Why is domain-wall fermion mathematically interesting?

Hidenori Fukaya (Osaka U.)

HF,. T Onogi, S. Yamaguchi PRD96(2017) no. 12, 125004 [arXiv:1710.03379]

M. Furuta (U. Tokyo), S. Matsuo (Nagoya U.),T. Onogi (Osaka U.), S. Yamaguchi (Osaka U.), M. Yamashita (U.Tokyo)

[arXiv: 19xx.xxxxx]

Menu : Atiyah-Patodi-Singer (APS) index theorem and domain-wall fermion

My talk (appetizer): perturbative finding that an APS index coincides with the eta-invariant of domain-wall Dirac op.

[F, Onogi, Yamaguchi2017]

Furuta's talk (main dish): mathematical proof that every APS index is equivalent to the eta-invariant of domain-wall Dirac op.

[F, Furuta, Matsuo, Onogi, Yamaguchi, Yamashita in progress.]

Menu : Atiyah-Patodi-Singer (APS) index and domain-wall fermion

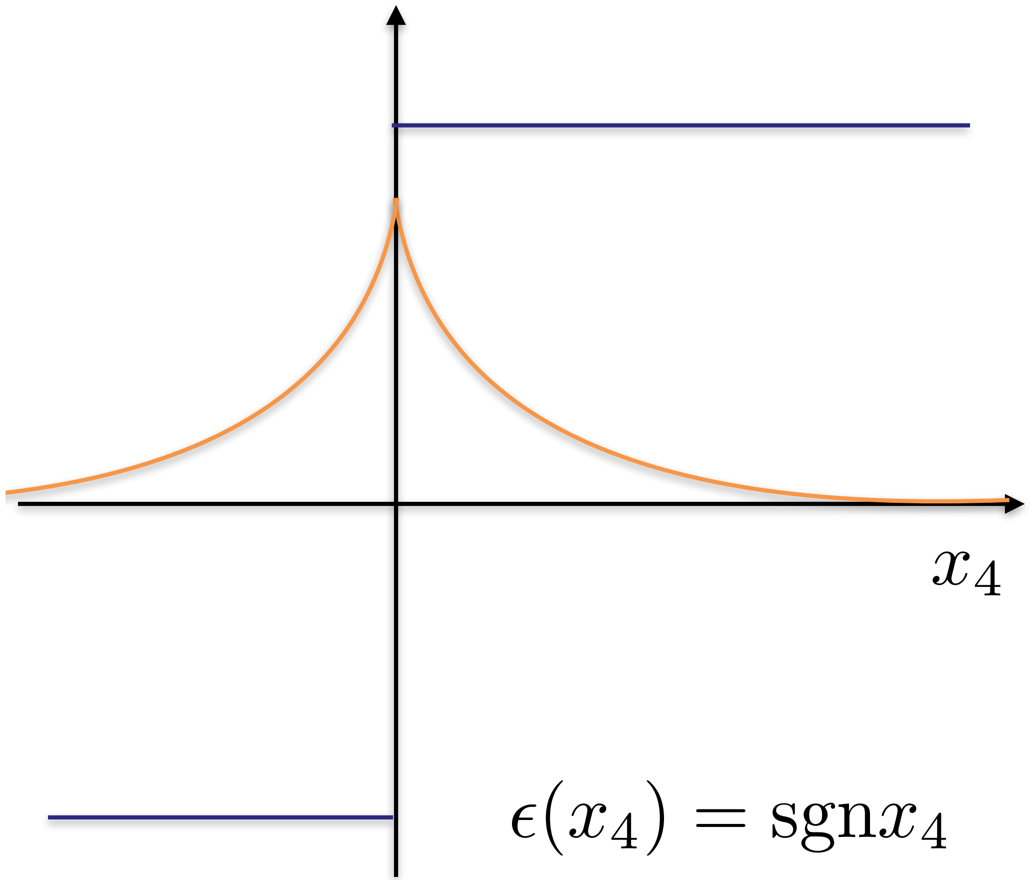
My talk (appetizer): 4D domain-wall fermion w/ SU(N) gauge field in the flat **continuum** Euclidean space with Pauli-Villars regulator.

Furuta's talk (main dish): More general set-up including curved metric.

APS index on a lattice? -> on going. Please wait for lattice 2019 conference [F, Kawai, Matsuki, Mori, Onogi, Yamaguchi in progress.].

4-dim. domain-wall fermion

$$D_{DW} = D + M\epsilon(x_4)$$



Massless Dirac fermion localized at 3-dim edge.

