hadronic vacuum polarization

$$\Pi_{\mu\nu}(q) = i \int d^4x e^{iqx} \langle 0| T j_\mu(x) j_\nu(0) |0\rangle = \left(q_\mu q_\nu - g_{\mu\nu} q^2\right) \Pi(q^2)$$

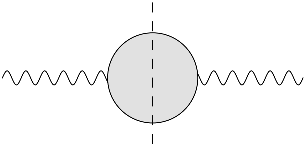
where $j^\mu(x) = \sum_i Q_i \bar{q}_i(x) \gamma^\mu q_i(x)$, $i = u, d, s$ is the em current

- Lorentz invariance: 2 structures
- gauge invariance: reduction to 1 structure
- Lorentz-tensor defined in such a way that the function $\Pi(q^2)$ does not have kinematic singularities or zeros
- $\bar{\Pi}(q^2) := \Pi(q^2) - \Pi(0)$ satisfies

$$\bar{\Pi}(q^2) = \frac{q^2}{\pi} \int_{4M_\pi^2}^\infty dt \frac{\mathrm{Im}\bar{\Pi}(t)}{t(t-q^2)}$$

Unitarity for HVP

For HVP the unitarity relation is simple and looks the same for all possible intermediate states



$$\mathrm{Im}\Pi(q^2) \propto \sigma(e^+e^- \to \mathrm{hadrons})$$

which implies ($H_i(t) = i$-th hadronic state)

$$\bar{\Pi}(q^2) = \sum_i \frac{q^2}{\pi} \int_{4M_\pi^2}^{\infty} dt \frac{\sigma(e^+e^- \to H_i(t))}{4\pi\alpha(t-q^2)}$$

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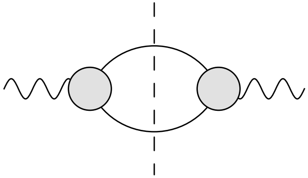

$$\mathrm{Im}\Pi(q^2) \propto \sigma(e^+e^- \to \mathrm{hadrons})$$

which implies $(H_i(t) = 2\pi)$

$$\bar{\Pi}_{2\pi}(q^2) = \frac{q^2}{\pi}\int_{4M_\pi^2}^{\infty} dt \frac{\sigma(e^+e^- \to 2\pi)}{4\pi\alpha(t-q^2)}$$

de Trocóniz, Ynduráin (01,04), Leutwyler, GC (02,03), Anthanarayan et al. (13,16)

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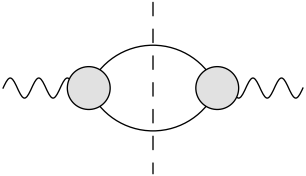

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$$\bar{\Pi}_{2\pi}(q^2) = \frac{q^2}{\pi}\int_{4M_\pi^2}^\infty dt \frac{\sigma(e^+e^- \to 2\pi)}{4\pi\alpha(t-q^2)} = \frac{q^2}{\pi}\int_{4M_\pi^2}^\infty dt \frac{\alpha\sigma_\pi(t)^3|F_V^\pi(t)|^2}{12t(t-q^2)}$$

de Trocóniz, Ynduráin (01,04), Leutwyler, GC (02,03), Anthanarayan et al. (13,16)

Analytic properties of pion form factors

Mathematical problem:

1. $F(t)$: analytic function except for a cut for $4M_\pi^2 \leq t < \infty$
2. $e^{-i\delta(t)}F(t) \in \mathbb{R}$ for $\mathrm{Im}(t) \to 0^+$, with $\delta(t)$ a known function

Exact solution:

Omnès (58)

$$F(t) = P(t)\Omega(t) = P(t) \exp\left\{ \frac{t}{\pi} \int_{4M_\pi^2}^\infty \frac{dt'}{t'} \frac{\delta(t')}{t'-t} \right\} ,$$

$P(t)$ a polynomial $\Leftrightarrow$ behaviour of $F(t)$ for $t \to \infty$
or presence of zeros

$\Omega(t)$ is called the Omnès function

Vector form factor of the pion

Pion vector form factor

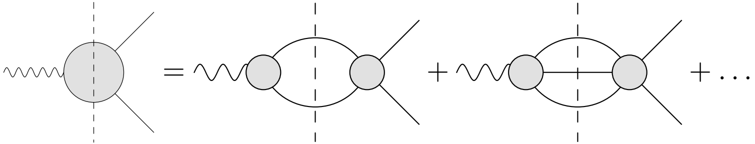
$$\langle \pi^i(p')|V_\mu^k(0)|\pi^l(p)\rangle = i\epsilon^{ikl}(p'+p)_\mu F_V^\pi(s) \qquad s=(p'-p)^2$$

- normalization fixed by gauge invariance:

$$F_V^\pi(0)=1 \qquad \overset{\text{no zeros}}{\Longrightarrow} \qquad P(t)=1$$

- $e^+e^- \to \pi^+\pi^-$ data $\Rightarrow$ free parameters in $\Omega(t)$

Omnès representation including isospin breaking



Omnès representation including isospin breaking

- Omnès representation

$$F_V^\pi(s) = \exp\left[\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\delta(s')}{s'(s'-s)}\right] \equiv \Omega(s)$$

- Split elastic from inelastic contributions

$$\delta = \delta_1^1 + \delta_{\text{in}} \quad \Rightarrow \quad F_V^\pi(s) = \Omega_1^1(s)\Omega_{\text{in}}(s)$$

Eidelman-Lukaszuk: unitarity bound on δ_{in}

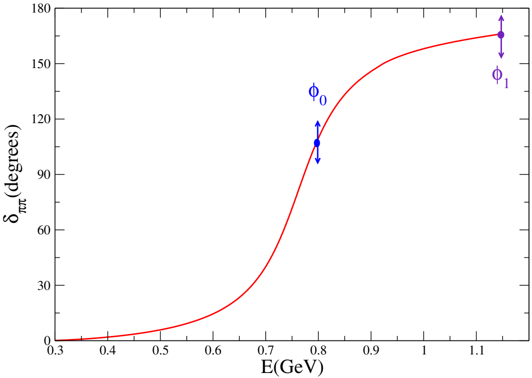
$$\sin^2\delta_{\text{in}} \le \frac{1}{2}\left(1-\sqrt{1-r^2}\right), \; r = \frac{\sigma^{I=1}_{e^+e^-\to\neq 2\pi}}{\sigma_{e^+e^-\to 2\pi}} \Rightarrow s_{\text{in}} = (M_\pi+M_\omega)^2$$

- $\rho-\omega$–mixing $\qquad F_V(s) = \Omega_{\pi\pi}(s)\cdot\Omega_{\text{in}}(s)\cdot G_\omega(s)$

$$G_\omega(s) = 1+\epsilon\frac{s}{s_\omega - s} \qquad \text{where} \qquad s_\omega = (M_\omega - i\Gamma_\omega/2)^2$$

Free parameters

$$\Omega_1^1(s) \quad \Rightarrow \quad \begin{cases} \phi_0 = \delta_{\pi\pi}((0.8\ \mathrm{GeV})^2) \\ \phi_1 = \delta_{\pi\pi}(68 M_\pi^2) \end{cases} \quad \text{[Roy eqs.]}$$



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$$G_\omega(s) \quad \Rightarrow \quad \begin{cases} \epsilon \qquad \omega - \rho \text{ mixing} \\ M_\omega \end{cases}$$

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$$G_\omega(s) \quad \Rightarrow \quad \begin{cases} \epsilon & \omega-\rho \text{ mixing} \\ M_\omega \end{cases}$$

$$\Omega_{\rm in}(s) \quad \Rightarrow \quad \begin{cases} c_1 \\ \vdots \\ c_P \end{cases} \qquad \mathrm{Im}\Omega_{\rm in}(s) = 0 \quad s \leq s_{\pi\omega}$$

$$\Omega_{\rm in}(s) = 1 + \sum_{k=1}^{N} c_k \left(z(s)^k - z(0)^k\right) \qquad z = \frac{\sqrt{s_{\pi\omega} - s_1} - \sqrt{s_{\pi\omega} - s}}{\sqrt{s_{\pi\omega} - s_1} + \sqrt{s_{\pi\omega} - s}}$$

