Long-distance Processes: from Flavor Physics to Nuclear Physics

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Workshop @ YITP, Kyoto U., 04/23/2019

Role of lattice QCD in flavor physics

Lattice QCD is powerful for observables such as



Flavor Lattice Averaging Group reported f_{κ}/f_{π} , $f_{+}(0)$ and B_{κ} [FLAG, S. Aoki et. al., arXiv:1902.08191]

	N _f	FLAG average	Frac. Err.
f_K/f_{π}	2 + 1 + 1	1.1932(19)	0.16%
$f_{+}(0)$	2 + 1 + 1	0.9706(27)	0.28%
Âκ	2 + 1	0.7625(97)	1.27%

lattice QCD calculations play important role in precision flavor physics

LD processes and non-local matrix elements $\langle f | O_1 O_2 | i \rangle$



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Example 1: $K^+ \rightarrow \pi^+ \nu \bar{\nu}$





$K^+ \rightarrow \pi^+ \nu \bar{\nu}$: Experiment vs Standard model



 $K^+ \rightarrow \pi^+ \nu \bar{\nu}$: largest contribution from top quark loop, thus theoretically clean

$$\mathcal{H}_{eff} \sim \frac{G_F}{\sqrt{2}} \cdot \underbrace{\frac{\alpha_{\rm EM}}{2\pi \sin^2 \theta_W} \lambda_t X_t(x_t)}_{\mathcal{N} \sim 2 \times 10^{-5}} \cdot (\bar{s}d)_{V-A} (\bar{\nu}\nu)_{V-A}}$$

Probe the new physics at scales of $\mathcal{N}^{-\frac{1}{2}}M_W = O(10 \text{ TeV})$

Past experimental measurement is 2 times larger than SM prediction

$$\begin{split} & \mathsf{Br}(K^+ \to \pi^+ \nu \bar{\nu})_{\mathsf{exp}} = 1.73^{+1.15}_{-1.05} \times 10^{-10} & \mathsf{arXiv:} 0808.2459 \\ & \mathsf{Br}(K^+ \to \pi^+ \nu \bar{\nu})_{\mathsf{SM}} = 9.11 \pm 0.72 \times 10^{-11} & \mathsf{arXiv:} 1503.02693 \end{split}$$

but still consistent with > 60% exp. error

New experiments

New generation of experiment: NA62 at CERN

- aims at observation of O(100) events
- 10%-precision measurement of ${\sf Br}({\cal K}^+ o \pi^+ \nu ar
 u)$



Fig: 09/2014, the final straw-tracker module is lowered into position in NA62

$K_L ightarrow \pi^0 u ar u$

- even more challenging since all the particles involved are neutral
- only upper bound was set by KEK E391a in 2010
- new KOTO experiment at J-PARC designed to observe K_L decays
 - one candidate event is found very recently [arXiv:1609.03637]

OPE: integrate out heavy fields Z, W, t, \cdots



$2^{\rm nd}\mbox{-}{\rm order}$ weak interaction and bilocal matrix element

Hadronic matrix element for the 2nd-order weak interaction

 $\int_{-T}^{T} dt \langle \pi^{+} \nu \bar{\nu} | T \left[Q_{A}(t) Q_{B}(0) \right] | K^{+} \rangle$ $= \sum_{n} \left\{ \frac{\langle \pi^{+} \nu \bar{\nu} | Q_{A}| n \rangle \langle n | Q_{B} | K^{+} \rangle}{M_{K} - E_{n}} + \frac{\langle \pi^{+} \nu \bar{\nu} | Q_{B}| n \rangle \langle n | Q_{A} | K^{+} \rangle}{M_{K} - E_{n}} \right\} \left(1 - e^{(M_{K} - E_{n})T} \right)$

• For $E_n > M_K$, the exponential terms exponentially vanish at large T

• For $E_n < M_K$, the exponentially growing terms must be removed

- \sum_{n} : principal part of the integral replaced by finite-volume summation
 - possible large finite volume correction when $E_n \rightarrow M_K$

[Christ, XF, Martinelli, Sachrajda, PRD 91 (2015) 114510]

Low lying intermediate states



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[Christ, XF, Portelli, Sachrajda, PRD 93 (2016) 114517]