No gauge anomaly, but T(or parity) anomaly.

good model for topological insulator.

Atiyah-Patodi-Singer index theorem

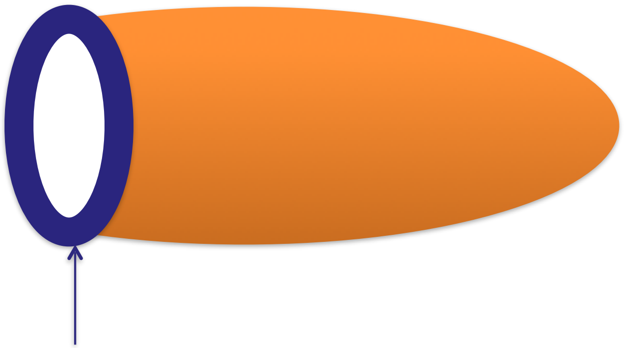
index on a manifold with boundary,

$$\lim_{\Lambda\to\infty} \mathrm{Tr}\gamma_5 e^{D^2_{4\mathrm{D}}/\Lambda^2} = \frac{1}{32\pi^2}\int_{x_4>0} d^4x \epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{3\mathrm{D}})}{2}$$

integer

non-integer

non-integer



$x_4 = 0$

$$\eta(iD^{3\mathrm{D}}) = \sum_{\lambda\geq 0}^{reg} - \sum_{\lambda<0}^{reg} = \sum_{\lambda}^{reg} \mathrm{sgn}\lambda$$

[Atiyah-Patodi-Singer 1975]

APS index in topological insulator

Witten 2015 : APS index is a key to understand bulk-edge correspondence in symmetry protected topological insulator:

fermion path integrals

$$Z_{\mathrm{edge}} \propto \exp(-i\pi\eta(iD^{\mathrm{3D}})/2)$$

T-anomalous

$$Z_{\mathrm{bulk}} \propto \exp\left(i\pi\frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}]\right)$$

T-anomalous

$$Z_{\mathrm{edge}}Z_{\mathrm{bulk}} \propto (-1)^{\mathfrak{I}}$$

➡ T is protected !

$$\mathfrak{I} = \frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{\mathrm{3D}})}{2}$$

[Related works: Metlitski 15, Seiberg-Witten 16, Tachikawa-Yonekura 16, Freed-Hopkins 16, Witten 16, Yonekura 16…]

What puzzled us

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3. No “physicist-friendly” description in the literature

   [except for Alvarez-Gaume et al. 1985 (bulk part is limitted to an integer due to adiabatic approximation, and boundary condition is obscure.)]

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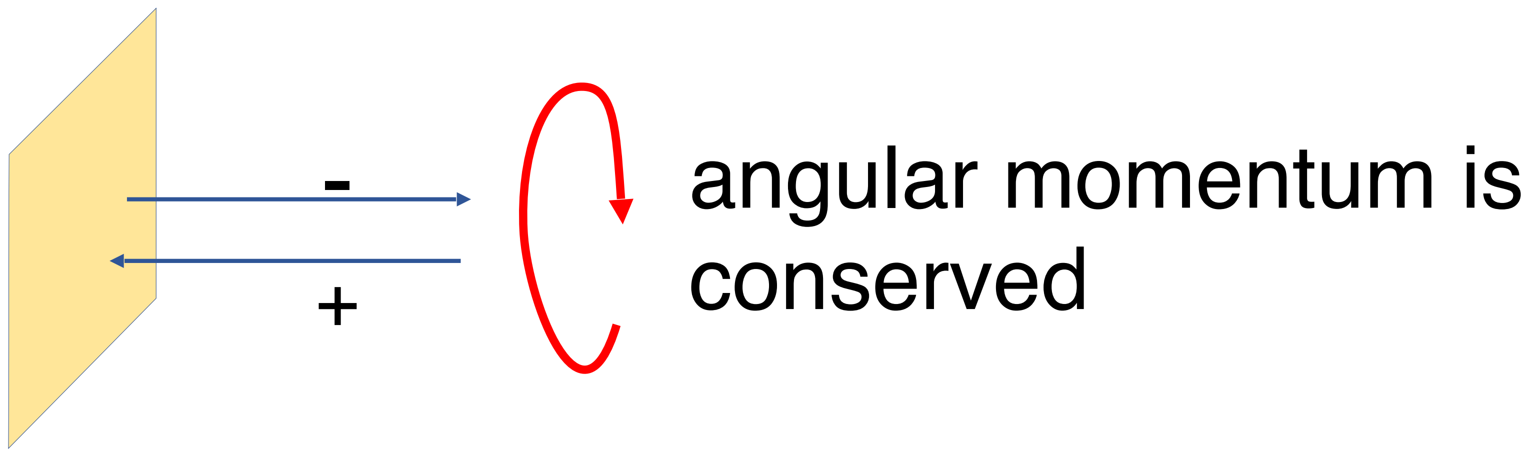
→ We launched a study group reading original APS paper and it took 3 months to translate it into “physics language”, and we propose an alternative expression.