Systematic effects

- input parameters for Roy equation solutions

Ananthanarayan et al. (01), Caprini, GC, Leutwyler (12)

- continuation of the phase above the region of validity of Roy equations
- ω width
- order N of the conformal polynomial, parameter s_1

Data sets and fit method

- timelike: SND, CMD-2, BaBar, KLOE, (BESIII)
  [using full covariance matrices of BaBar and KLOE]
- spacelike: NA7
- Eidelman-Łukasuk bound on the inelastic phase

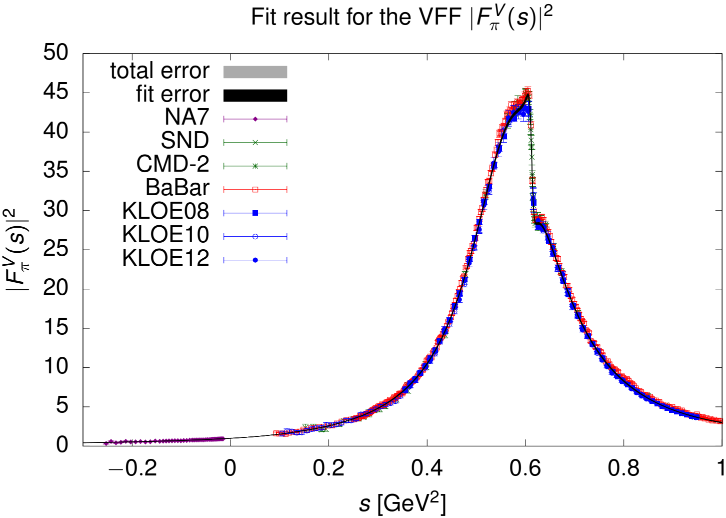
  Eidelman-Łukasuk (04)
- energy rescaling parameter ξ_i for each experiment, within the declared systematic uncertainty for energy calibration
- we apply an iterative fit routine to avoid the D'Agostini bias

  D'Agostini (94), Ball et al. (NNPDF) (10)

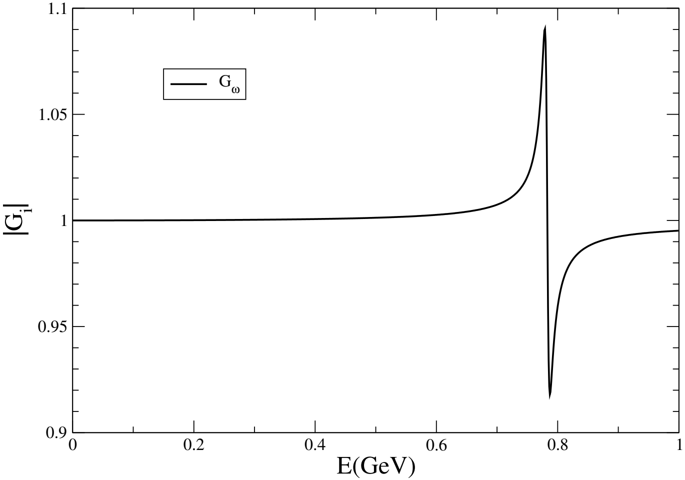
Fit results

	χ^2/dof	M_ω [MeV]	$10^3 \times \xi_j$	$\delta_1^1(s_0)$ [°]	$\delta_1^1(s_1)$ [°]	$10^3 \times \epsilon_\omega$
SND	1.40	781.49(32)(2)	0.0(6)(0)	110.5(5)(8)	165.7(0.3)(2.4)	2.03(5)(2)
CMD-2	1.18	781.98(29)(1)	0.0(6)(0)	110.5(5)(8)	166.4(0.4)(2.4)	1.88(6)(2)
BaBar	1.14	781.86(14)(1)	0.0(2)(0)	110.4(3)(7)	165.7(0.2)(2.5)	2.04(3)(2)
KLOE	1.36	781.82(17)(4)	{ 0.6(2)(0); −0.3(2)(0); −0.2(3)(0)	110.4(2)(6)	165.6(0.1)(2.4)	1.97(4)(2)
KLOE″	1.20	781.81(16)(3)	{ 0.5(2)(0); −0.3(2)(0); −0.2(3)(0)	110.3(2)(6)	165.6(0.1)(2.4)	1.98(4)(1)
Energy scan	1.28	781.75(22)(1)		110.4(3)(8)	166.0(0.2)(2.4)	1.97(4)(2)
All e^+e^-	1.31	781.68(9)(4)		110.5(1)(7)	165.8(0.1)(2.4)	2.02(2)(3)
All e^+e^-, NA7	1.29	781.68(9)(3)		110.4(1)(7)	165.8(0.1)(2.4)	2.02(2)(3)

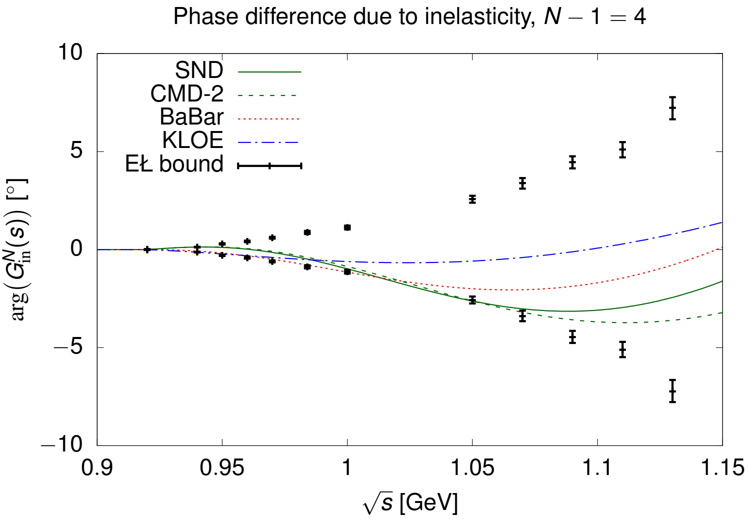
Fit results


Fit result for the VFF $|F_\pi^V(s)|^2$
total error
fit error
NA7
SND
CMD-2
BaBar
KLOE08
KLOE10
KLOE12
$|F_\pi^V(s)|^2$
s [GeV^2]
0
5
10
15
20
25
30
35
40
45
50
−0.2
0
0.2
0.4
0.6
0.8
1

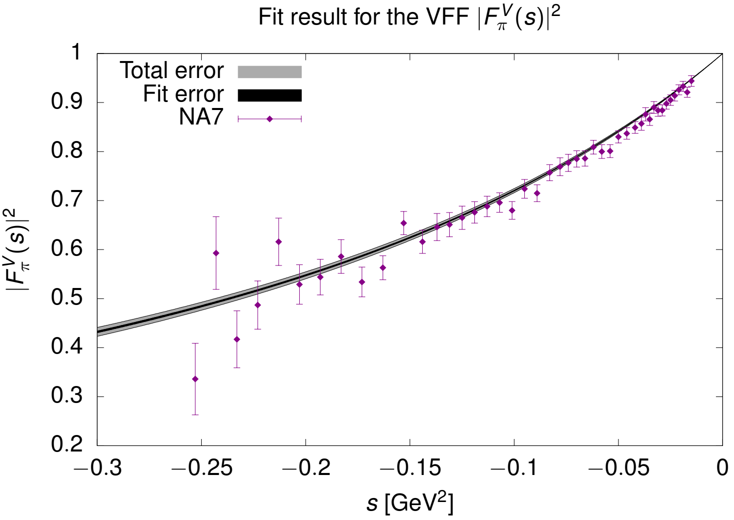
Fit results



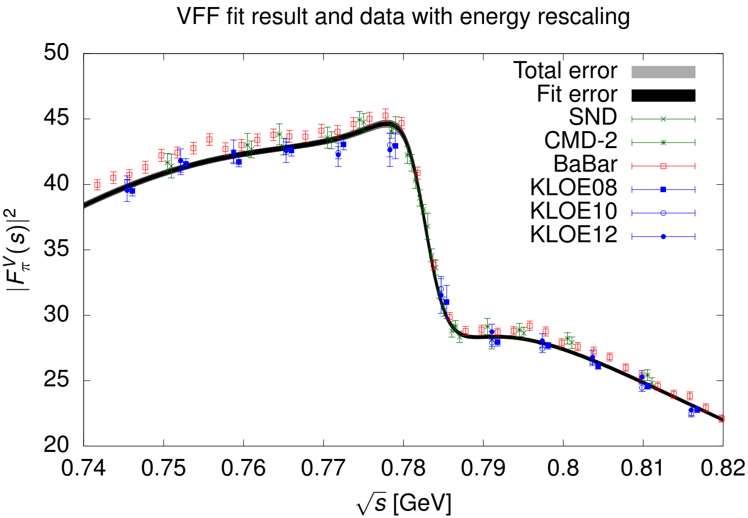
Fit results


Phase difference due to inelasticity, $N-1=4$
SND
CMD-2
BaBar
KLOE
EŁ bound
10
5
0
−5
−10
0.9
0.95
1
1.05
1.1
1.15
$\sqrt{s}$ [GeV]
$\arg(G_{\text{in}}^N(s))$ [°]