SD divergence appears in $Q_A(x)Q_B(0)$ when $x \to 0$

• Introduce a counter term $X \cdot Q_0$ to remove the SD divergence



The coefficient X is determined in the RI/(S)MOM scheme

• The bilocal operator in the $\overline{\mathrm{MS}}$ scheme can be written as

$$\begin{split} &\left\{ \int d^4 x \ T[Q_A^{\overline{\text{MS}}}(x)Q_B^{\overline{\text{MS}}}(0)] \right\}^{\overline{\text{MS}}} \\ &= Z_A Z_B \left\{ \int d^4 x \ T[Q_A^{\text{lat}}Q_B^{\text{lat}}] \right\}^{\text{lat}} + \left(-X^{\text{lat} \to \text{RI}} + Y^{\text{RI} \to \overline{\text{MS}}} \right) Q_0(0) \end{split}$$

► $X^{\text{lat} \rightarrow \text{RI}}$ is calculated using NPR and $Y^{\text{RI} \rightarrow \overline{\text{MS}}}$ calculated using $\text{PT}_{10/44}$

Lattice results

Step 1: @ $m_{\pi} = 420$ MeV, $m_c = 860$ MeV

[Bai, Christ, XF, Lawson, Portelli, Schrajda, PRL 118 (2017) 252001]

 $P_c = 0.2529(\pm 13)_{\rm stat}(\pm 32)_{\rm scale}(-45)_{\rm FV}$



Lattice QCD is now capable of first-principles calculation of rare kaon decay

• The remaining task is to control various systematic effects

Momentum dependence

Step 2: calculation @ $m_{\pi} = 170$ MeV, $m_c = 750$ MeV, $L^3 \times T = 32^3 \times 64$



Momentum dependence is mild at near-physical pion mass

Move on to Step 3: @ both physical pion and charm quark mass

Example 2: E&M corrections in leptonic decay



 $m_{\gamma} = 0 \Rightarrow$ long-range propagator enclosed in the lattice box \Rightarrow power-law finite-size effects

Remove zero mode - QED_L

Infinite volume propagator \Rightarrow finite-volume propagator

$$S_{\infty}(x) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{ipx}}{p^2} = \frac{1}{4\pi^2 x^2} \quad \Rightarrow \quad S_L(x) = \frac{1}{VT} \sum_p \frac{e^{ipx}}{p^2}, \quad p = \frac{2\pi}{L} n \neq 0$$



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Infinite volume reconstruction method (I)

XF, Luchang Jin [arXiv:1812.09817]

QED self energy



• We start with infinite volume

$$\mathcal{I} = rac{1}{2}\int d^4x\,\mathcal{H}_{\mu,
u}(x)S^{\gamma}_{\mu,
u}(x)$$

• The hadronic part $\mathcal{H}_{\mu,\nu}(x)$ is given by

$$\mathcal{H}_{\mu,\nu}(x) = \mathcal{H}_{\mu,\nu}(t,\vec{x}) = \langle N | T[J_{\mu}(x)J_{\nu}(0)] | N \rangle$$

Insert hadronic intermediate state

$$\int \frac{d^3 \vec{p}}{(2\pi)^3} \langle N | J_{\mu}(x) | N(\vec{p}) \rangle \frac{1}{2E_{\vec{p}}} \langle N(p) | J_{\nu}(0)] | N \rangle$$

$$\sim \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{e^{i \vec{p} \cdot \vec{x}} e^{-E_{\vec{p}}t} e^{Mt}}{2E_{\vec{p}}} \sim \int \frac{d^4 p}{(2\pi)^4} \frac{e^{i p x}}{p^2 + M^2} e^{Mt}$$

For large |t|, we shall have:

$$\mathcal{H}_{\mu,
u}(t,ec{x})\sim e^{-M\left(\sqrt{t^2+ec{x}^2}-t
ight)}\sim e^{-Mrac{ec{x}^2}{2t}}\sim O(1)$$

Infinite volume reconstruction method (II)

XF, Luchang Jin [arXiv:1812.09817]

For large |t|, we shall have:

$$\mathcal{H}_{\mu,
u}(t,ec{x})\sim e^{-M\left(\sqrt{t^2+ec{x}^2}-t
ight)}\sim e^{-Mrac{ec{x}^2}{2t}}\sim O(1)$$

- One cannot neglect the contribution of $\mathcal{H}_{\mu,\nu}(t,\vec{x})$ for large \vec{x}
- Lattice QCD cannot provide $\mathcal{H}_{\mu,\nu}(t,\vec{x})$ for $\vec{x} > L$

How to solve this problem?