Contents

Difficulty with boundary

If we impose **local** and **Lorentz (rotation)** invariant boundary condition, + and – chirality sectors do not decouple any more.



n_+, n_- and the index do not make sense.

Atiyah-Patodi-Singer boundary condition [Atiyah, Patodi, Singer 75]

Gives up the locality and rotational symmetry but keeps the chirality.

Eg. 4 dim $x^4 \geq 0$ $A_4 = 0$ gauge

$$D = \gamma^4 \partial_4 + \gamma^i D_i = \gamma^4(\partial_4 + \underbrace{\gamma^4 \gamma^i D_i}_{A})$$

They impose a non-local b.c.

$$(A + |A|)\psi|_{x^4=0} = 0$$

$$\text{index} = n_+ - n_-$$

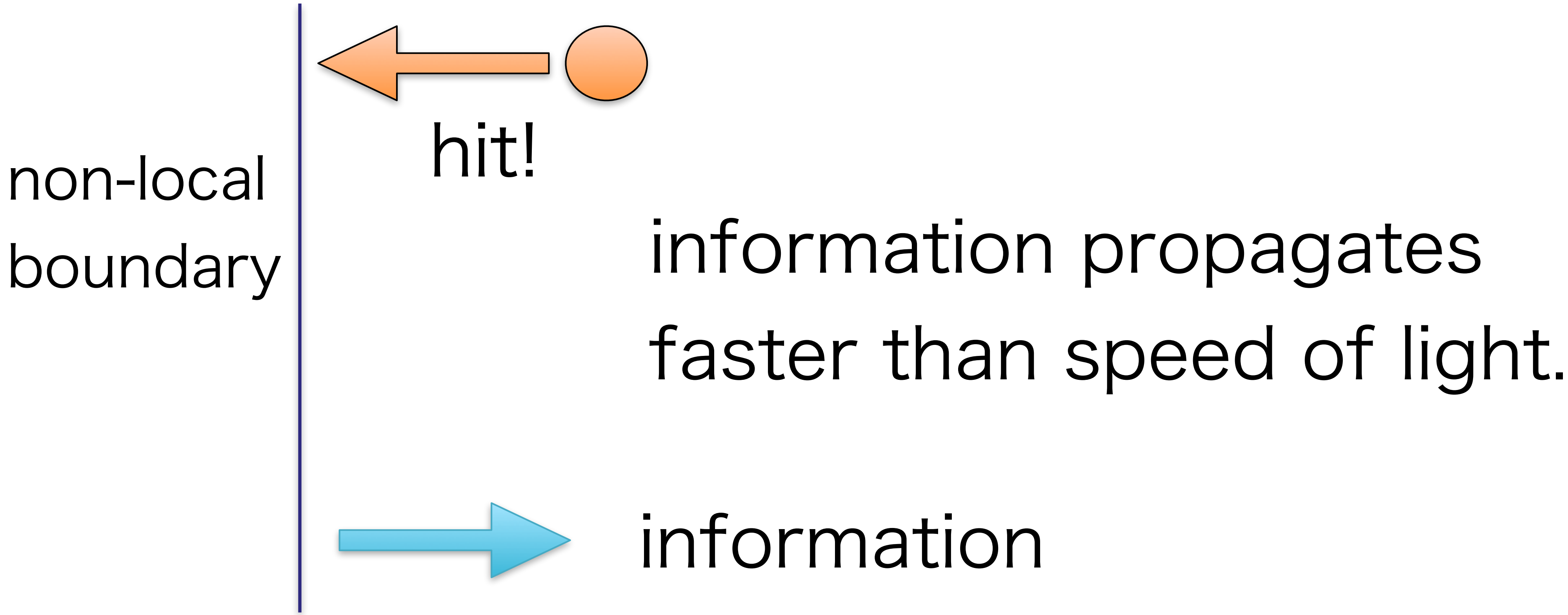
Beautiful!

But physicist-unfriendly.

Locality >> chirality for physicists

Locality (=causality) is essential.

We cannot accept APS condition even if it is beautiful.