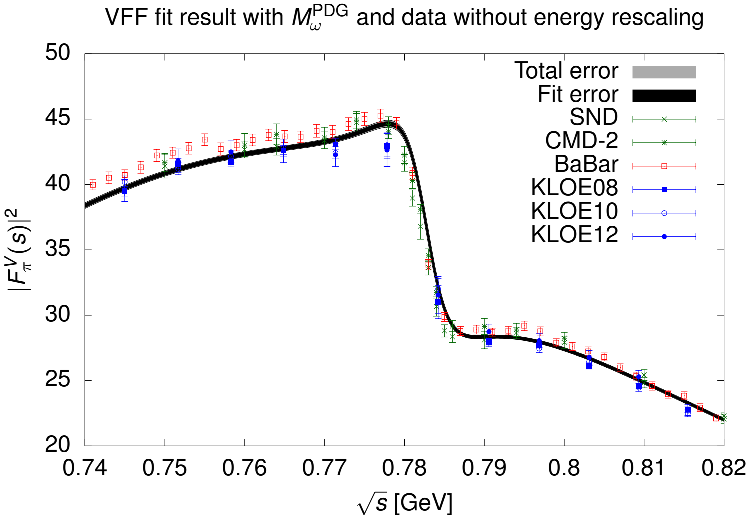
Fit results


Fit result for the VFF $|F_\pi^V(s)|^2$
Total error
Fit error
NA7
$|F_\pi^V(s)|^2$
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
−0.3
−0.25
−0.2
−0.15
−0.1
−0.05
0
s [GeV^2]

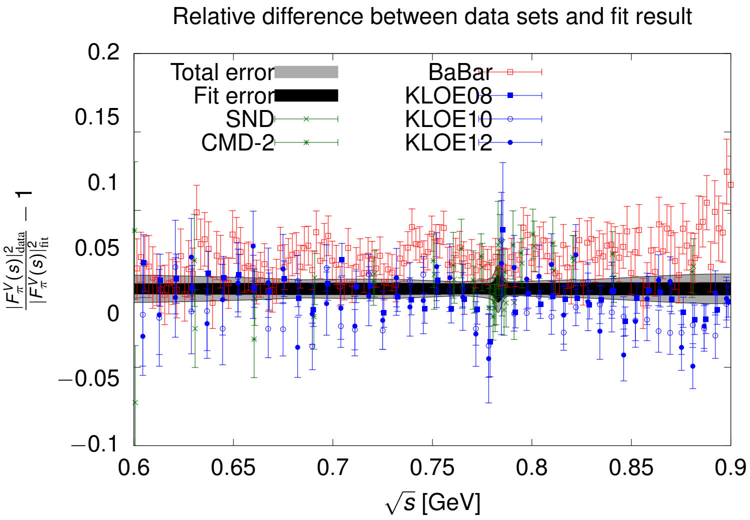
Fit results


VFF fit result and data with energy rescaling
$|F_\pi^V(s)|^2$
$\sqrt{s}$ [GeV]
Total error
Fit error
SND
CMD-2
BaBar
KLOE08
KLOE10
KLOE12
50
45
40
35
30
25
20
0.74
0.75
0.76
0.77
0.78
0.79
0.8
0.81
0.82

Fit results


VFF fit result with M_ω^{PDG} and data without energy rescaling
Total error
Fit error
SND
CMD-2
BaBar
KLOE08
KLOE10
KLOE12
$|F_\pi^V(s)|^2$
$\sqrt{s}$ [GeV]
50
45
40
35
30
25
20
0.74
0.75
0.76
0.77
0.78
0.79
0.8
0.81
0.82

Fit results



Relative difference between data sets and fit result
Total error
Fit error
SND
CMD-2
BaBar
KLOE08
KLOE10
KLOE12
$\frac{|F_\pi^V(s)|^2_{\text{data}}}{|F_\pi^V(s)|^2_{\text{fit}}} - 1$
0.2
0.15
0.1
0.05
0
−0.05
−0.1
0.6
0.65
0.7
0.75
0.8
0.85
0.9
$\sqrt{s}$ [GeV]

Results for $(g-2)_\mu$

- Low energy:

$$a_\mu^{\text{HVP},\pi\pi}{}_{|\leq 0.63\text{GeV}} = 132.8(0.4)(1.0)\cdot 10^{-10}$$

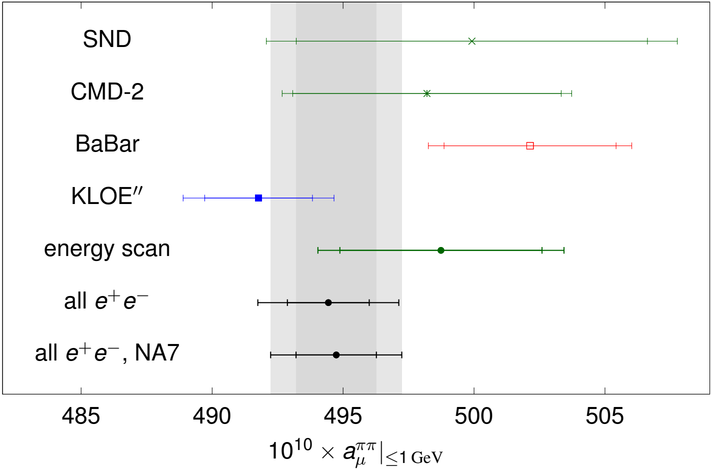
[in agreement with 132.9(8) Ananthanarayan et al. (16)]

- Full range:

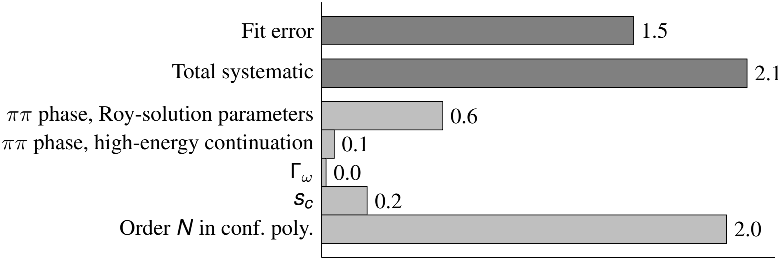
$$a_\mu^{\text{HVP},\pi\pi}{}_{|\leq 1\text{GeV}} = 495.0(1.5)(2.1)\cdot 10^{-10}$$

Results for $(g-2)_\mu$

Result for $a_\mu^{\pi\pi}|_{\leq 1\,\mathrm{GeV}}$ from the VFF fits to single experiments and combinations



Results for $(g-2)_\mu$



Uncertainties on $a_\mu^{\pi\pi}|_{\leq 1\,\mathrm{GeV}}$ in combined fit to all experiments

