Gradient flow and the EMT on the lattice

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Gradient flow (Narayanan–Neuberger, Lüscher)

- One-parameter $t \geq 0$ (the flow time) deformation of the gauge field $A_\mu(x)$,

$$A_\mu(x) \to B_\mu(t,x), \qquad B_\mu(t=0,x) = A_\mu(x),$$

according to (the flow equation)

$$\partial_t B_\mu(t,x) = -g_0^2 \frac{\delta S_{\mathrm{YM}}[B]}{\delta B_\mu(t,x)}, = D_\nu G_{\nu\mu}(t,x) = \Delta B_\mu(t,x) + \cdots ,$$

- Here, S_{YM} is the Yang–Mills action and the RHS is the gradient in functional space. So the name of the Yang–Mills gradient flow.
- Since

$$D_\mu = \partial_\mu + [B_\mu, \cdot], \quad G_{\mu\nu}(t,x) = \partial_\mu B_\nu(t,x) - \partial_\nu B_\mu(t,x) + [B_\mu(t,x), B_\nu(t,x)],$$

this is a diffusion-type equation with the diffusion length,

$$x \sim \sqrt{8t}.$$

The flow time t has the mass dimension -2.

Yang–Mills gradient flow

- Yang–Mills gradient flow (continuum)

$$\partial_t B_\mu(t,x) = -g_0^2 \frac{\delta S_{\mathrm{YM}}[B]}{\delta B_\mu(t,x)}, \qquad B_\mu(t=0,x) = A_\mu(x).$$

- Wilson flow (lattice)

$$\partial_t V(t,x,\mu) V(t,x,\mu)^{-1} = -g_0^2 \partial_{x,\mu} S_{\mathrm{Wilson}}[V], \qquad V(t=0,x,\mu) = U(x,\mu).$$

- Applications in lattice gauge theory (the citation of the Lüscher's original paper is $\gtrsim 500$)
  - Topological charge
  - Scale setting
  - Non-perturbative gauge coupling constant
  - Chiral condensate
  - Various renormalized operators, including the energy–momentum tensor
  - Supersymmetric theory
  - Chiral gauge theory
  - etc.

Finiteness of the gradient flow (Lüscher, Weisz (2011))

- Correlation function of the flowed gauge field,

$$\langle B_{\mu_1}(t_1, x_1) \cdots B_{\mu_n}(t_n, x_n) \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}A_\mu \, B_{\mu_1}(t_1, x_1) \cdots B_{\mu_n}(t_n, x_n) \, e^{-S_{\text{YM}}[A]},$$

  when expressed in terms of renormalized coupling,

$$g^2 = g_0^2 \mu^{-2\varepsilon} Z^{-1},$$

  is UV finite without the wave function renormalization.

- This is quite contrast to the conventional gauge field, for which

$$\langle A_{\mu_1}(x_1) \cdots A_{\mu_n}(x_n) \rangle \, ,$$

  requires the wave function renormalization

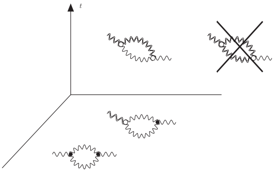
$$(A_R)^a_\mu = Z^{-1/2} Z_3^{-1/2} A^a_\mu.$$

Finiteness of the gradient flow

- This finiteness persists even for the equal-point product,

$$\langle B_{\mu_1}(t_1,x_1)B_{\mu_2}(t_1,x_1)\cdots B_{\mu_n}(t_n,x_n)\rangle\,, \qquad t_1>0,\ldots,t_n>0.$$

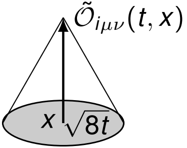
- Any composite operator of the flowed gauge field is automatically UV finite.
- All order proof of the finiteness uses a local $D+1$-dimensional field theory:



- Because of the gaussian damping factor $\sim e^{-tp^2}$ in the propagator $\Rightarrow$ No bulk ($t>0$) counterterm.
- BRS symmetry $\Rightarrow$ No boundary ($t=0$) counterterm besides Yang–Mills ones.

Small flow-time expansion (Lüscher, Weisz (2011))

- Generally, the relation between a composite operator in $t > 0$ and that in 4D can be quite complicated.
- The relation becomes tractable, however, in the small flow time limit $t \to 0$.
- Small flow-time expansion



$$\tilde{\mathcal{O}}_{i\mu\nu}(t,x) = \langle \tilde{\mathcal{O}}_{i\mu\nu}(t,x) \rangle \mathbb{1} + \sum_j \zeta_{ij}(t) \left[\mathcal{O}_{Rj\mu\nu}(x) - \text{VEV} \right] + O(t).$$

- This is quite analogous to the OPE, but the continuous flow time t is more suitable for lattice gauge theory.

Small flow-time expansion

- Small flow-time expansion:

$$\tilde{\mathcal{O}}_{i\mu\nu}(t,x) = \left\langle \tilde{\mathcal{O}}_{i\mu\nu}(t,x) \right\rangle \mathbb{1} + \sum_j \zeta_{ij}(t) \left[\mathcal{O}_{Rj\mu\nu}(x) - \mathrm{VEV} \right] + O(t).$$

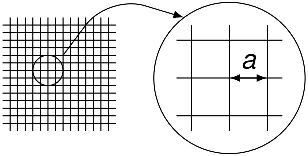
- Inverting this,

$$\mathcal{O}_{Ri\mu\nu}(x) - \mathrm{VEV} = \lim_{t\to 0} \left\{ \sum_j \left(\zeta^{-1}\right)_{ij}(t) \left[\tilde{\mathcal{O}}_{j\mu\nu}(t,x) - \left\langle \tilde{\mathcal{O}}_{j\mu\nu}(t,x) \right\rangle \mathbb{1} \right] \right\},$$

  we have a representation of the (renormalized) operator in terms of flowed field.
- Furthermore, the $t \to 0$ behavior of the coefficients $\zeta_{ij}(t)$ can be determined by perturbation theory, thanks to the asymptotic freedom (cf. OPE).
- We use these facts to find a universal representation of the EMT.