Realizing at large $t > t_s$ we have ground state dominance:

$$\mathcal{H}_{\mu,\nu}(t,ec{x}) = \int rac{d^3ec{p}}{(2\pi)^3} \langle N|J_{\mu}|N(ec{p})\rangle \langle N(ec{p})|J_{\nu}|N
angle e^{ec{p}\cdotec{x}}e^{-(E_{ec{p}}-M)t}$$

• Reconstruct $\mathcal{H}_{\mu,\nu}(t,\vec{x})$ at large t using $\mathcal{H}_{\mu,\nu}(t_s,\vec{x})$ at modest t_s

XF, Luchang Jin [arXiv:1812.09817]

We then split the integral \mathcal{I} into two parts

$$\mathcal{I} = \mathcal{I}^{(s)} + \mathcal{I}^{(l)} \mathcal{I}^{(s)} = \frac{1}{2} \int_{-t_s}^{t_s} \int d^3 \vec{x} \, \mathcal{H}_{\mu,\nu}(x) S^{\gamma}_{\mu,\nu}(x) \mathcal{I}^{(l)} = \int_{t_s}^{\infty} \int d^3 \vec{x} \, \mathcal{H}_{\mu,\nu}(x) S^{\gamma}_{\mu,\nu}(x) = \int d^3 \vec{x} \, \mathcal{H}_{\mu,\nu}(t_s, \vec{x}) L_{\mu,\nu}(t_s, \vec{x})$$

At $t \leq t_s$,

$$\mathcal{H}_{\mu,
u}(t,ec{x}) \quad \Rightarrow \quad \mathcal{H}^{\mathsf{L}}_{\mu,
u}(t,ec{x})$$

The replacement only amounts for exponentially suppressed FV effects

Move to $\pi o \mu u(\gamma)$

Norman Christ, XF, Luchang Jin, Chris Sachrajda

• 0 photon emission





• 1 photon emission



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1 photon emission



• Decay amplitude: split into hadronic and leptonic parts

$$\mathcal{M} = \int d^4x \underbrace{\langle 0 | \mathcal{T}[J^W_\mu(0) J_\rho(x)] | \phi \rangle}_{\mathcal{H}^{(1\gamma)}_{\mu\rho}(x)} e^{-ikx} \epsilon^*_\rho(\lambda, k) \cdot \underbrace{\bar{u}(p) \gamma_\mu(1 - \gamma_5) v(-p - k)}_{L_\mu(p, k)}$$

with photon momentum k and muon momentum p

• Hadronic part

.

$$\int_{0}^{\infty} dt \int d^{3}\vec{x} \, H_{\mu\rho}^{(1\gamma)}(x) e^{-ikx} + \int_{-\infty}^{0} dt \int d^{3}\vec{x} \, H_{\mu\rho}^{(1\gamma)}(x) e^{-ikx}$$

$$= -\sum_{n} \frac{\langle 0|J_{\rho}(0)|n(\vec{k})\rangle\langle n(\vec{k})|J_{\mu}^{W}(0)|\phi\rangle}{k - E_{n}} + \sum_{n} \underbrace{\frac{\langle 0|J_{\mu}^{W}(0)|n(-\vec{k})\rangle\langle n(-\vec{k})|J_{\rho}(0)|\phi\rangle}{k + E_{n} - m_{\phi}}}_{\text{singular when } k \to 0}$$

IR singularity from 1 photon emission

• Similar as self energy, we split the time integral over $(-\infty,0]$ into two parts

$$\int_{t_s}^{0} dt \int d^3 \vec{x} \, H^{(1\gamma)}_{\mu\rho}(x) e^{-ikx} + \int_{-\infty}^{t_s} dt \int d^3 \vec{x} \, H^{(1\gamma)}_{\mu\rho}(x) e^{-ikx}$$