Results for $(g-2)_\mu$

Comparison to other recent analyses

Energy range	our work	KNT18	DHMZ17	ACD18
$\leq 0.6\,\text{GeV}$	110.1(9)	108.7(9)	110.2(1.0)	
$\leq 0.7\,\text{GeV}$	214.8(1.7)	213.0(1.2)	214.7(1.3)	
$\leq 0.8\,\text{GeV}$	413.2(2.3)	411.6(1.7)	414.0(1.9)	
$\leq 0.9\,\text{GeV}$	479.8(2.6)	478.1(1.9)	481.2(2.2)	
$\leq 1.0\,\text{GeV}$	495.0(2.6)	493.4(1.9)	496.7(2.2)	
$[0.6, 0.7]\,\text{GeV}$	104.7(7)	104.3(5)	104.5(5)	
$[0.7, 0.8]\,\text{GeV}$	198.3(9)	198.6(8)	199.3(9)	
$[0.8, 0.9]\,\text{GeV}$	66.6(4)	66.5(3)	67.2(4)	
$[0.9, 1.0]\,\text{GeV}$	15.3(1)	15.3(1)	15.5(1)	
$\leq 0.63\,\text{GeV}$	132.8(1.1)	131.2(1.1)	132.8(1.1)	132.9(8)
$[0.6, 0.9]\,\text{GeV}$	369.6(1.7)	369.4(1.3)	371.0(1.5)	
$\left[\sqrt{0.1}, \sqrt{0.95}\right]\,\text{GeV}$	490.7(2.6)	489.0(1.9)	492.2(2.2)	

KNT18 = Keshavarzi et al. (18), DHMZ17 = Davier et al. (17), ACD18 = Ananthanarayan et al. (18)

A puzzle: the ω mass

- the ω mass is very well known:

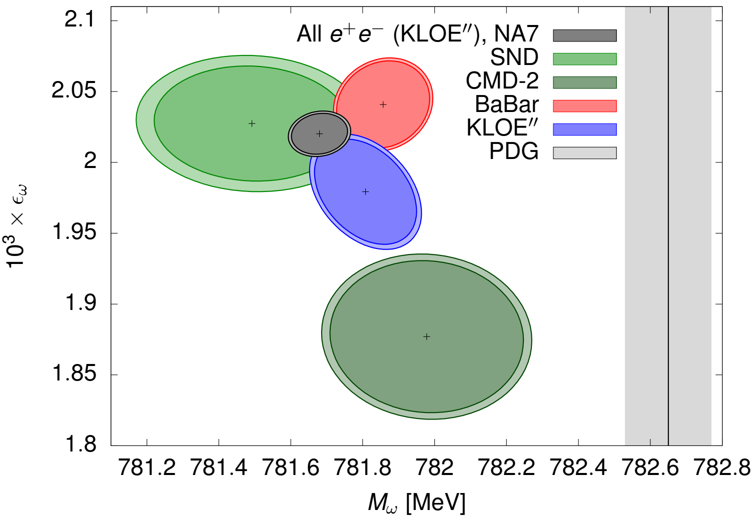
$$M_\omega = 782.65(12)\text{MeV} \quad \text{[PDG]}$$

[3π (dominant) and $\pi\gamma$ channels]

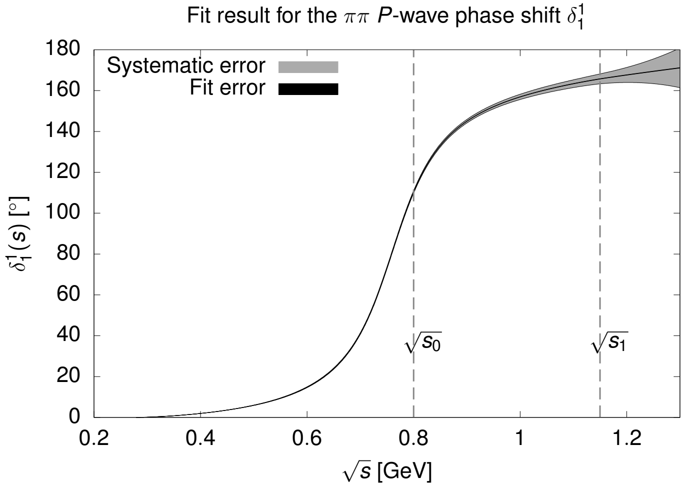
- if we keep it fixed at this value, we get terrible fits
- systematic uncertainties in the energy calibration(s) not enough to explain the discrepancy
- our combined fit gives:

$$M_\omega = 781.69(9)(3)\text{MeV}$$

A puzzle: the ω mass



P wave $\pi\pi$ phase shift



$$\delta_1^1(s_0) = 110.4(1)(7)^\circ, \qquad \delta_1^1(s_1) = 165.7(0.1)(2.4)^\circ$$

Pion charge radius

Definition of the charge radius:

$$F_\pi^V(s) = 1 + \frac{1}{6}\langle r_\pi^2\rangle s + \mathcal{O}(s^2)$$

Dispersive representation for $F_\pi^V(s) \Rightarrow$ sum rule:

$$\langle r_\pi^2\rangle = \frac{6}{\pi}\int_{4M_\pi^2}^\infty ds \frac{\mathrm{Im} F_\pi^V(s)}{s^2}$$

which we evaluate to

$$\langle r_\pi^2\rangle = 0.429(1)(4)\ \mathrm{fm}^2 = 0.429(4)\ \mathrm{fm}^2$$

Compare to PDG 2018:

$$\langle r_\pi^2\rangle = 0.452(11)\ \mathrm{fm}^2 \quad \overset{\text{update 19}}{\Longrightarrow} \quad 0.434(5)\ \mathrm{fm}^2$$

Outline

Different analytic evaluations of HLbL

Jegerlehner-Nyffeler 2009

Contribution	BPaP(96)	HKS(96)	KnN(02)	MV(04)	BP(07)	PdRV(09)	N/JN(09)
π^0, η, η'	85±13	82.7±6.4	83±12	114±10	–	114±13	99±16
π, K loops	−19±13	−4.5±8.1	–	–	–	−19±19	−19±13
" " + subl. in N_c	–	–	–	0±10	–	–	–
axial vectors	2.5±1.0	1.7±1.7	–	22± 5	–	15±10	22± 5
scalars	−6.8±2.0	–	–	–	–	−7± 7	−7± 2
quark loops	21± 3	9.7±11.1	–	–	–	2.3	21± 3
total	83±32	89.6±15.4	80±40	136±25	110±40	105±26	116±39

Legenda: B=Bijnens Pa=Pallante P=Prades H=Hayakawa K=Kinoshita S=Sanda Kn=Knecht N=Nyffeler M=Melnikhov V=Vainshtein dR=de Rafael J=Jegerlehner

- large uncertainties (and differences among calculations) in individual contributions
- pseudoscalar pole contributions most important
- second most important: pion loop, *i.e.* two-pion cuts (*Ks are subdominant*, see below)
- heavier single-particle poles decreasingly important

Advantages of the dispersive approach

- model independent
- unambiguous definition of the various contributions
- makes a data-driven evaluation possible
  (in principle)
- if data not available: use theoretical calculations of subamplitudes, short-distance constraints etc.

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- First attempts: GC, Hoferichter, Procura, Stoffer (14)

  Pauk, Vanderhaeghen (14)
- similar philosophy, with a different implementation: Schwinger sum rule Hagelstein, Pascalutsa (17)
- why hasn't this been adopted before?