Lattice gauge theory (LGT) and the energy–momentum tensor (EMT)

- LGT is very nice...



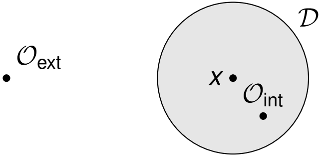
  This however breaks spacetime symmetries (translation, Poincaré, SUSY, ...) for $a \neq 0$.
- For $a \neq 0$, one cannot define the Noether current associated with the translational invariance, EMT $T_{\mu\nu}(x)$.
- Even for the continuum limit $a \to 0$, this is difficult, because EMT is a composite operator which generally contains UV divergences:

$$a \times \frac{1}{a} \overset{a \to 0}{\to} 1.$$

EMT in LGT?

- We want to construct EMT on the lattice, which becomes the correct EMT, automatically in the continuum limit $a \to 0$.
- The correct EMT is characterized by the translation Ward–Takahashi relation

$$\left\langle \mathcal{O}_{\text{ext}} \int_{\mathcal{D}} d^D x \, \partial_\mu T_{\mu\nu}(x) \, \mathcal{O}_{\text{int}} \right\rangle = - \langle \mathcal{O}_{\text{ext}} \, \partial_\nu \mathcal{O}_{\text{int}} \rangle \, .$$



- This contains the correct normalization and the conservation law.
- Applications to physics related to spacetime symmetries:
  QCD thermodynamics, transport coefficients in gauge theory, momentum/spin structure of baryons, conformal field theory, dilaton physics, ...

Conventional approach (Caracciolo et al. (1989–))

- Under the hypercubic symmetry, the operator reproducing the correct EMT of QCD for $a \to 0$ is given by

$$T_{\mu\nu}(x) = \sum_{i=1}^{7} Z_i \mathcal{O}_{i\mu\nu}(x)\big|_{\text{lattice}} - \text{VEV},$$

where

$$\mathcal{O}_{1\mu\nu}(x) \equiv \sum_{\rho} F^a_{\mu\rho}(x) F^a_{\nu\rho}(x), \qquad \mathcal{O}_{2\mu\nu}(x) \equiv \delta_{\mu\nu} \sum_{\rho,\sigma} F^a_{\rho\sigma}(x) F^a_{\rho\sigma}(x),$$

$$\mathcal{O}_{3\mu\nu}(x) \equiv \bar{\psi}(x) \left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu \right) \psi(x), \quad \mathcal{O}_{4\mu\nu}(x) \equiv \delta_{\mu\nu} \bar{\psi}(x) \overleftrightarrow{\not{D}} \psi(x),$$

$$\mathcal{O}_{5\mu\nu}(x) \equiv \delta_{\mu\nu} m_0 \bar{\psi}(x) \psi(x),$$

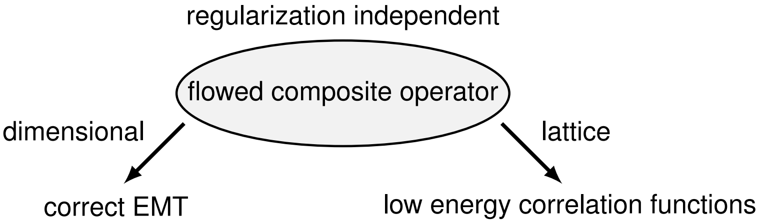
and, Lorentz non-covariant ones:

$$\mathcal{O}_{6\mu\nu}(x) \equiv \delta_{\mu\nu} \sum_{\rho} F^a_{\mu\rho}(x) F^a_{\mu\rho}(x), \qquad \mathcal{O}_{7\mu\nu}(x) \equiv \delta_{\mu\nu} \bar{\psi}(x) \gamma_\mu \overleftrightarrow{D}_\mu \psi(x)$$

- Seven non-universal coefficients Z_i must be determined by lattice perturbation theory or non-perturbatively.

Our approach (arXiv:1304.0533)

- We bridge lattice regularization and dimensional regularization, which preserves the translational invariance, by the gradient flow.
- Schematically,



EMT in dimensional regularization

- Vector-like gauge theory:

$$S = -\frac{1}{2g_0^2}\int d^Dx\,\mathrm{tr}\left[F_{\mu\nu}(x)F_{\mu\nu}(x)\right] + \int d^Dx\,\bar{\psi}(x)(\not{D} + m_0)\psi(x).$$

- By the Noether method,

$$T_{\mu\nu}(x) = \frac{1}{g_0^2}\left\{\mathcal{O}_{1\mu\nu}(x) - \frac{1}{4}\mathcal{O}_{2\mu\nu}(x)\right\} + \frac{1}{4}\mathcal{O}_{3\mu\nu}(x) - \frac{1}{2}\mathcal{O}_{4\mu\nu}(x) - \mathcal{O}_{5\mu\nu}(x) - \mathrm{VEV},$$

where

$$\mathcal{O}_{1\mu\nu}(x) \equiv \sum_\rho F^a_{\mu\rho}(x)F^a_{\nu\rho}(x), \qquad \mathcal{O}_{2\mu\nu}(x) \equiv \delta_{\mu\nu}\sum_{\rho,\sigma} F^a_{\rho\sigma}(x)F^a_{\rho\sigma}(x),$$

$$\mathcal{O}_{3\mu\nu}(x) \equiv \bar{\psi}(x)\left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu\right)\psi(x), \quad \mathcal{O}_{4\mu\nu}(x) \equiv \delta_{\mu\nu}\bar{\psi}(x)\overleftrightarrow{\not{D}}\psi(x),$$

$$\mathcal{O}_{5\mu\nu}(x) \equiv \delta_{\mu\nu} m_0\bar{\psi}(x)\psi(x).$$

- Under the dimensional regularization, this simple combination is the correct EMT.