• For $t < t_s <$ 0, $H^{(1\gamma)}_{\mu
ho}(t,ec{x})$ can be written as

$$\int d^{3}\vec{x} H_{\mu\rho}^{(1\gamma)}(t,\vec{x}) e^{-ikx} \approx \int d^{3}\vec{x} H_{\mu\rho}^{(1\gamma)}(t_{s},\vec{x}) e^{-i\vec{k}\cdot\vec{x}} e^{(E_{\phi}-m_{\phi})(t-t_{s})} e^{kt}$$
$$\approx \underbrace{\langle 0|J_{\mu}^{W}(0)|n(-\vec{k})\rangle\langle n(-\vec{k})|J_{\rho}(0)|\phi\rangle}_{H_{\mu\rho}^{1\gamma}(\vec{k})} e^{(E_{\phi}-m_{\phi})t} e^{kt}$$

• IR singularity in decay width

$$\int \frac{d^3k}{(2\pi)^3} \frac{1}{2k} \sum_{\lambda} |\mathcal{M}|^2 \sim \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k} \frac{H_{\mu\rho}^{1\gamma}(\vec{k}) H_{\nu\rho}^{1\gamma*}(\vec{k})}{(k+E_{\phi}-m_{\phi})^2} \cdot L_{\mu}(p,k) L_{\nu}^*(p,k)$$

• We use $H^{1\gamma}_{\mu
ho}(0>t\geq t_s,ec{x})$ and $H^{1\gamma}_{\mu
ho}(t_s,ec{x})$ as inputs for SD and LD parts

LD contribution from 1 photon emission

LD contribution to the decay width is given by

$$\Gamma^{\rm LD}_{(1\gamma)} = \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{1}{2k} \int d^3 \vec{x} \int d^3 \vec{y} \, \frac{\langle H^{(1\gamma)}_{\mu\rho}(t_s, \vec{x}) \rangle \langle H^{(1\gamma)}_{\nu\rho}(t_s, \vec{y}) \rangle^*}{(k + E_{\phi} - m_{\phi})^2} \\ e^{-i \vec{k} \cdot (\vec{x} - \vec{y})} e^{2kt_s} \cdot L_{\mu}(p, k) L^*_{\nu}(p, k)$$

We split the LD contribution into two pieces

0 photon emission



• Correlation function

$$\frac{1}{2}\int d^4x\int d^4y\,\langle 0|J^W_\nu(0)J_\rho(x)J_\sigma(y)\phi(-t)|0\rangle S^\gamma_{\rho\sigma}(x,y)$$

• The IR divergence appears in

 $\frac{1}{2}\int dt_{x}\int dt_{y}\int d^{3}\vec{x}\langle 0|J^{W}_{\mu}(0)|\phi\rangle\underbrace{\langle\phi|J_{\rho}(t_{x},\vec{x})J_{\sigma}(t_{y},0)|\phi\rangle}_{\sim e^{(E_{\phi}-m_{\phi})(t_{x}-t_{y})}}S^{\gamma}_{\rho\sigma}(t_{x},\vec{x};t_{y},\vec{0})$

- Integral over t_x and t_y yields $\frac{1}{(E_{\phi}+k-m_{\phi})^2}$
- Photon propagator produces another factor of $\frac{1}{2E_{\gamma}} = \frac{1}{2k}$
- Define a hadronic matrix element as

$$H^{(0\gamma)}_{
ho\sigma}(t,ec{x})=\langle \phi|J_{
ho}(t,ec{x})J_{\sigma}(0,ec{0})|\phi
angle$$

For $|t| > t_s$, $H^{(0\gamma)}_{
ho\sigma}(t, \vec{x})$ can be replaced by $H^{(0\gamma)}_{
ho\sigma}(t_s, \vec{x})$

Long-distance contribution to decay width





 $\langle 0|J^W_{\mu}|\phi\rangle H^{0\gamma}_{\rho\sigma}(t,\vec{x})S^{\gamma}_{\rho\sigma}(t,\vec{x}) \qquad \langle 0|J^W_{\nu}|\phi\rangle$

Long-distance contribution to decay width is

$$\Gamma^{\rm LD}_{(0\gamma)} = -\int \frac{d^3\vec{k}}{(2\pi)^3} \int d^3\vec{x} \, \frac{\langle 0|J^W_{\mu}|\phi\rangle\langle 0|J^W_{\nu}|\phi\rangle^* H^{0\gamma}_{\rho\rho}(t_s,\vec{x})}{(E_{\phi}+k-m_{\phi})^2} \\ e^{-i\vec{k}\cdot\vec{x}} e^{-(m_{\phi}-E_{\phi})t_s} \frac{1}{2k} L_{\mu}(p,0) L_{\nu}^*(p,0)$$