The HLbL tensor

HLbL tensor:

$$\Pi^{\mu\nu\lambda\sigma} = i^3 \int dx \int dy \int dz\, e^{-i(x\cdot q_1 + y\cdot q_2 + z\cdot q_3)} \langle 0 | T\{ j^\mu(x) j^\nu(y) j^\lambda(z) j^\sigma(0) \} | 0 \rangle$$

$$q_4 = k = q_1 + q_2 + q_3 \qquad k^2 = 0$$

General Lorentz-invariant decomposition:

$$\Pi^{\mu\nu\lambda\sigma} = g^{\mu\nu} g^{\lambda\sigma} \Pi^1 + g^{\mu\lambda} g^{\nu\sigma} \Pi^2 + g^{\mu\sigma} g^{\nu\lambda} \Pi^3 + \sum_{i,j,k,l} q_i^\mu q_j^\nu q_k^\lambda q_l^\sigma \Pi^4_{ijkl} + \ldots$$

consists of 138 scalar functions $\{\Pi^1, \Pi^2, \ldots\}$, but in $d = 4$ only 136 are linearly independent Eichmann *et al.* (14)

Constraints due to gauge invariance? (see also Eichmann, Fischer, Heupel (2015))

$\Rightarrow$ Apply the Bardeen-Tung (68) method+Tarrach (75) addition

Gauge-invariant hadronic light-by-light tensor

Applying the Bardeen-Tung-Tarrach method to $\Pi^{\mu\nu\lambda\sigma}$ one ends up with: GC, Hoferichter, Procura, Stoffer (2015)

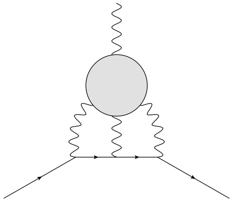
- 43 basis tensors (BT) — in $d = 4$: 41=no. of helicity amplitudes
- 11 additional ones (T) — to guarantee basis completeness everywhere
- of these 54 only 7 are distinct structures
- all remaining 47 can be obtained by crossing transformations of these 7: manifest crossing symmetry
- the dynamical calculation needed to fully determine the LbL tensor concerns these 7 scalar amplitudes

$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i$$

Master Formula

$$a_\mu^{\mathrm{HLbL}} = -e^6 \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{\sum_{i=1}^{12} \hat{T}_i(q_1, q_2; p) \hat{\Pi}_i(q_1, q_2, -q_1 - q_2)}{q_1^2 q_2^2 (q_1 + q_2)^2 [(p + q_1)^2 - m_\mu^2][(p - q_2)^2 - m_\mu^2]}$$

- $\hat{T}_i$: known kernel functions
- $\hat{\Pi}_i$: linear combinations of the Π_i
- the Π_i are amenable to a dispersive treatment: their imaginary parts are related to measurable subprocesses
- 5 integrals can be performed with Gegenbauer polynomial techniques



GC, Hoferichter, Procura, Stoffer (2015)

Master Formula

After performing the 5 integrations:

$$a_\mu^{\text{HLbL}} = \frac{2\alpha^3}{48\pi^2} \int_0^\infty dQ_1^4 \int_0^\infty dQ_2^4 \int_{-1}^1 d\tau \sqrt{1-\tau^2} \sum_{i=1}^{12} T_i(Q_1, Q_2, \tau) \bar{\Pi}_i(Q_1, Q_2, \tau)$$

where Q_i^μ are the Wick-rotated four-momenta and τ the four-dimensional angle between Euclidean momenta:

$$Q_1 \cdot Q_2 = |Q_1||Q_2|\tau$$

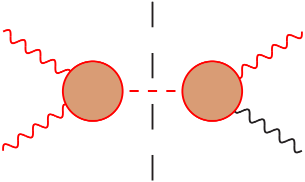
The integration variables $Q_1 := |Q_1|$, $Q_2 := |Q_2|$.

GC, Hoferichter, Procura, Stoffer (2015)

Setting up the dispersive calculation

We split the HLbL tensor as follows:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\pi\text{-box}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$



Pion pole: imaginary parts = δ-functions

Projection on the BTT basis: easy ✓

Our master formula=explicit expressions in the literature ✓

Input: pion transition form factor — Hoferichter et al. (18)

First results of direct lattice calculations — Gerardin, Meyer, Nyffeler (16,19)

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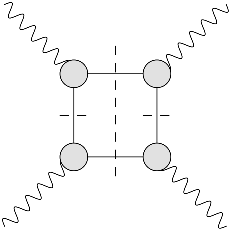
π-box with the BTT set:

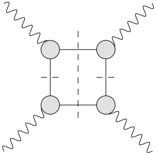


– we have constructed a Mandelstam representation for the contribution of the 2-pion cut with LHC due to a pion pole
– we have explicitly checked that this is identical to sQED multiplied by $F_V^\pi(s)$ (FsQED)

Setting up the dispersive calculation

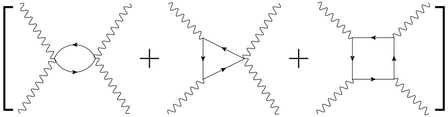
We split the HLbL tensor as follows:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\pi\text{-box}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$



$$\equiv F_\pi^V(q_1^2) F_\pi^V(q_2^2) F_\pi^V(q_3^2)$$

$\times$



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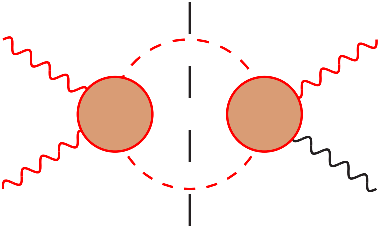


The "rest" with 2π intermediate states has cuts only in one channel and will be calculated dispersively after partial-wave expansion

Setting up the dispersive calculation

We split the HLbL tensor as follows:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\pi\text{-box}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$

E.g. $\gamma^*\gamma^* \to \pi\pi$ S-wave contributions

$$\hat{\Pi}_4^S = \frac{1}{\pi}\int_{4M_\pi^2}^\infty ds' \frac{-2}{\lambda_{12}(s')(s'-q_3^2)^2}\Big(4s' \,\mathrm{Im} h^0_{++,++}(s') - (s'+q_1^2-q_2^2)(s'-q_1^2+q_2^2)\,\mathrm{Im} h^0_{00,++}(s')\Big)$$

$$\hat{\Pi}_5^S = \frac{1}{\pi}\int_{4M_\pi^2}^\infty dt' \frac{-2}{\lambda_{13}(t')(t'-q_2^2)^2}\Big(4t' \,\mathrm{Im} h^0_{++,++}(t') - (t'+q_1^2-q_3^2)(t'-q_1^2+q_3^2)\,\mathrm{Im} h^0_{00,++}(t')\Big)$$

$$\hat{\Pi}_6^S = \frac{1}{\pi}\int_{4M_\pi^2}^\infty du' \frac{-2}{\lambda_{23}(u')(u'-q_1^2)^2}\Big(4u' \,\mathrm{Im} h^0_{++,++}(u') - (u'+q_2^2-q_3^2)(u'-q_2^2+q_3^2)\,\mathrm{Im} h^0_{00,++}(u')\Big)$$

$$\hat{\Pi}_{11}^S = \frac{1}{\pi}\int_{4M_\pi^2}^\infty du' \frac{4}{\lambda_{23}(u')(u'-q_1^2)^2}\Big(2\,\mathrm{Im} h^0_{++,++}(u') - (u'-q_2^2-q_3^2)\,\mathrm{Im} h^0_{00,++}(u')\Big)$$

$$\hat{\Pi}_{16}^S = \frac{1}{\pi}\int_{4M_\pi^2}^\infty dt' \frac{4}{\lambda_{13}(t')(t'-q_2^2)^2}\Big(2\,\mathrm{Im} h^0_{++,++}(t') - (t'-q_1^2-q_3^2)\,\mathrm{Im} h^0_{00,++}(t')\Big)$$

$$\hat{\Pi}_{17}^S = \frac{1}{\pi}\int_{4M_\pi^2}^\infty ds' \frac{4}{\lambda_{12}(s')(s'-q_3^2)^2}\Big(2\,\mathrm{Im} h^0_{++,++}(s') - (s'-q_1^2-q_2^2)\,\mathrm{Im} h^0_{00,++}(s')\Big)$$

Setting up the dispersive calculation

We split the HLbL tensor as follows:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\pi\text{-box}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$

Contributions of cuts with anything else other than one and two pions in intermediate states are neglected in first approximation

of course, the η, η' and other pseudoscalars pole contribution, or the kaon-box/rescattering contribution can be calculated within the same formalism

Pion-pole contribution

- Expression of this contribution in terms of the pion transition form factor already known Knecht-Nyffeler (01)
- Both transition form factors (TFF) must be included:

$$\bar{\Pi}_1 = \frac{F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) F_{\pi^0\gamma^*\gamma^*}(q_3^2, 0)}{q_3^2 - M_{\pi^0}^2}$$

[dropping one bc short-distance not correct Melnikov-Vainshtein (04)]