EMT from the gradient flow

- We consider following composite operators of flowed fields:

$$\tilde{\mathcal{O}}_{1\mu\nu}(t,x) \equiv G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x),$$
$$\tilde{\mathcal{O}}_{2\mu\nu}(t,x) \equiv \delta_{\mu\nu}G^a_{\rho\sigma}(t,x)G^a_{\rho\sigma}(t,x),$$
$$\tilde{\mathcal{O}}_{3\mu\nu}(t,x) \equiv \mathring{\bar{\chi}}(t,x)\left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu\right)\mathring{\chi}(t,x),$$
$$\tilde{\mathcal{O}}_{4\mu\nu}(t,x) \equiv \delta_{\mu\nu}\mathring{\bar{\chi}}(t,x)\overleftrightarrow{\not{D}}\mathring{\chi}(t,x),$$
$$\tilde{\mathcal{O}}_{5\mu\nu}(t,x) \equiv \delta_{\mu\nu}m\mathring{\bar{\chi}}(t,x)\mathring{\chi}(t,x),$$

and then the small flow-time expansion reads,

$$\tilde{\mathcal{O}}_{i\mu\nu}(t,x) = \left\langle\tilde{\mathcal{O}}_{i\mu\nu}(t,x)\right\rangle\mathbb{1} + \sum_j \zeta_{ij}(t)\left[\mathcal{O}_{j\mu\nu}(x) - \langle\mathcal{O}_{j\mu\nu}(x)\rangle\mathbb{1}\right] + O(t).$$

- We compute $\zeta_{ij}(t)$ with dimensional regularization. We then substitute

$$\mathcal{O}_{i\mu\nu}(x) - \langle\mathcal{O}_{i\mu\nu}(x)\rangle\mathbb{1} = \lim_{t\to0}\left\{\sum_j\left(\zeta^{-1}\right)_{ij}(t)\left[\tilde{\mathcal{O}}_{j\mu\nu}(t,x) - \left\langle\tilde{\mathcal{O}}_{j\mu\nu}(t,x)\right\rangle\mathbb{1}\right]\right\},$$

in the expression of the EMT in dimensional regularization.

Fermion flow

- We also introduce the fermion flow (Lüscher (2013))

$$\partial_t \chi(t,x) = [\Delta - \alpha_0 \partial_\mu B_\mu(t,x)]\,\chi(t,x), \qquad \chi(t=0,x) = \psi(x),$$
$$\partial_t \bar{\chi}(t,x) = \bar{\chi}(t,x)\left[\overleftarrow{\Delta} + \alpha_0 \partial_\mu B_\mu(t,x)\right], \qquad \bar{\chi}(t=0,x) = \bar{\psi}(x),$$

where

$$\Delta = D_\mu D_\mu, \qquad D_\mu = \partial_\mu + B_\mu,$$
$$\overleftarrow{\Delta} = \overleftarrow{D}_\mu \overleftarrow{D}_\mu, \qquad \overleftarrow{D}_\mu \equiv \overleftarrow{\partial}_\mu - B_\mu.$$

- It turns out that the flowed fermion field requires the wave function renormalization:

$$\chi_R(t,x) = Z_\chi^{1/2} \chi(t,x), \qquad \bar{\chi}_R(t,x) = Z_\chi^{1/2} \bar{\chi}(t,x),$$
$$Z_\chi = 1 + \frac{g^2}{(4\pi)^2} C_2(R) 3 \frac{1}{\epsilon} + O(g^4).$$

- Still, any composite operators of $\chi_R(t,x)$ are UV finite.

Ringed fermion fields

- To avoid the complication associated with the wave function renormalization, we introduce the variable,

$$\mathring{\chi}(t,x) = \mathcal{C}\frac{\chi(t,x)}{\sqrt{t^2\left\langle\bar{\chi}(t,x)\overleftrightarrow{\not{D}}\chi(t,x)\right\rangle}} = \chi_R(t,x) + O(g^2),$$

  where

$$\mathcal{C} \equiv \sqrt{\frac{-2\dim(R)}{(4\pi)^2}},$$

  and similarly for $\bar{\chi}(t,x)$.
- Since Z_χ is canceled out in $\mathring{\chi}(t,x)$, any composite operators of $\mathring{\chi}(t,x)$ and $\mathring{\bar{\chi}}(t,x)$ are UV finite.