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Locality (=causality) is essential.

We cannot accept APS condition even if it is beautiful.

→ need to give up chirality and consider L/R mixing

(massive case)

$$\cancel{n_+ - n_-} = \frac{1}{32\pi^2}\int_{x_4>0} d^4x \epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{\mathrm{3D}})}{2}$$

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Can we still make a fermionic integer (even if it is ugly)?

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Can we still make a fermionic integer (even if it is ugly)?

Our answer is "Yes, we can".

Contents

Massive Dirac operator

$$D + M = \begin{pmatrix} M & D_{LR} \\ D_{RL} & M \end{pmatrix}$$

Anti-Hermitian   Hermitian

Let's consider a Hermitian operator:

$$H = \gamma_5(D + M) \qquad \gamma_5 = i\gamma_1\gamma_2\gamma_3\gamma_4.$$

on a manifold without boundary.

Zero-modes & non-zero modes

$$H = \gamma_5(D + M)$$

Zero-modes of D = still eigenstates of H:

$$H\phi_0 = \gamma_5 M\phi_0 = \pm M\phi_0.$$

Non-zero modes make ± pairs

$$H\phi_i = \lambda_i\phi_i$$

$$HD\phi_i = -DH\phi_i = -\lambda_i D\phi_i$$

Eta invariant of massive Dirac operator

$$\eta(H) = \sum_i \mathrm{sgn}\lambda_i$$

$$H = \gamma_5(D+M)$$

$$= \# \text{ of } +M - \# \text{ of } -M$$

coincides with the original AS index!

Eta invariant of massive Dirac operator

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coincides with the original AS index!

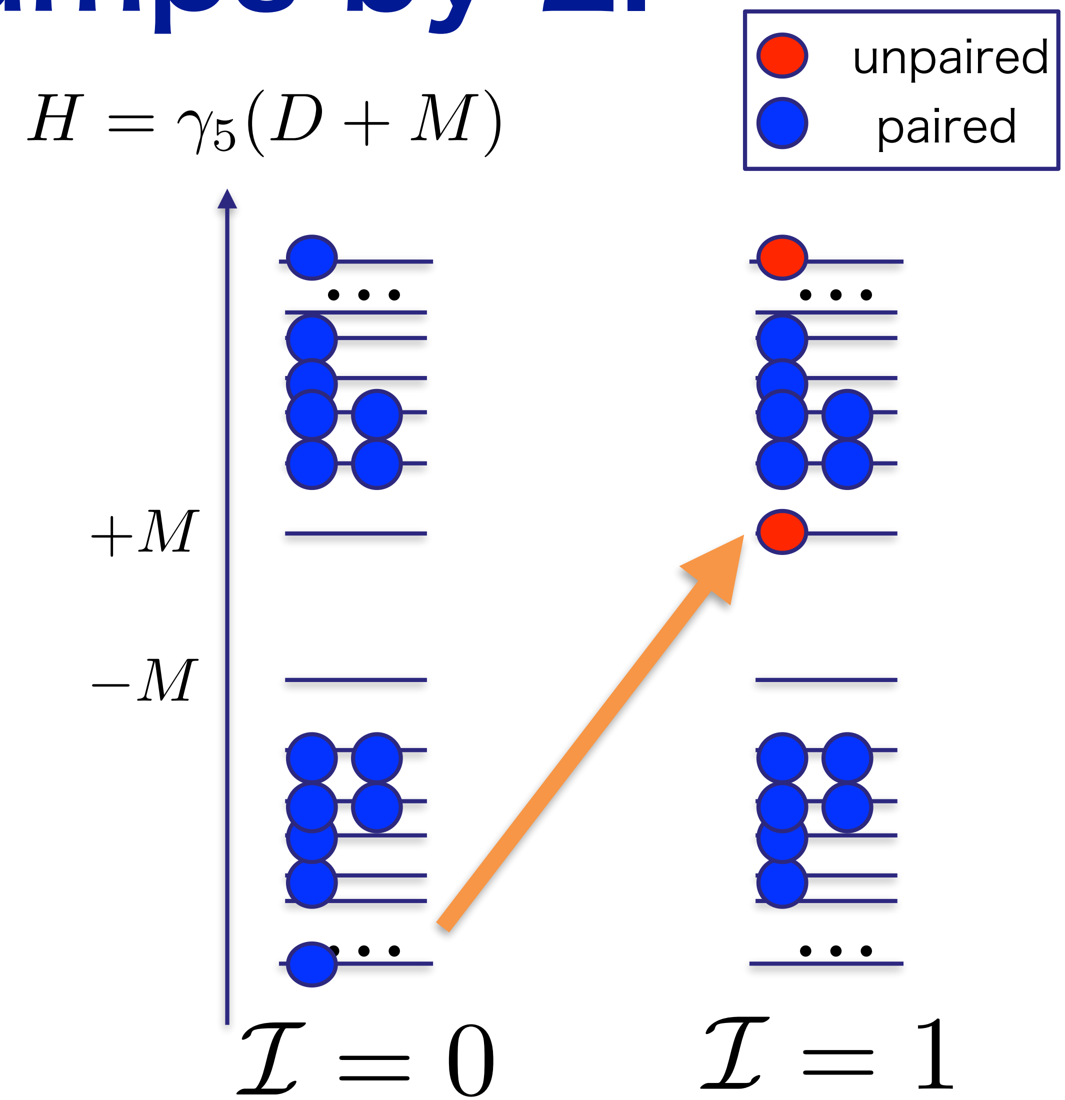
In fact, we need a factor 1/2.