- data on singly-virtual TFF available CELLO, CLEO, BaBar, Belle, BESIII
- several calculations of the transition form factors in the literature Masjuan & Sanchez-Puertas (17), Eichmann et al. (17), Guevara et al. (18)
- dispersive approach works here too Hoferichter et al. (18)
- quantity where lattice calculations can have a significant impact Gerardin, Meyer, Nyffeler (16,19)

Pion-pole contribution

Latest complete analyses:

- Dispersive calculation of the pion TFF — Hoferichter et al. (18)

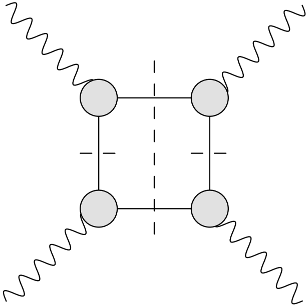
$$10^{11} a_\mu^{\pi^0} = 62.6(1.7)_{F_{\pi\gamma\gamma}}(1.1)_{\text{disp}}\left({}^{2.2}_{1.4}\right)_{\text{BL}}(0.5)_{\text{asym}} = 62.6^{+3.0}_{-2.5}$$

- Padé-Canterbury approximants — Masjuan & Sanchez-Puertas (17)

$$10^{11} a_\mu^{\pi^0} = 63.6(1.3)_{\text{stat}}(0.6)_{a_{P;1,1}}(2.3)_{\text{sys}} = 63.6(2.7)$$

Pion-box contribution

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$



Pion-box contribution

The only ingredient needed for the pion-box contribution is the vector form factor

$$\hat{\Pi}_i^{\pi\text{-box}} = F_\pi^V(q_1^2)F_\pi^V(q_2^2)F_\pi^V(q_3^2)\frac{1}{16\pi^2}\int_0^1 dx \int_0^{1-x} dy\, I_i(x,y),$$

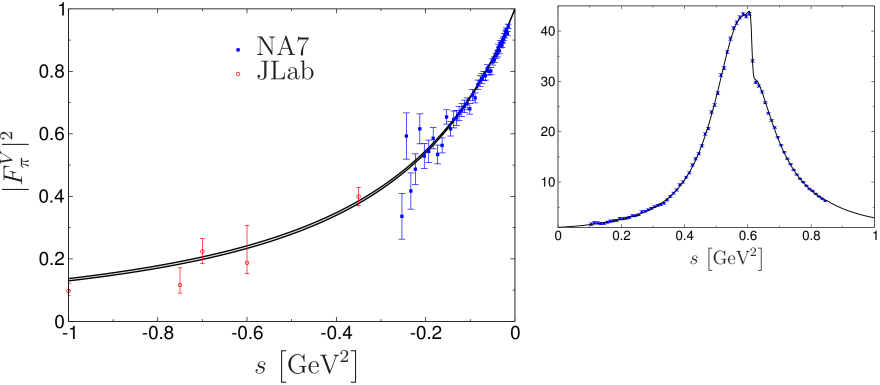
where

$$I_1(x,y) = \frac{8xy(1-2x)(1-2y)}{\Delta_{123}\Delta_{23}},$$

and analogous expressions for $I_{4,7,17,39,54}$ and

$$\Delta_{123} = M_\pi^2 - xyq_1^2 - x(1-x-y)q_2^2 - y(1-x-y)q_3^2,$$
$$\Delta_{23} = M_\pi^2 - x(1-x)q_2^2 - y(1-y)q_3^2$$

Pion-box contribution



Uncertainties are negligibly small:

$$a_\mu^{\text{FsQED}} = -15.9(2) \cdot 10^{-11}$$

Pion-box contribution

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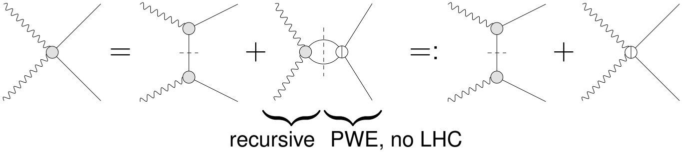
First evaluation of S- wave 2π-rescattering

Omnès solution for $\gamma^*\gamma^* \to \pi\pi$ provides the following:



Based on:

- taking the pion pole as the only left-hand singularity
- $\Rightarrow$ pion vector FF to describe the off-shell behaviour
- $\pi\pi$ phases obtained with the inverse amplitude method
  [realistic only below 1 Gev: accounts for the $f_0(500)$ + unique and well defined extrapolation to ∞]
- numerical solution of the $\gamma^*\gamma^* \to \pi\pi$ dispersion relation

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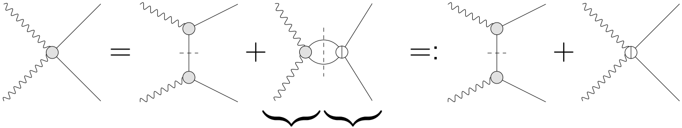

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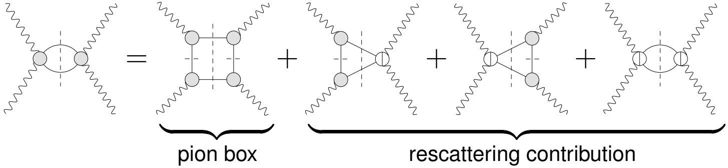
S-wave contributions : $a_{\mu,J=0}^{\pi\pi,\pi\text{-pole LHC}} = -8(1) \times 10^{-11}$

Two-pion contribution to $(g-2)_\mu$ from HLbL

Two-pion contributions to HLbL:



$$a_\mu^{\pi-\text{box}} + a_{\mu,J=0}^{\pi\pi,\pi\text{-pole LHC}} = -24(1)\cdot 10^{-11}$$

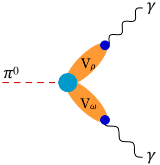
$\gamma^*\gamma^* \to \pi\pi$ contribution from other partial waves

- formulae get significantly more involved with several subtleties in the calculation
- in particular sum rules which link different partial waves must be satisfied by different resonances in the narrow width approximation Danilkin, Pascalutsa, Pauk, Vanderhaeghen (12,14,17)
- data and dispersive treatments available for on-shell photons e.g. Dai & Pennington (14,16,17)
- dispersive treatment for the singly-virtual case and check with forthcoming data is very important

Short-distance contraints

- short-distance constraints on n-point functions in QCD is a well known issue
- low- and intermediate-energy representation in terms of hadronic states doesn't typically extrapolate to the right high-energy limit
- requiring that the latter be satisfied is often essential to obtain a description of spectral functions which leads to correct integrals over them vast literature [de Rafael, Goltermann, Peris,...]
- implementing such an approach for HLbL not very simple, but it works GC, Hagelstein, Laub, work in progress

A Regge-like large-N_C inspired model



$$F_{\pi\gamma^*\gamma^*}(q_1^2,q_2^2) = \sum_{V_\rho, V_\omega} \frac{F_{V_\rho}(q_1^2)\, F_{V_\omega}(q_2^2)\, G_{\pi V_\rho V_\omega}(q_1^2,q_2^2)}{(q_1^2+M_{V_\rho}^2)(q_2^2+M_{V_\omega}^2)} + \left\{q_1 \leftrightarrow q_2\right\}$$

where

$$M_{V_{\rho,\omega}}^2 = M_{\rho,\omega}^2(i_{\rho,\omega}) = M_{\rho,\omega}^2(0) + i_{\rho,\omega}\,\sigma_{\rho,\omega}^2$$

Masjuan, Broniowski, Ruiz Arriola (12)

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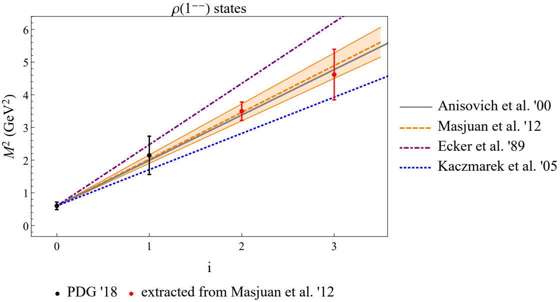