Universal formula for EMT

- In this way, (Makino, H.S., arXiv:1403.4772)

$$T_{\mu\nu}(x) = \lim_{t\to 0}\Big\{c_1(t)\left[\tilde{\mathcal{O}}_{1,\mu\nu}(t,x) - \frac{1}{4}\tilde{\mathcal{O}}_{2,\mu\nu}(t,x)\right] + c_2(t)\tilde{\mathcal{O}}_{2,\mu\nu}(t,x)$$
$$+ c_3(t)\left[\tilde{\mathcal{O}}_{3,\mu\nu}(t,x) - 2\tilde{\mathcal{O}}_{4,\mu\nu}(t,x)\right]$$
$$+ c_4(t)\tilde{\mathcal{O}}_{4,\mu\nu}(t,x) + c_5(t)\tilde{\mathcal{O}}_{5,\mu\nu}(t,x) - \text{VEV}\Big\},$$

where, to the one-loop order ($T_F = Tn_f$)

$$c_1(t) = \frac{1}{g(\mu)^2} + \left[-\beta_0 L(\mu,t) - \frac{7}{3}C_A + \frac{3}{2}T_F\right]\frac{1}{(4\pi)^2},$$
$$c_2(t) = \frac{1}{4}\left(\frac{11}{6}C_A + \frac{11}{6}T_F\right)\frac{1}{(4\pi)^2},$$
$$c_3(t) = \frac{1}{4} + \left[\frac{1}{4}\left(\frac{3}{2} + \ln 432\right)C_F\right]\frac{g(\mu)^2}{(4\pi)^2},$$
$$c_4(t) = \frac{3}{4}C_F\frac{g(\mu)^2}{(4\pi)^2},$$
$$c_5(t) = -1 - \left[3L(\mu,t) + \frac{7}{2} + \ln 432\right]C_F\frac{g(\mu)^2}{(4\pi)^2},$$

where $\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F$ and $L(\mu,t) = \ln(2\mu^2 t) + \gamma_E$. We set $\mu \propto 1/\sqrt{t} \to \infty$.

Universal formula for EMT

- This is manifestly finite, as it should be for EMT!
- This is universal: $c_i(t)$ are independnt of the regularization. In the limit of infinite cutoff, the formula holds irrespective of the regularization.
- We have to first take the continuum limit $a \to 0$ and then the small flow time limit $t \to 0$.
- Practically, we cannot simply take $a \to 0$ and may take t as small as possible in the fiducial window,

$$a \ll \sqrt{8t} \ll \frac{1}{\Lambda}.$$

  The usefulness with presently-accessible lattice parameters is not obvious a priori. . .
- In the last year, $c_i(t)$ were obtained to the two-loop order! (Harlander, Kluth, Lange, arXiv:1808.09837)

First trial: Thermodynamics in the quenched QCD (FlowQCD Collaboration, arXiv:1312.7492)

- The finite temperature expectation value of the EMT, $T_{\mu\nu}(x)$.
- The entropy density as the traceless part:

$$\varepsilon + p = -\frac{4}{3}\left\langle T_{00}(x) - \frac{1}{4}T_{\mu\nu}(x)\right\rangle,$$

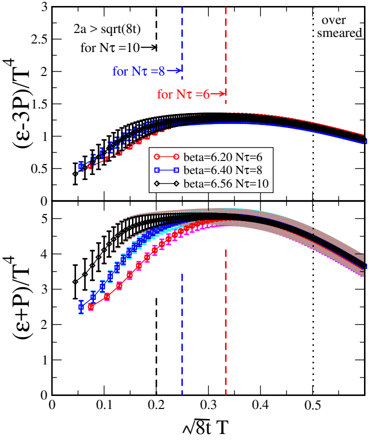
and the "trace anomaly" as the trace part:

$$\varepsilon - 3p = -\langle T_{\mu\mu}(x)\rangle.$$

- Considered $T = 0.99T_c$, $1.24T_c$, and $1.65T_c$ by $32^3 \times (6, 8, 10)$ lattices. 300 configurations for each temperature. 32^4 lattice for the vacuum.
- For the quenched QCD, the two-loop order coefficient for the trace part is available.

FlowQCD Collaboration, arXiv:1312.7492

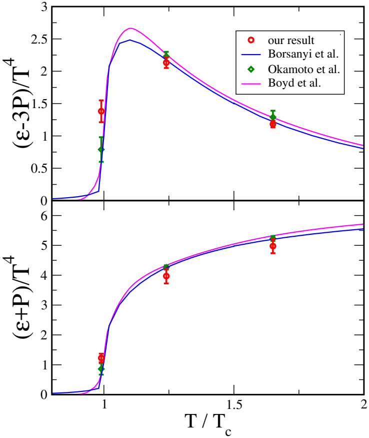
- Thermal expectation values as a function of the flow time $\sqrt{8t}$ for $T = 1.65T_c$:



- Stable behavior in the fiducial window, $2a < \sqrt{8t} < 1/(2T)$.

FlowQCD Collaboration, arXiv:1312.7492

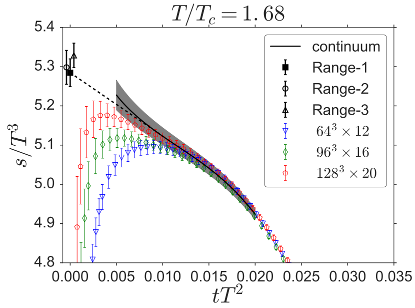
- In the continuum limit, from the values at $\sqrt{8t}T = 0.40$,

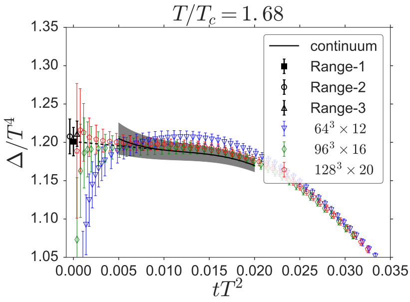


- Although the error bars were rather large, this encouraged us very much!

FlowQCD Collaboration, arXiv:1610.07810

- More systematic study: $a = 0.013$–0.061 fm, $N_s = 64$–128, $N_\tau = 12$–24, ~ 1000–2000 configurations:





- The gray band: the continuum limit at each flow time.