$$\mathrm{Index}(D) = \frac{1}{2}\eta(H)^{reg}.$$

$\eta(H)$ always jumps by 2.

To increase + modes,
we have to borrow
one from - (UV) modes.

Good regularizations
(e.g. Pauli-Villars, lattice)
respect this fact.



$$\text{Index}(D) = \frac{1}{2}\eta(H).$$

Perturbative "proof" (in physics sense)

using Pauli-Villars regulator

$$\frac{1}{2}\eta(H)^{reg} = \frac{1}{2}\left[\eta(H) - \eta(H_{PV})\right].$$

$$H = \gamma_5(D + M)$$

$$H_{PV} = \gamma_5(D + \Lambda), \quad \Lambda \gg M$$

$$\eta(H) = \lim_{s\to 0} \mathrm{Tr}\frac{H}{(\sqrt{H^2})^{1+s}} = \frac{1}{\sqrt{\pi}}\int_0^\infty dt t^{-1/2}\mathrm{Tr} H e^{-tH^2}$$

$$= \frac{1}{\sqrt{\pi}}\int_0^\infty dt' t'^{-1/2}\mathrm{Tr}\gamma_5\left(\mathrm{sgn}M + \frac{D}{M}\right) e^{-t' D^\dagger D/M^2} e^{-t'},$$

$(t' = M^2 t)$

Fujikawa-method

does not contribute.

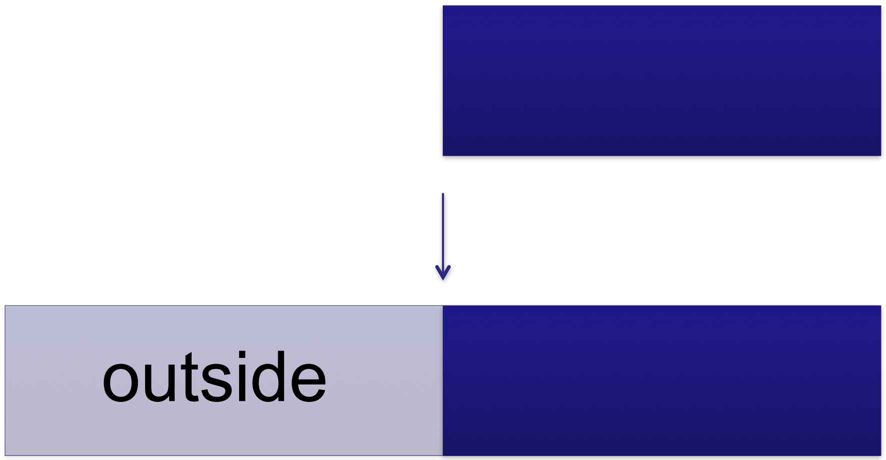
$$= \mathrm{sgn}M \frac{1}{32\pi^2}\int d^4x\, \epsilon_{\mu\nu\rho\sigma}\mathrm{tr}_c F^{\mu\nu}F^{\rho\sigma} + \mathcal{O}(1/M^2).$$

Contents

More physical set-up?

In physics,

1. Any boundary has “outside”: manifold + boundary → domain-wall.



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In physics,

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More physical set-up?

In physics,

1. Any boundary has "outside": manifold + boundary → domain-wall.
2. Boundary should not preserve helicity but keep angular-mom: massless → massive (in bulk)
3. Boundary condition should not be put by hand → but automatically chosen.
4. Edge-localized modes play the key role.