We split the contribution into two pieces

$$\Gamma^{\text{LD,finite}}_{(0\gamma)} = -\int \frac{d^{3}\vec{k}}{(2\pi)^{3}} \int d^{3}\vec{x} \frac{\langle 0|J_{\mu}^{W}|\phi\rangle\langle 0|J_{\nu}^{W}|\phi\rangle^{*}H_{\rho\rho}^{0\gamma}(t_{s},\vec{x})}{(E_{\phi}+k-m_{\phi})^{2}} \\ \left(e^{-i\vec{k}\cdot\vec{x}}-1\right)e^{-(m_{\phi}-E_{\phi})t_{s}}\frac{1}{2k}L_{\mu}(p,0)L_{\nu}^{*}(p,0) \\ \Gamma^{\text{LD,div}}_{(0\gamma)} = -\int \frac{d^{3}\vec{k}}{(2\pi)^{3}} \frac{\langle 0|J_{\mu}^{W}|\phi\rangle\langle 0|J_{\nu}^{W}|\phi\rangle^{*}\langle\phi|J_{\rho}|\phi\rangle\langle\phi|J_{\rho}|\phi\rangle^{*}}{(E_{\phi}+k-m_{\phi})^{2}} \\ e^{-(m_{\phi}-E_{\phi})t_{s}}\frac{1}{2k}L_{\mu}(p,0)L_{\nu}^{*}(p,0) \\ 2^{3/44} \end{cases}$$

Cancellation between the divergent parts yields

$$\begin{split} & \Gamma^{\mathrm{LD},\mathrm{div}}_{(1\gamma)} + \Gamma^{\mathrm{LD},\mathrm{div}}_{(0\gamma)} \\ &= \langle 0|J^W_{\mu}|\phi\rangle\langle 0|J^W_{\nu}|\phi\rangle^*\langle\phi|J_{\rho}|\phi\rangle\langle\phi|J_{\rho}|\phi\rangle^* \\ &\times \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2k} \frac{e^{2kt_s}L_{\mu}(p,k)L^*_{\nu}(p,k) - e^{-(m_{\phi}-E_{\phi})t_s}L_{\mu}(p,0)L^*_{\nu}(p,0)}{(k+E_{\phi}-m_{\phi})^2} \end{split}$$

Similar cancellation can happens for example in



Example 3: $0\nu 2\beta$ decays



Neutrinoless double beta decay

0 uetaeta decay

- The easiest way to determine whehter ν is a Majorana fermion
- Give the information on the absolute mass scale of u
- Provide the evidence of lepton number violation
- It can be measured by using tons of materials such as TeO_2

	CANDLES	Ca-48	60 CaF ₂ crystals in liq.	6 kg	Construction
			scint		
	CARVEL	Ca-48	⁴⁸ CaWO ₄ crystal scint.	100 kg	
1	COBRA	Cd-116,	CdZnTe detectors	10 kg	R&D
		Te-130			
I	CUROICINO	Te-130	TeO ₂ Bolometer	11 kg	Operating
	CUORE	Te-130	TeO ₂ Bolometer	206 kg	Construction
	DCBA	Nd-150	Nd foils & tracking	20 kg	R&D
chambers					
	EXO200	Xe-136	Xe TPC	200 kg	Construction
	EXO	Xe-136	Xe TPC	1-10t	R&D
1	GEM	Ge-76	Ge diodes in LN	1 t	
I	GERDA	Ge-76	Seg. and UnSeg. Ge in	35-40 kg	Construction

• 4 Exp. (Majorana, EXO, CUORE, GERDA) reached $T_{1/2}^{0\nu} > 10^{25}$ year

 $\bullet~1$ Exp. (KamLAND-Zen) exceeded the level of 1×10^{26} year

Light-neutrino exchange in 0 uetaeta decay

Minimal extension of SM – exchange of three light Majorana neutrinos

• Effective Lagrangian for β decay

 $\mathcal{L}_{eff} = 2\sqrt{2}G_F V_{ud}(\overline{u}_L \gamma_\mu d_L)(\overline{e}_L \gamma_\mu \nu_{eL})$

• Effective Hamlitonian for 2β decay

$$\mathcal{H}_{eff}^{2\beta} = rac{1}{2!} \int d^4 x \, \mathcal{L}_{eff}(x) \mathcal{L}_{eff}(0)$$

• Neutrino flavor eigenstate mixes with three mass eigenstates

$$\overline{e}_L \gamma_\mu \nu_{eL} \rightarrow \sum_k \overline{e}_L \gamma_\mu U_{ek} \nu_{kL}$$

 U_{ek} is the mixing matrix element.