FlowQCD Collaboration, arXiv:1610.07810

- The double limit, $a \to 0$ first and then $t \to 0$ yields

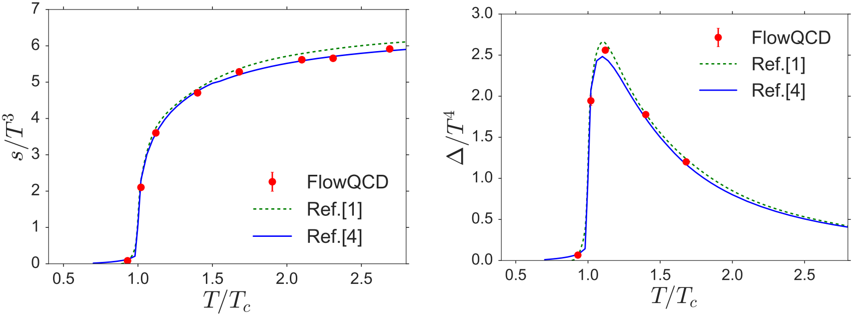
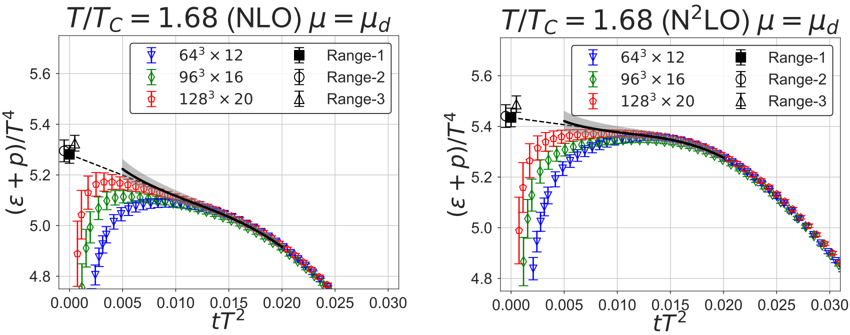


Figure: [1] Boyd, et al., hep-lat/9602007. [4] Borsanyi, et al., arXiv:1204.6184.

- It appears that no room for doubt.

More recently, Iritani, Kitazawa, H.S., Takaura, arXiv:1812.06444

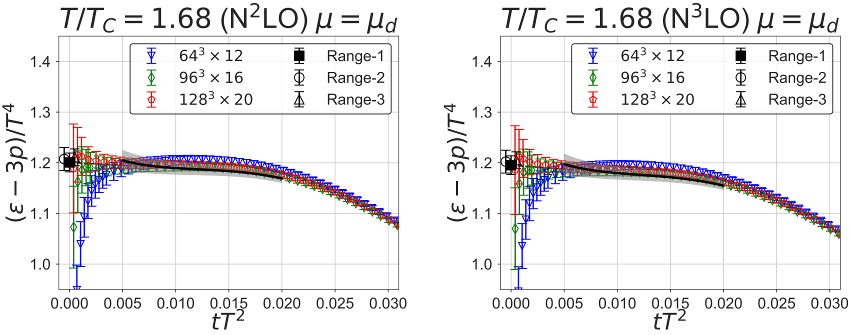
- Same lattice data as arXiv:1610.07810, but with the higher order coefficients! (Harlander, Kluth, Lange, arXiv:1808.09837).
- For the entropy density



- The higher order coefficient renders the behavior more stable $\Rightarrow$ Less sensitive to the method of the $t \to 0$ extrapolation.

Iritani, Kitazawa, H.S., Takaura, arXiv:1812.06444

- For the trace anomaly:



- The two-loop coefficient already gives a well-stable behavior.

Iritani, Kitazawa, H.S., Takaura, arXiv:1812.06444

- Already the field of a precise determination:

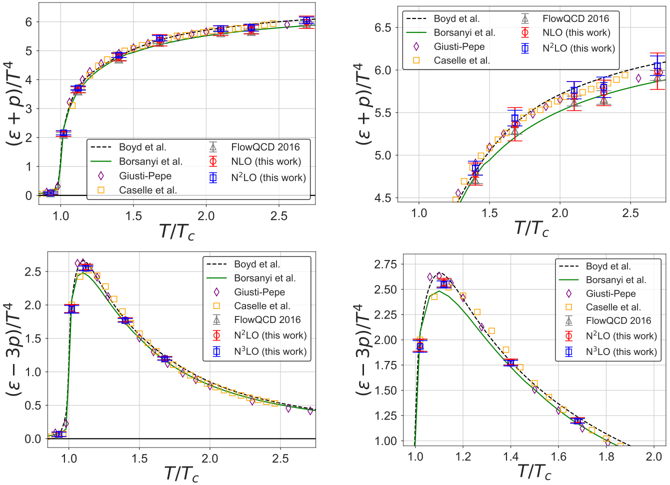


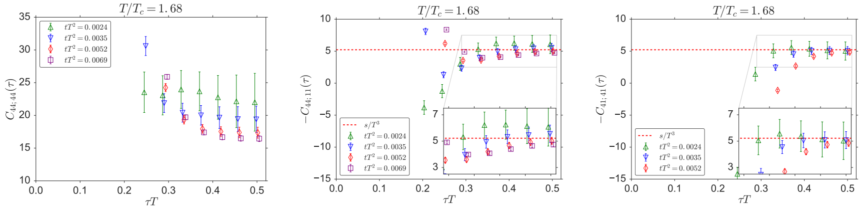
Figure: Boyd et al., Borsanyi et al.: Integral method, Giusti, Pepe: Moving frame method, Caselle et al.: Jarzynski's equality.