Domain-wall Dirac operator

Let us consider

$$D_{\rm 4D} + M\epsilon(x_4), \quad \epsilon(x_4) = {\rm sgn} x_4$$

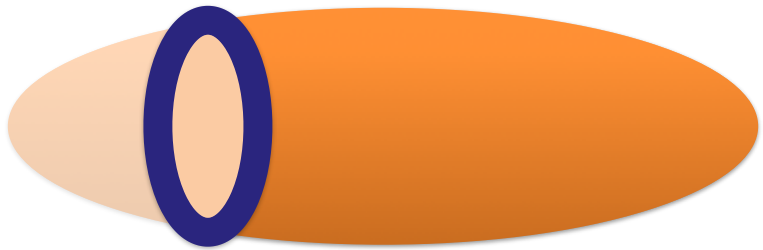
[Jackiw-Rebbi 1976, Callan-Harvey 1985, Kaplan 1992]

on a closed manifold

with sign flipping mass,

without assuming any

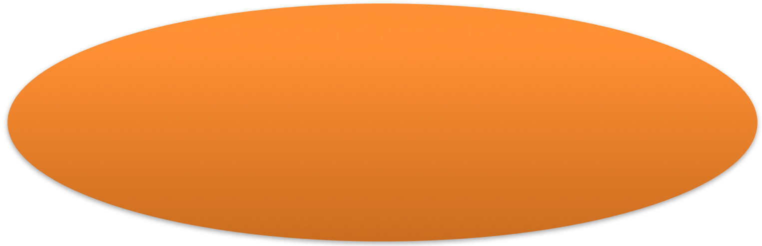
boundary condition



(we expect it dynamically given.).

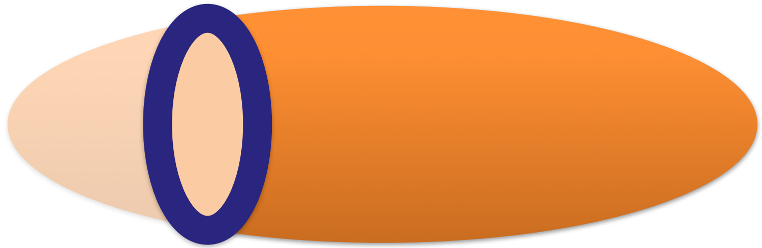
“new” APS index [F-Onogi-Yamaguchi 2017]

$$\frac{1}{2}\eta(\gamma_5(D+M))^{reg} = \text{AS index}$$



$$\Downarrow$$

$$\frac{1}{2}\eta(\gamma_5(D+M\epsilon(x_4)))^{reg}$$



$$= \frac{1}{2}\eta(H_{DW}) - \frac{1}{2}\eta(H_{PV})$$

$$H_{DW} = \gamma_5(D_{\rm 4D} + M\epsilon(x_4))$$

$$H_{PV} = \gamma_5(D_{\rm 4D} - M_2)$$

We will show this coincides with APS index!

PV part = Atiyah-Singer index

$$\eta(H_{PV}) = \lim_{s\to 0} \mathrm{Tr}\frac{H_{PV}}{(\sqrt{H_{PV}^2})^{1+s}} = \frac{1}{\sqrt{\pi}}\int_0^\infty dt t^{-1/2}\mathrm{Tr} H_{PV} e^{-tH_{PV}^2}$$

$$= \frac{1}{\sqrt{\pi}}\int_0^\infty dt' t'^{-1/2}\mathrm{Tr}\gamma_5\left(-1+\frac{D}{M}\right) e^{-t'D^\dagger D/M^2} e^{-t'},$$

Fujikawa-method

does not contribute.

$(t' = M^2 t)$

$$= -\frac{1}{32\pi^2}\int d^4x\, \epsilon_{\mu\nu\rho\sigma}\mathrm{tr}_c F^{\mu\nu}F^{\rho\sigma} + \mathcal{O}(1/M^2).$$

$$H_{PV} = \gamma_5(D_{4\mathrm{D}} - M_2)$$

Domain-wall fermion part

Now let's compute

$$\eta(H_{DW}) = \lim_{s\to 0} \mathrm{Tr}\frac{H_{DW}}{(\sqrt{H_{PV}^2})^{1+s}} = \lim_{s\to 0} \frac{1}{\Gamma(\frac{1+s}{2})}\int_0^\infty dt t^{(s-1)/2}\mathrm{Tr} H_{DW} e^{-tH_{DW}^2}$$

$$H_{DW} = \gamma_5(D_{4\mathrm{D}} + M\epsilon(x_4))$$

In the free fermion case,

$$H_{DW}^2 = -\partial_\mu^2 + M^2 - 2M\gamma_4\delta(x_4).$$

→ eigenvalue problem = Schrodinger equation with δ-function-like potential.