• These neutrinos are very light

Long-distance contribution dominated

Light-neutrino exchange in 0 uetaeta decay

Assume that $0\nu\beta\beta$ is mediated by exchange of light Majorana neutrinos

$$\sum_{k} \overline{e}_{L}(x)\gamma_{\mu}U_{ek}\nu_{kL}(x)\overline{e}_{L}(0)\gamma_{\nu}U_{ek}\nu_{kL}(0)$$

$$= -\sum_{k} \overline{e}_{L}(x)\gamma_{\mu}U_{ek}\nu_{kL}(x)\overline{\nu_{kL}^{c}}(0)\gamma_{\nu}U_{ek}e_{L}^{c}(0)$$

$$= -\sum_{k} \overline{e}_{L}(x)\gamma_{\mu}U_{ek}P_{L}\left(\int \frac{d^{4}q}{(2\pi)^{4}}\frac{-i\not{q}+m_{k}}{q^{2}+m_{k}^{2}}e^{iqx}\right)P_{L}\gamma_{\nu}U_{ek}e_{L}^{c}(0)$$

$$\approx -m_{\beta\beta}\int \frac{d^{4}q}{(2\pi)^{4}}\frac{e^{iqx}}{q^{2}}\overline{e}_{L}(x)\gamma_{\mu}\gamma_{\nu}e_{L}^{c}(0)$$

In the last step, q vanishes and m_k enters into the effective mass m_{etaeta}

$$m_{\beta\beta} = \sum_{k} m_k U_{ek}^2$$

 $0\nu2\beta$ decay amplitude is proportional to the absolute neutrino mass

Double β decay: generic difficulties

At present, lattice QCD mainly targets on light nuclei

• For nucleus A:
$$\frac{\text{signal}}{\text{noise}} \sim \exp\left[-A(M_N - 3/2m_\pi)t\right] \Rightarrow$$
 a sign problem!

For nuclear matrix element, various models yield O(100%) discrepancies



Single β decay of nuclei

Coupling of currents to nuclei in nuclear EFT [Detmold, talk at Lat18]

• One body coupling dominates



• Two nucleon contributions are subleading but non-negligible



A promising way to provide few-body inputs to ab initio many-body calculations

Start with $\pi^-\pi^- \rightarrow ee$

Our setup

$\pi\pi ightarrow ee$

- Two pions stay at rest
- Two electrons carry spatial momentum $\vec{p}_1 = -\vec{p}_2$, $\vec{p}_{1,2} = E_{\pi\pi}/2$



• Bilocal matrix element under investigation

$$\mathcal{A} = rac{1}{2!}\int d^4x \left\langle e_1 e_2 | \mathcal{L}_{eff}(x) \mathcal{L}_{eff}(0) | \pi \pi
ight
angle$$

• It can be separated into three parts

$$\mathcal{A} = \underbrace{T_{\text{lept}}}_{\text{kinematic factor}} \int d^4 x \underbrace{\mathcal{H}(x)}_{\text{hadronic part}} \underbrace{S_0(x)(e^{-ip_1 \cdot x} + e^{-ip_2 \cdot x})}_{\text{neutrino propagator}}$$
with $T_{\text{lept}} = 4G_F^2 V_{ud}^2 m_{\beta\beta} \bar{u}_L(p_1) u_L^c(p_2)$

Implementation of neutrino propagator

Scalar propagator $S_0(x, y)e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}$ with $\vec{k} = \vec{p}_{1,2}$ can be implemented

$$S_{0}(x,y)e^{-i\vec{k}\cdot(\vec{x}-\vec{y})} = \int \frac{d^{4}q}{(2\pi)^{4}} \frac{e^{iq(x-y)}}{q_{t}^{2}+(\vec{q}+\vec{k})^{2}}$$

$$\Rightarrow \frac{1}{VT} \sum_{\vec{q},q_{t}} \frac{e^{iq(x-y)}}{\hat{q}_{t}^{2}+\sum_{i} \widehat{q}_{i}+k_{i}^{2}}$$