The two point functions (Kitazawa, Iritani, Asakawa, Hatsuda, arXiv:1708.01415)

- The connected part

$$C_{\mu\nu;\rho\sigma}(\tau) \equiv \frac{1}{T^5} \int_V d^3x \, \langle \delta T_{\mu\nu}(x) \delta T_{\rho\sigma}(0) \rangle \,,$$

where $\delta T_{\mu\nu}(x) \equiv T_{\mu\nu}(x) - \langle T_{\mu\nu}(x) \rangle$.



- Indicating the conservation law of the EMT, $\partial_\tau C_{\mu\nu;\rho\sigma}(\tau) = 0$!!!
- Confirms the linear response relations, s.t,

$$\frac{\varepsilon + p}{T^4} = \frac{1}{T^3} \frac{dp}{dT} = -C_{44,;11}(\tau).$$

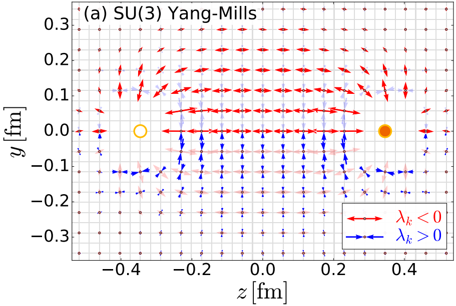
Stress tensor distribution around the static quark–anti-quark pair (Yanagihara, Iritani, Kitazawa, Asakawa, Hatsuda, arXiv:1803.05656)

- The EMT around the static quark–anti-quark pair:

$$\mathcal{T}_{\mu\nu}(x) \equiv \langle T_{\mu\nu}(x) \rangle_{Q\bar{Q}} = \lim_{T\to\infty} \frac{\langle T_{\mu\nu}(x) W(R,T) \rangle}{\langle W(R,T) \rangle}.$$

- Eigenvectors:

$$\mathcal{T}_{ij} n_j^{(k)} = \lambda_k n_j^{(k)}$$

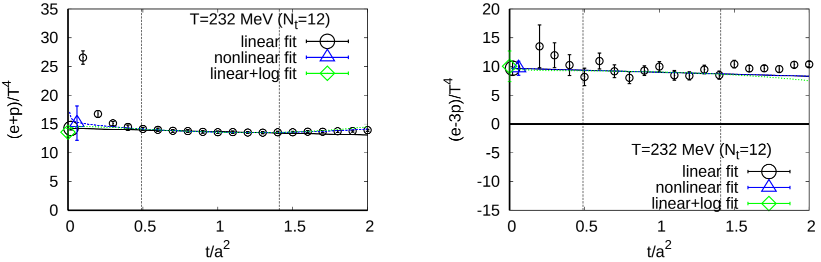


WHOT-QCD Collaboration: Baba, Ejiri, Iwami, Kanaya, Kitazawa, Shimojo, Shirogane, A. Suzuki, H.S., Taniguchi, Umeda

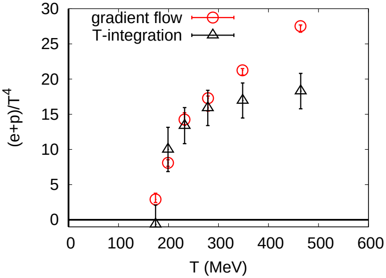
- For Full QCD?
- We are studying the $N_f = 2 + 1$ QCD by using the NP $O(a)$-improved Wilson quark action and the RG improved Iwasaki gauge action.
- Somewhat heavy ud quarks ($m_\pi / m_\rho \simeq 0.63$, $m_{\eta_{ss}} / m_\phi \simeq 0.74$)
  - $a = 0.0701(29)$ fm, $28^3 \times 56$ (JLQCD), $32^3 \times N_t$ ($N_t = 6, 8, \dots, 16$)
  - $a = 0.0970(26)$ fm, $32^3 \times 40$, $32^3 \times N_t$ ($N_t = 8, 10, 11, 12$)
  - [$a = 0.04976$ fm, $40^3 \times 80$]
- Aiming at the test of the methodology, the continuum limit.
- Physical mass quarks
  - $a = 0.08995(40)$ fm, $32^3 \times 64$ (PACS-CS), $32^3 \times N_t$ ($N_t = 4, 5, 6, \dots, 14, 16, [18]$)
- Physical prediction on the EoS etc. . .

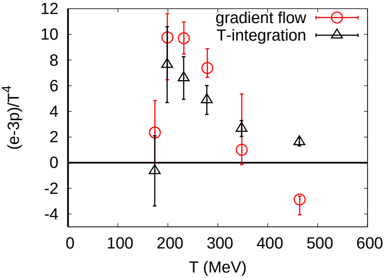
Somewhat heavy ud quarks, $a \simeq 0.07$ fm, arXiv:1609.01417

- Typical $t \to 0$ extrapolation ($N_t = 12$)



Somewhat heavy ud quarks, $a \simeq 0.07$ fm, arXiv:1609.01417

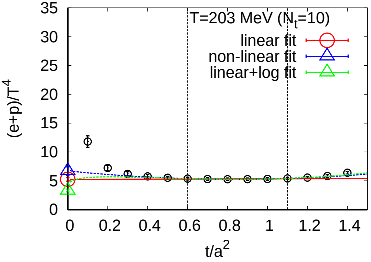


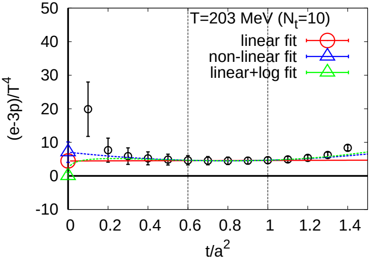


- Comparison to Umeda et al. (WHOT-QCD), arXiv:1202.4719.
- Indicating $a \simeq 0.07$ fm is fine enough for $T \lesssim 300$ MeV.
- Disagreement for $T \gtrsim 350$ MeV ($N_t \leq 8$) may be attributed to $O((aT)^2 = 1/N_t^2)$ error.
- It appears that the method is basically working.