Complete set in the free case

Solutions to $\left(-\partial_{x_4}^2 + \omega^2 - 2M\gamma_4\delta(x_4)\right)\varphi = 0$

are

$$\varphi^{\omega}_{\pm,o}(x_4) = \frac{1}{\sqrt{4\pi}}\left(e^{i\omega x_4} - e^{-i\omega x_4}\right),$$

$$\varphi^{\omega}_{\pm,e}(x_4) = \frac{1}{\sqrt{4\pi(\omega^2+M^2)}}\left((i\omega \mp M)e^{i\omega|x_4|} + (i\omega \pm M)e^{-i\omega|x_4|}\right),$$

$$\varphi^{\text{edge}}_{+,e}(x_4) = \sqrt{M}e^{-M|x_4|},$$ → Edge mode appears !

where $\omega = \sqrt{\boldsymbol{p}^2 + M^2 - \lambda_{4D}^2}$ and $\gamma_4\varphi^{\omega,\text{edge}}_{\pm,e/o} = \pm\varphi^{\omega,\text{edge}}_{\pm,e/o}$

3D direction = conventional plane waves.

“Automatic” boundary condition

We didn’t put any boundary condition by hand. But

$$\left[\frac{\partial}{\partial x_4} \pm M\epsilon(x_4)\right] \varphi^{\omega,\mathrm{edge}}_{\pm,e}(x_4)\Big|_{x_4=0} = 0, \quad \varphi^{\omega}_{\pm,o}(x_4=0) = 0.$$

is automatically satisfied due to the δ-function-like potential.

This condition is LOCAL and PRESERVES angular-momentum in x_4 direction but DOES NOT keep chirality.

Fujikawa-method

$$\eta(H_{DW}) = \frac{1}{\Gamma(\frac{1+s}{2})}\int_0^\infty dt' t'^{\frac{s-1}{2}} \mathrm{Tr}\gamma_5 \left(\epsilon(x_4) + \frac{D}{M}\right) e^{-t' H_{DW}^2/M^2} e^{-t'},$$

Perturbative expansion

We insert our complete set $\{\varphi^{\omega,\mathrm{edge}}_{\pm,e/o}(x_4) \times e^{i\boldsymbol{p}\cdot\boldsymbol{x}}\}$

(See our paper for the details.)

100% edge-mode effect

$$= \frac{1}{32\pi^2}\int d^4x\, \epsilon(x_4)\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}_c F^{\mu\nu}F^{\rho\sigma} - \eta(iD^{3\mathrm{D}})$$

$\epsilon(x_4) = \mathrm{sgn} x_4$ $\qquad\qquad$ $\eta(iD^{3\mathrm{D}}) = (CS \text{ mod integer})$

Total index

$$
\begin{aligned}
\mathfrak{I} &= \frac{\eta(H_{DW}))}{2} - \frac{\eta(H_{PV})}{2} \\
&= \frac{1}{2}\left[\frac{1}{32\pi^2}\int d^4x\ \epsilon(x_4)\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}_c F^{\mu\nu}F^{\rho\sigma} - \eta(iD^{\mathrm{3D}})\right. \\
&\left. + \frac{1}{32\pi^2}\int d^4x\ \epsilon_{\mu\nu\rho\sigma}\mathrm{tr}_c F^{\mu\nu}F^{\rho\sigma}\right] \\
&= \frac{1}{32\pi^2}\int_{x_4>0} d^4x\ \epsilon_{\mu\nu\rho\sigma}\mathrm{tr}_c F^{\mu\nu}F^{\rho\sigma} - \frac{1}{2}\eta(iD^{\mathrm{3D}})
\end{aligned}
$$

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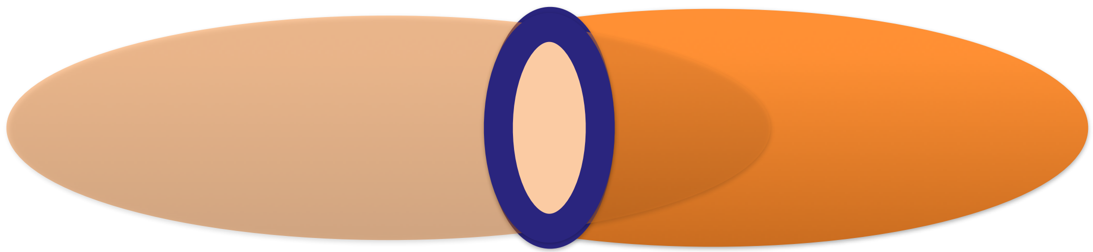
We don't need any boundary condition.