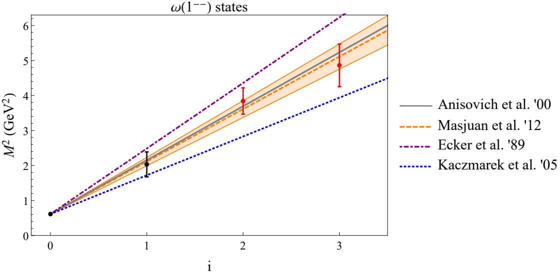

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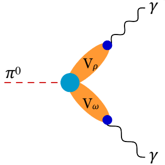

$$F_{\pi^{(n)}\gamma^*\gamma^*}(q_1^2, q_2^2) = \sum_{V_\rho, V_\omega} \frac{F_{V_\rho}(q_1^2)\, F_{V_\omega}(q_2^2)\, G_{\pi^{(n)} V_\rho V_\omega}(q_1^2, q_2^2)}{(q_1^2 + M_{V_\rho}^2)(q_2^2 + M_{V_\omega}^2)} + \left\{ q_1 \leftrightarrow q_2 \right\}$$

where

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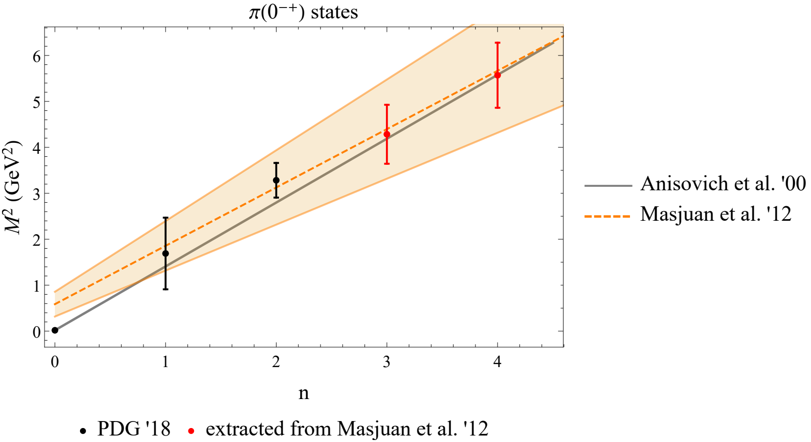
Masjuan, Broniowski, Ruiz Arriola (12)

similarly for “excited pions”, described by a Regge-like model:

$$m_\pi^2(n) = \begin{cases} m_{\pi^0}^2 & n = 0, \\ m_0^2 + n\,\sigma_\pi^2 & n \geq 1, \end{cases}$$

A Regge-like large-N_C inspired model



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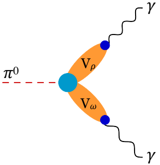

$$F_{\pi^{(n)}\gamma^*\gamma^*}(q_1^2,q_2^2) = \sum_{V_\rho,V_\omega} \frac{F_{V_\rho}(q_1^2)\, F_{V_\omega}(q_2^2)\, G_{\pi^{(n)}V_\rho V_\omega}(q_1^2,q_2^2)}{(q_1^2+M_{V_\rho}^2)(q_2^2+M_{V_\omega}^2)} + \left\{q_1 \leftrightarrow q_2\right\}$$

coupling between pions, and rho's and omega's taken diagonal for simplicity:

$$G_{\pi^{(n)}V_\rho V_\omega}(q_1^2,q_2^2) \propto \delta_{n\, i_\rho}\delta_{n\, i_\omega}$$

Satisfying short-distance constraints

$$\lim_{Q_3\to\infty}\lim_{\tilde{Q}\to\infty}\sum_{n=0}^{\infty}\frac{F_{\pi^{(n)}\gamma^*\gamma^*}(\tilde{Q}^2,\tilde{Q}^2)\,F_{\pi^{(n)}\gamma\gamma^*}(Q_3^2)}{Q_3^2+m_{\pi^{(n)}}^2}=$$
$$=\frac{1}{6\pi^2}\frac{1}{\tilde{Q}^2}\frac{1}{Q_3^2}+\mathcal{O}\left(\tilde{Q}^{-2}Q_3^{-4}\right),$$

where $F_{\pi^{(n)}\gamma^*\gamma^*}$ is the TFF of the n-th radially-excited pion

The infinite sum over excited pions changes the large-Q_3^2 behaviour from Q_3^{-4} (single pion pole) to Q_3^{-2}

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$$=\frac{1}{6\pi^2}\frac{1}{\tilde{Q}^2}\frac{1}{Q_3^2}+\mathcal{O}\left(\tilde{Q}^{-2}Q_3^{-4}\right),$$

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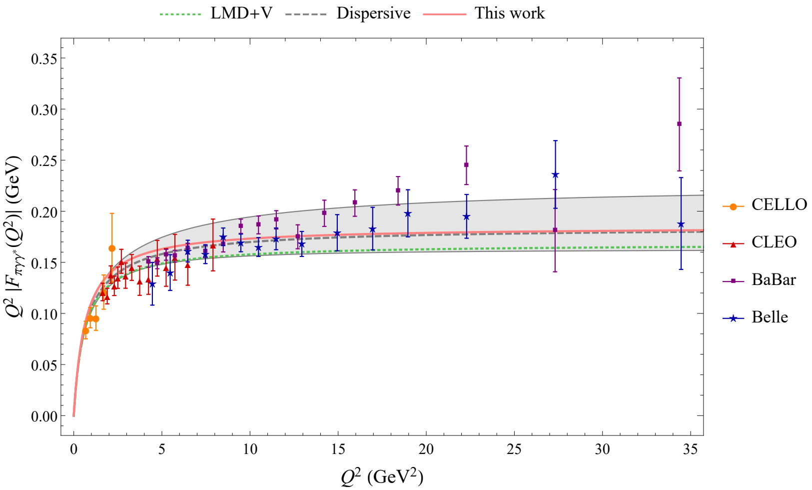
The infinite sum over excited pions changes the large-Q_3^2 behaviour from Q_3^{-4} (single pion pole) to Q_3^{-2}

Is this a realistic model? Can it satisfy all theory constraints (anomaly, Brodsky-Lepage, etc.)?

Comparing our Regge-like model to phenomenology



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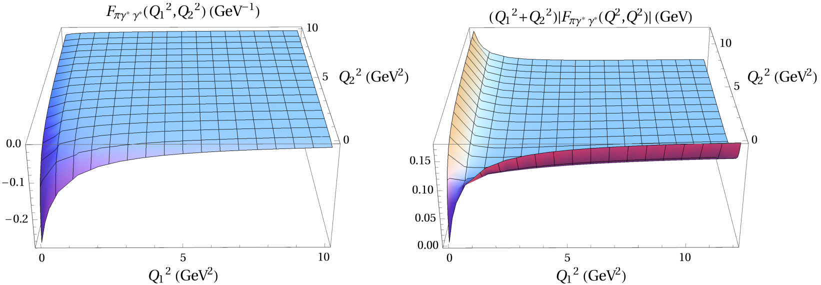
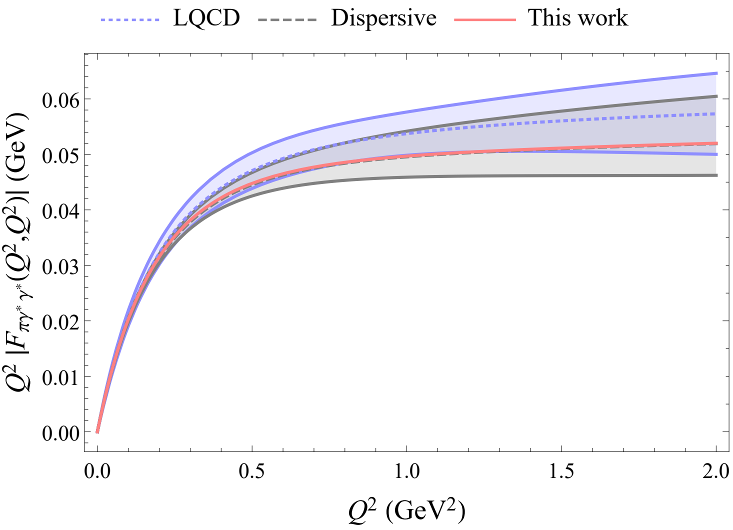
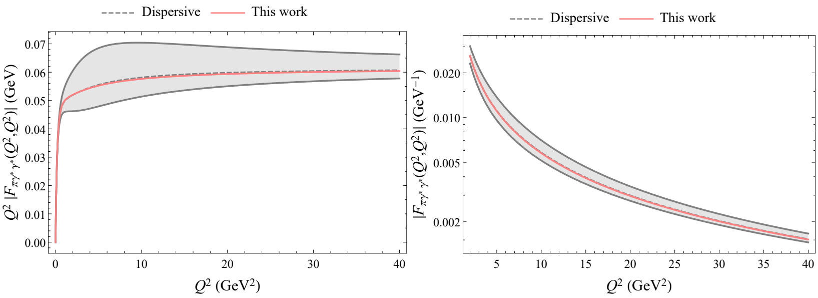