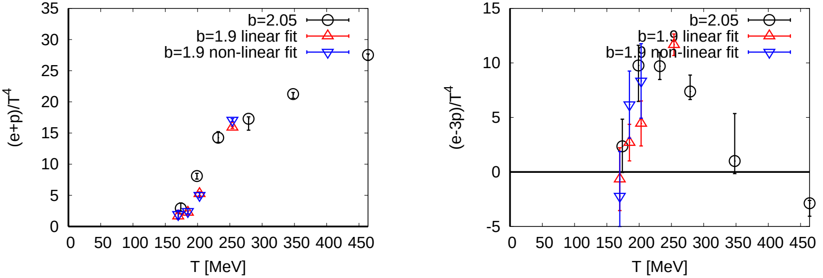
Somewhat heavy ud quarks, $a \simeq 0.097\,\text{fm}$ (Preliminary)

- Typical $t \to 0$ extrapolation ($N_t = 10$). The "linear region" becomes smaller, as expected.





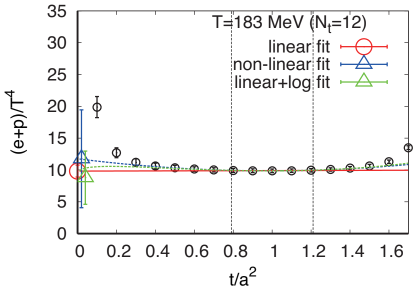
Somewhat heavy ud quarks, $a \simeq 0.07$ fm and $a \simeq 0.097$ fm (Preliminary)

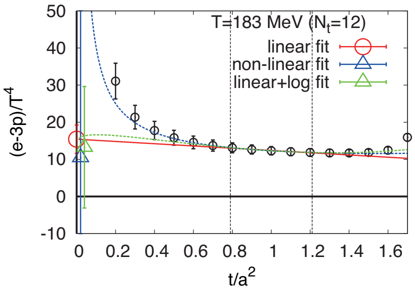


- It appears that the a dependence is fairly small.
- Systematic fit is ongoing

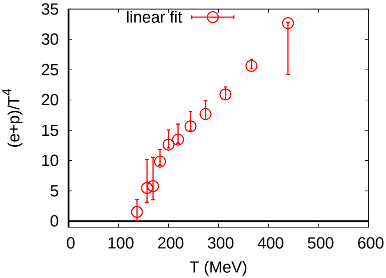
Physical mass ud, $a \simeq 0.09\,\mathrm{fm}$, arXiv:1710.10015, plus new $N_t = 16$ (Preliminary)

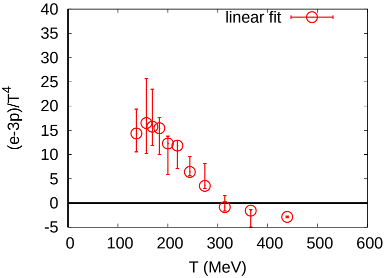
- Typical $t \to 0$ extrapolation ($N_t = 12$)





Physical mass ud, $a \simeq 0.09\,\mathrm{fm}$, arXiv:1710.10015, plus new $N_t = 16$ (Preliminary)



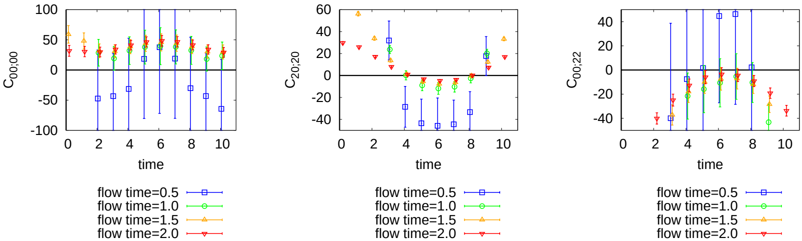


- Entropy seems to be consistent with that by the staggered fermion.
- Trace anomaly is much larger compared with the staggered.
- Increasing the statistics and a lower temperature are ongoing.
- Finer lattices, the continuum limit are future problem.

Two point functions, the somewhat heavy ud case, $a \simeq 0.07\,\text{fm}$, arXiv:1711.02262

- The connected part ($\delta T_{\mu\nu}(x) \equiv T_{\mu\nu}(x) - \langle T_{\mu\nu}(x) \rangle$):)

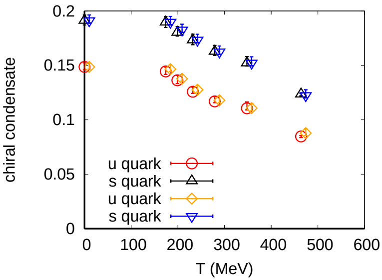
$$C_{\mu\nu;\rho\sigma}(\tau) \equiv \frac{1}{T^5} \int_V d^3x \, \langle \delta T_{\mu\nu}(x) \delta T_{\rho\sigma}(0) \rangle .$$

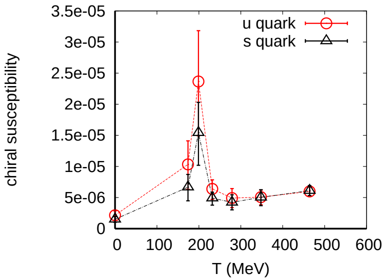


- Indicating the conservation law, restoration of the rotational symmetry, and the linear response relations.
- Shear viscosity from $C_{12;12}$, $\eta/s = 0.145(51)$ @ $T = 232\,\text{MeV}$ (Preliminary), (JPS meeting @ Shinshu).