The kink structure automatically chooses a local and rotationally symmetric boundary condition,

and extension from AS index is simple:

$$\frac{1}{2}\eta(\gamma_5(D+M)) \rightarrow \frac{1}{2}\eta(\gamma_5(D+M\epsilon(x)))$$

massive fermion = chiral symmetry is NOT important.

The lattice fermion "knew" this fact:

$$Ind(D_{ov}) = \frac{1}{2}\mathrm{Tr}\gamma_5\left(1 - \frac{aD_{ov}}{2}\right) \quad D_{ov} = \frac{1}{a}\left(1 + \gamma_5\frac{H_W}{\sqrt{H_W^2}}\right)$$

$$= -\frac{1}{2}\mathrm{Tr}\frac{H_W}{\sqrt{H_W^2}} = -\frac{1}{2}\eta(\gamma_5(D_W - M))!$$

If the original AS index were given by

$$-\frac{1}{2}\eta(\gamma_5(D - M))$$

we should have known the lattice index theorem much before Hasenfratz 1998.

Massless vs. massive

index theorem with massless Dirac

	continuum	lattice
AS	$\mathrm{Tr}\gamma^5 e^{-D^2/M^2}$	$\mathrm{Tr}\gamma^5(1-aD_{ov}/2)$
APS	$\mathrm{Tr}\gamma^5 e^{-D^2/M^2}$ w/ APS b.c.	not known.

index theorem with massive Dirac

	continuum	lattice
AS	$-\frac{1}{2}\eta(\gamma_5(D-M))$	$-\frac{1}{2}\eta(\gamma_5(D_W-M))$
APS	$-\frac{1}{2}\eta(\gamma_5(D-M\epsilon(x)))$	$-\frac{1}{2}\eta(\gamma_5(D_W-M\epsilon(x)))$?

we will report at Lat19, Wuhan

Contents

✔ 1. Introduction

✔ 2. What is APS index theorem?
mathematically O.K. but unphysical.

✔ 3. AS index from massive Dirac operator
$\mathfrak{I} = \eta(\gamma_5(D+M))^{reg}/2$ coincides with the AS index.

✔ 4. New index from domain-wall operator
$\mathfrak{I} = \eta(\gamma_5(D+M\epsilon(x_4)))^{reg}/2$ coincides with the APS index.

✔ 5. What's good with eta-invariant
gives a united view without chiral symmetry.

6. Summary

Summary

1. APS index describes bulk-edge correspondence (of anomaly) of topological insulators.
2. APS (as well as AS) index can be reformulated as the eta-invariant of massive domain-wall Dirac operator.
3. Eta-invarant with massive Dirac operator gives a united view of inde theorems, where chiral symmetry is not important.

The original APS and Domain-wall fermion are totally different !

APS

1. massless Dirac
   (even in bulk)
2. non-local boundary cond.
   (depending on gauge fields)
3. SO(2) rotational sym. on boundary is lost.
4. no edge mode appears.
5. manifold + boundary

Domain-wall fermion

1. massive Dirac in bulk
   (massless mode at edge)
2. local boundary cond.
3. SO(2) rotational sym. on boundary is kept.
4. Edge mode describes eta-invariant.
5. closed manifold + domain-wall

Next talk by Furuta-san = A mathematical proof for

$$Ind(D_{\mathrm{APS}}) = \frac{1}{2}\eta(H_{DW}^{reg})$$

on general even-dimensional manifold.

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Backup slides

Example : 1+1d bulk + 0+1d edge Majorana fermion coupled to gravity

APS index tells

$$Z \propto \exp\left(2\pi i \frac{n}{8}\right)$$

consistent with Z_8 classification of Kitaev's interacting Majorana chain.

Eta invariant = Chern Simons term + integer (non-local effect)

$$\frac{\eta(iD^{\rm 3D})}{2} = \frac{CS}{2\pi} + \text{integer}$$

$$CS \equiv \frac{1}{4\pi}\int_Y d^3x \; \mathrm{tr}_c\left[\epsilon_{\nu\rho\sigma}\left(A^\nu\partial^\rho A^\sigma + \frac{2i}{3}A^\nu A^\rho A^\sigma\right)\right],$$

= surface term.

$$\mathfrak{I} = \frac{1}{32\pi^2}\int_{x_4>0} d^4x \epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{\rm 3D})}{2}$$