Comparing our model to the dispersive representation



Comparing our model to the dispersive representation



Contribution to $(g-2)_\mu$

The π^0-pole contribution to $(g-2)_\mu$ evaluated with our model is:

$$a_\mu^{\pi^0} = 64.1 \cdot 10^{-11}$$

very close to the value obtained with the dispersive representation for the pion TFF ($62.6^{+3.0}_{-2.5} \cdot 10^{-11}$)

After resumming the contribution of all pion excitations we get:

$$\Delta a_\mu^\pi := \sum_{n=1}^{\infty} a_\mu^{\pi^{(n)}} = 5.8(5) \cdot 10^{-11}$$

Much smaller than the shift obtained by Melnikov-Vainshtein by dropping the pion TFF at the outer $\pi^0\gamma^*\gamma$ vertex:

$$\Delta a_\mu^\pi(\text{M-V}) = 13.5 \cdot 10^{-11}$$

Contribution to $(g-2)_\mu$

The π^0-pole contribution to $(g-2)_\mu$ evaluated with a second model (not described here) is:

$$a_\mu^{\pi^0} = 64.1 \cdot 10^{-11}$$

very close to the value obtained with the dispersive representation for the pion TFF ($62.6^{+3.0}_{-2.5} \cdot 10^{-11}$)

After resumming the contribution of all pion excitations we get:

$$\Delta a_\mu^\pi := \sum_{n=1}^{\infty} a_\mu^{\pi^{(n)}} = 9.1(5) \cdot 10^{-11}$$

Much smaller than the shift obtained by Melnikov-Vainshtein by dropping the pion TFF at the outer $\pi^0\gamma^*\gamma$ vertex:

$$\Delta a_\mu^\pi(\text{M-V}) = 13.5 \cdot 10^{-11}$$

Effect due to short-distance constraints

Melnikov-Vainshtein's solution to satisfy (longitudinal) SDC: drop the π-TFF at the outer $\pi^0\gamma^*\gamma$ vertex. Effect is significant:

$$\Delta a_\mu^\pi(\text{M-V}) = 13.5 \cdot 10^{-11}$$

With two different models which satisfy the SDC, agree w/ data on the π^0 TFF and with the dispersive representation we obtain:

$$\Delta a_\mu^\pi(\text{our model}) 6(3) \cdot 10^{-11}$$

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Work on the transverse SDC is in progress, but M-V estimate (axials) seems to be an overestimate (for various reasons)

Our models will be matched to the quark loop (in progress)

Improvements obtained with the dispersive approach

Contribution	BPaP(96)	HKS(96)	KnN(02)	MV(04)	BP(07)	PdRV(09)	N/JN(09)
π^0, η, η'	85±13	82.7±6.4	83±12	114±10	–	114±13	99±16
π, K loops	−19±13	−4.5±8.1	–	–	–	−19±19	−19±13
" " + subl. in N_c	–	–	–	0±10	–	–	–
axial vectors	2.5±1.0	1.7±1.7	–	22± 5	–	15±10	22± 5
scalars	−6.8±2.0	–	–	–	–	−7± 7	−7± 2
quark loops	21± 3	9.7±11.1	–	–	–	2.3	21± 3
total	83±32	89.6±15.4	80±40	136±25	110±40	105±26	116±39

Results with the dispersive approach:

Pion pole:	$62.6^{+3.0}_{-2.6}$	
Pion box:	-15.9 ± 0.2	
Kaon box (VMD):	~ -0.5	(prelim. Hoferichter, Stoffer)
Pion S-wave rescatt.:	-8 ± 1	
Longitudinal SDC (π^0):	~ 6	(prelim. $\eta^{(\prime)}$ in progr.)

Outline

Conclusions: HVP

- the 2π contribution to HVP is the dominant one to a_μ
- below 1 GeV this contribution is essentially determined by the P wave $\pi\pi$ phase shift
- the latter is strongly constrained by analyticity, unitarity and crossing symmetry
- implementing these constraints we have analyzed $\sigma(e^+e^- \to \pi^+\pi^-)$ data and obtained:

$$a_\mu^{\mathrm{HVP},\pi\pi}\Big|_{\leq 1\mathrm{GeV}} = 495.0(2.6)\cdot 10^{-10}$$

- by-product: new precise determinations of:
  - ω mass (puzzle)
  - P wave $\pi\pi$ phase shift
  - pion charge radius

Conclusions: HLbL

- The HLbL contribution to $(g-2)_\mu$ can be expressed in terms of measurable quantities in a dispersive approach
- master formula: HLbL contribution to a_μ as triple-integral over scalar functions which satisfy dispersion relations
- the relevant measurable quantity entering the dispersion relation depends on the intermediate state:
  - single-pion contribution: pion transition form factor
  - pion-box contribution: pion vector form factor
  - 2-pion rescattering: $\gamma^*\gamma^{(*)} \to \pi\pi$ helicity amplitudes

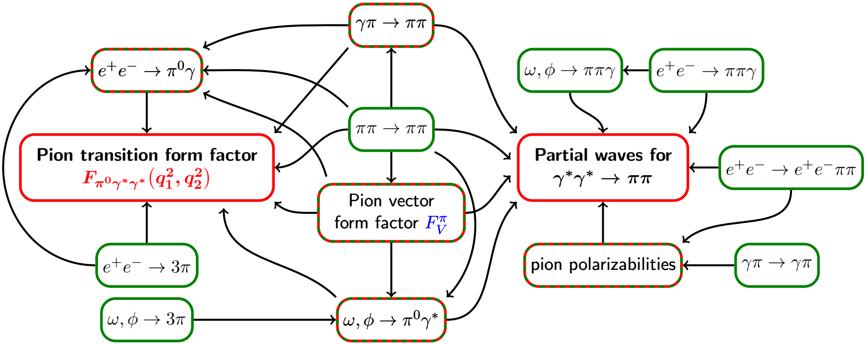
  these three contributions (S-wave for the latter) have been calculated with remarkably small uncertainties
- work on calculating other contributions and estimating missing pieces is in progress

Outlook: HLbL

- More work is needed to complete the evaluation of contributions of 2π intermediate states esp. for $\ell \geq 2$
  - take into account experimental constraints on $\gamma^{(*)}\gamma \to \pi\pi$
  - estimate the dependence on the q^2 of the second photon (theoretically, there are no data on $\gamma^*\gamma^* \to \pi\pi$ – Lattice?)
  - $\Rightarrow$ solve the dispersion relation for the helicity amplitudes of $\gamma^*\gamma^* \to \pi\pi$, including a full treatment of the LHC
- same formulae apply to heavier $n \leq 2$ intermediate states ($\eta^{(\prime)}$ or $\bar{K}K$); for $n > 2$ the formalism must be extended;
- implementation of short-distance constraints is in progress: effect seems to be somewhat smaller than estimated so far

Hadronic light-by-light: a roadmap

GC, Hoferichter, Kubis, Procura, Stoffer `arXiv:1408.2517` (PLB '14)



Artwork by M. Hoferichter

A reliable evaluation of the HLbL requires many different contributions by and a collaboration among (lattice) theorists and experimentalists