• $\hat{q}_i = 2\sin(q_i/2)$ are the lattice discretized momenta

Zero mode $(\vec{q} = 0)$ of the propagator is computed exactly

$$\frac{1}{VT}\sum_{q_t}\frac{e^{iq_t(t_x-t_y)}}{\widehat{q_t}^2+\sum_i\widehat{k_i}^2}$$

Non-zero modes $(\vec{q} \neq \vec{0})$ of the propagator can be constructed as

$$\frac{1}{N_r}\sum_{r=1}^{N_r}\phi_r(x)\phi_r^*(y), \quad \phi_r(x) = \frac{1}{\sqrt{VT}}\sum_{\vec{q}\neq\vec{0},q_t}\frac{\xi_r(q)e^{iqx}}{\sqrt{\widehat{q}_t^2 + \sum_i \widehat{q_i + k_i}^2}}.$$

Here the stochastic sources $\xi_r(q)$ satisfy

$$\lim_{N_r\to\infty}\frac{1}{N_r}\sum \xi_r(q)\xi_r^*(q')=\delta_{q,q'}.$$

Decay amplitude for double beta decay

The amplitude \mathcal{A}

$$\mathcal{A}=\int dt\,\mathcal{M}(t)$$



At large |t|, $\mathcal{M}(t)$ is saturated by ground intermediate state - $ear{
u}\pi$

$$\mathcal{M}(t) \xrightarrow{|t|\gg 0} - \mathcal{T}_{ ext{lept}} rac{1}{V} rac{2\langle 0|J_{\mu L}|\pi
angle_{ ext{V}} \langle \pi|J_{\mu L}|\pi\pi
angle_{ ext{V}}}{(2m_{\pi})(2E_{
u})} e^{-m_{\pi}|t|}$$

 $\pi\pi
ightarrow ee$ decay amplitude @ $m_{\pi} = 420$ MeV



 $\pi\pi
ightarrow ee$ decay amplitude @ $m_{\pi}=140$ MeV



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XF, L.-C. Jin, X.-Y. Tuo, S.-C. Xia, PRL 122 (2019) 022001

m_{π} [MeV]	$t_a - t_{\pi\pi}$	$\mathcal{A}^{(g)}$	$\mathcal{A}^{(e)}$	$\mathcal{A}^{(g)} + \mathcal{A}^{(e)}$
	6		0.055(13)	1.517(13)
420	7	1.462(10)	0.060(13)	1.522(13)
	8		0.052(14)	1.514(14)
	6		-0.0664(70)	1.8199(63)
140	7	1.8863(50)	-0.0660(73)	1.8203(62)
	8		-0.0665(70)	1.8199(60)

We obtain the result with sub-percent statisitcal errors:

$$egin{aligned} rac{\mathcal{A}(\pi\pi
ightarrow ee)}{F_{\pi}^2 \ T_{
m lept}} igg|_{m_{\pi}=420 \ {
m MeV}} &= 1.517(13) \ rac{\mathcal{A}(\pi\pi
ightarrow ee)}{F_{\pi}^2 \ T_{
m lept}} igg|_{m_{\pi}=140 \ {
m MeV}} &= 1.820(6). \end{aligned}$$

Comparison with chiral perturbation theory

Chiral perturbation theory [Cirigliano, Dekens, Mereghetti, Walker-Loud, PRC97 (2018) 065501]

$$-\frac{\mathcal{A}(\pi\pi\to ee)}{F_{\pi}^2 T_{\rm lept}} = 2\left[1 - \frac{m_{\pi}^2}{(4\pi F_{\pi})^2} \left(3\ln\frac{\mu^2}{m_{\pi}^2} + \frac{7}{2} + \frac{\pi^2}{4} + \frac{5}{6}g_{\nu}^{\pi\pi}(\mu)\right)\right]$$

At m_{π} = 420 and 140 MeV, $A(\pi\pi \rightarrow ee)$ from LQCD are 24% and 9% smaller than the leading-order ChPT predication

$$rac{\mathcal{A}^{
m LO}(\pi\pi
ightarrow ee)}{\mathcal{F}_{\pi}^2 \, \mathcal{T}_{
m lept}} = 2$$

We obtain

An estimate from sum rule yields $g_{
u}^{\pi\pi}(\mu=m_{
ho})=-7.6$

1.1

Second process on $\pi^- \rightarrow e^- e^- \pi^+$ (or $\pi^- e^+ \rightarrow \pi^+ e^-$)