Chiral condensate

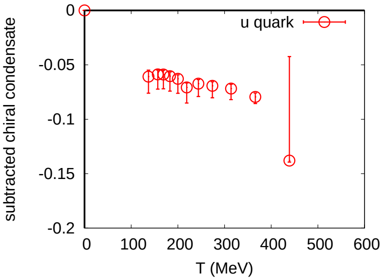
- Gradient flow can be employed also to construct the (renormalized) scalar operator to compute the chiral condensate and (disconnected) chiral susceptibility.
- For the somewhat heavy ud quarks, $a \simeq 0.07\,\text{fm}$, arXiv:1609.01417.
- $T_{pc} \simeq 190\,\text{MeV}$?

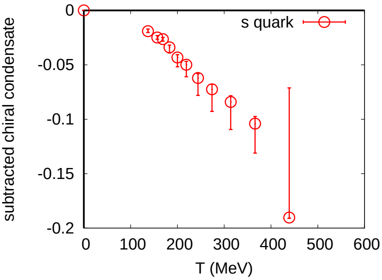




Chiral condensate

- For the physical mass ud, $a \simeq 0.09\,\mathrm{fm}$, arXiv:1710.10015, plus new $N_t = 16$ (Preliminary).
- VEV extracted chiral condensate.
- It appears that sharper for ud quarks





3D scalar theory (Morikawa, Sonoda, H.S., work in progress)

- 3D N-component scalar theory

$$S = \int d^D x \left[\frac{1}{2} \partial_\mu \phi^I \partial_\mu \phi^I + \frac{m_0^2}{2} \phi^I \phi^I + \frac{\lambda_0}{8N} \left(\phi^I \phi^I \right)^2 \right]$$

- The flow equation

$$\partial_t \varphi^I(t,x) = \partial_\mu \partial_\mu \varphi^I(t,x), \qquad \varphi^I(t=0,x) = \phi^I(x).$$

- A universal formula for EMT ($\mathcal{C} = 3.844365111074$):

$$T_{\mu\nu} = \partial_\mu \varphi^I \partial_\nu \varphi^I - \delta_{\mu\nu} \left[\frac{1}{2} \partial_\rho \varphi^I \partial_\rho \varphi^I + \frac{m^2}{2} \varphi^I \varphi^I + \frac{\lambda}{8N} \left(\varphi^I \varphi^I \right)^2 \right]$$
$$- \delta_{\mu\nu} \left(\frac{\lambda}{4\pi} \left(1 + \frac{2}{N} \right) \left(-\frac{1}{3} \right) (8\pi t)^{-1/2} \right.$$
$$\left. + \frac{\lambda^2}{(4\pi)^2} \left\{ \left(1 + \frac{2}{N} \right)^2 \left(-\frac{1}{4\pi} \right) + \frac{1}{N} \left(1 + \frac{2}{N} \right) \left(-\frac{1}{8} \right) \left[\ln(8\pi\mu^2 t) - \frac{1}{3} + \mathcal{C} \right] \right\} \right) \varphi^I \varphi^I .$$

3D scalar theory (Morikawa, Sonoda, H.S., work in progress)

- The theory around the Wilson–Fisher fixed point can be realized as the long-distance limit,

$$\langle \phi(x_1)\dots\phi(x_n)\rangle_{g_E} = \lim_{\tau\to\infty} e^{nx_h\tau} \langle \phi(e^\tau x_1)\dots\phi(e^\tau x_n)\rangle_{m^2,\lambda},$$

  where

$$m^2 = m_{\text{cr}}^2(\lambda) + g_E e^{-y_E\tau}.$$

  ($m^2 = m_{\text{cr}}^2(\lambda)$ is the critical line).
- The theory with $g_E = 0$ flows to a CFT in IR.
- It can be interesting to explore the GF fixed point and the critical exponents by using the universal formula.
- cf. in the large N limit,

$$x_h = \frac{1}{2}, \qquad y_E = 1,$$

  and

$$m_{\text{cr}}^2(\lambda) = 0.$$

Summary and prospects

- We wrote down a universal formula for the EMT in vector-like gauge theories by employing the gradient flow.
- The formula can be used in nonperturbative lattice simulations.
- The window problem:

$$a \ll \sqrt{8t} \ll \frac{1}{\Lambda}.$$

- Numerical experiments so far show encouraging results; the method appears usable practically.

Summary and prospects

- Asymptotic form in $t \to 0$? (work in progress).
- Push applications further: EoS of QCD, viscosities in gauge theory, momentum/spin structure of baryons, critical exponents in low-energy conformal field theory, dilaton physics, ...
- Further theoretical study, including the equal-point correction. The axial $U(1)_A$ anomaly in gravitational field is not automatically reproduced (Morikawa, H.S., arXiv:1803.04132),

$$\partial_\alpha^x \langle j_{5\alpha}(x) T_{\mu\nu}(y) T_{\rho\sigma}(z) \rangle$$
$$\neq \int_{p,q} e^{ip(x-y)} e^{iq(x-z)} \frac{1}{(4\pi)^2} \frac{1}{6} \epsilon_{\mu\rho\beta\gamma} p_\beta q_\gamma (q_\nu p_\sigma - \delta_{\nu\sigma} pq) + (\mu \leftrightarrow \nu, \rho \leftrightarrow \sigma),$$

  but requires a correction by a "local counterterm" $\propto \delta(x-y)\delta(x-z)$.
- Other Noether currents, such as the axial and super currents (partially already done).