Similarity between $\pi^- \rightarrow \pi^+ ee$ and $\pi^+ - \pi^0$ mass splitting

 $\pi^-
ightarrow \pi^+ ee:$





 $m_{\pi^+} - m_{\pi^0}$:



Decay amplitude for $\pi^- \rightarrow \pi^+ ee$



Low energy constants

Chiral perturbation theory for $\pi^- \rightarrow \pi^+ ee$

[Cirigliano, Dekens, Mereghetti, Walker-Loud, PRC97 (2018) 065501]

$$\frac{\mathcal{A}(\pi^- \to \pi^+ ee)}{F_\pi^2 \, T_{\rm lept}} = 2 \left[1 + \frac{m_\pi^2}{(4\pi F_\pi)^2} \left(3 \ln \frac{\mu^2}{m_\pi^2} + 6 + \frac{5}{6} g_\nu^{\pi\pi}(\mu) \right) \right]$$

Preliminary results



Move to dibaryon



Outlook

It is exciting time for lattice QCD

- Standard quantity: expect the precision significantly enhanced
- Non-standard quantity, such as LD processes: worthwhile for study

For flavor physics:

- lattice QCD provides useful low-energy QCD information
- plays important role in high-precision frontier

The techniques developed in flavor physics can be used in nuclear physics

- help to study the rare processes related to nuclear matter
- Can one day, nuclear physics become a new flavor physics?

Backup slides

Three types of power-law finite-volume effects

- Generic FV effects associated to long-distance processes i → n → f
 N. Christ, XF, G. Martinelli, C. Sachrajda, PRD91 (2015), 114510
 - n is the multi-particle intermediate state
 - Energy of n is smaller than initial-state energy
- FV effects caused by $\pi\pi\text{-rescattering}$ in the initial state

$$|\pi\pi\rangle_{\infty} = \left(2\pi \frac{E_{\pi\pi}}{k^3}\right)^{\frac{1}{2}} \left(q\frac{d\phi}{dq} + k\frac{d\delta}{dk}\right)^{\frac{1}{2}} |\pi\pi\rangle_{\mathrm{V}}$$

At threshold

$$\frac{2\pi}{k^3}\left(q\frac{d\phi}{dq}+k\frac{d\delta}{dk}\right)=V\left[1+d_1\frac{a_{\pi\pi}}{L}+d_2\left(\frac{a_{\pi\pi}}{L}\right)^2+d_3\left(\frac{a_{\pi\pi}}{L}\right)^3-2\pi\frac{a_{\pi\pi}^2r_{\pi\pi}}{L^3}+O(L^{-4})^3\right]$$

- FV effects due to the long-range property of neutrino propagator
 - e.g FV effects relevant for the $e\bar{\nu}\pi$ -intermediate state

Three types of power-law finite-volume effects

• FV effects relevant for the $e\bar{\nu}\pi$ -intermediate state

$$\Delta_{\rm FV} = \left(\frac{1}{V}\sum_{\vec{p}} -\int \frac{d^3\vec{p}}{(2\pi)^3}\right) \frac{\langle 0|J_{\mu L}|\pi(\vec{p})\rangle\langle\pi(\vec{p})|J_{\mu L}|\pi\pi\rangle}{E_{\nu}E_{\pi}(E_{\pi}+E_{\nu}+E_{e}-E_{\pi\pi})}$$

• Integrand is singular at $E_{\nu} = |\vec{p}_{\nu}| = |\vec{p} + \vec{p}_{e}|$

- Define the regular part $f(\vec{p}) \equiv \frac{\langle 0|J_{\mu L}|\pi(\vec{p})\rangle\langle\pi(\vec{p})|J_{\mu L}|\pi\pi\rangle}{E_{\pi}(E_{\pi}+E_{\nu}+E_{e}-E_{\pi\pi})}$ and write $f(\vec{p}) = f(-\vec{p}_{e}) + [f(\vec{p}) - f(-\vec{p}_{e})]$
- Finally we have

$$\Delta_{
m FV} = f(-ec{p_e}) \left(rac{1}{V} \sum_{ec{p}} - \int rac{d^3 ec{p}}{(2\pi)^3}
ight) rac{1}{|ec{p} + ec{p_e}|}$$