SO(10) chiral gauge theory and the SM on the lattice with exact gauge invariance

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based on :
Y.K., arXiv:1710.11618[hep-lat], arXiv:1710.11101[hep-lat]

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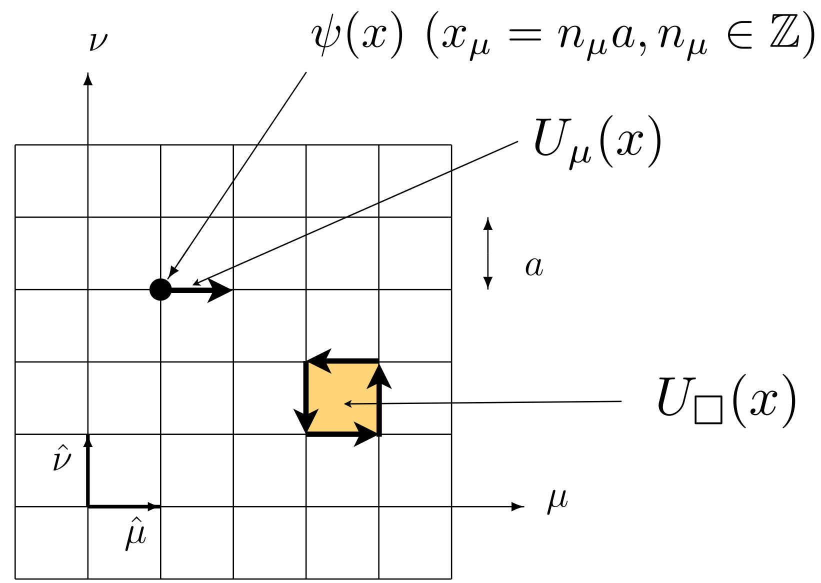
Plan

- the Standard Model / SO(10) chiral gauge theory on the lattice
- Schwinger-Keldysh formalism for lattice gauge theories
  real-time, non-equilibrium dynamics / finite-temperature・density
- Lefschetz-Thimble methods
  sign problem
  generalized method(GLTM), tempered method(tLTM)

Lattice Gauge Theory (QCD)



Action & Path Integral measure

$$S_G = \frac{1}{g^2}\sum_{x\mu\nu} \mathrm{ReTr}\left(1 - U_\Box(x)\right)$$

$$S_F = a^4 \sum_x \bar{\psi}(x)\, D\, \psi(x)$$

$$\mathcal{D}[U_\mu(x)] = \prod_{x,\mu} dU_\mu(x) \qquad \mathcal{D}[\psi]\mathcal{D}[\bar{\psi}] = \prod_x d\psi(x) d\bar{\psi}(x)$$

Gauge symmetry

$$\psi(x) \longrightarrow g(x)\psi(x) \qquad g(x) \in G$$

$$U_\mu(x) \to g(x)U_\mu(x)g^{-1}(x+\hat{\mu}) \qquad U_\mu(x) \in G$$

$$\nabla_\mu \psi(x) = \frac{1}{a}\left(U_\mu(x)\psi(x+\hat{\mu}a) - \psi(x)\right)$$

$$[\nabla_\mu, \nabla_\nu]\psi(x) = \Big(1 - U_\Box(x)\Big)U_\mu(x)U_\nu(x+\hat{\mu}a)\psi(x+\hat{\mu}a+\hat{\nu}a)$$

$$U_\Box(x) = U_\mu(x)U_\nu(x+\hat{\mu}a)U_\mu(x+\hat{\nu}a)^{-1}U_\nu(x)^{-1}$$

Chiral symmetry (NN theor., GWrel.)

$$\gamma_5 D + D\gamma_5 = 2aD\gamma_5 D$$

$$\delta_\alpha \psi(x) = i\alpha\, \gamma_5(1 - 2aD)\psi(x)$$

$$\delta_\alpha \bar{\psi}(x) = i\alpha\, \bar{\psi}(x)\gamma_5$$

$$D = \frac{1}{2a}\left(1 + X\frac{1}{\sqrt{X^\dagger X}}\right), \qquad X = aD_{\mathrm{w}} - m_0$$

$$D_{\mathrm{w}} = \sum_{\mu=1}^{4}\left\{\gamma_\mu \frac{1}{2}\left(\nabla_\mu - \nabla_\mu^\dagger\right) + \frac{a}{2}\nabla_\mu \nabla_\mu^\dagger\right\}$$

the Standard Model / SO(10)

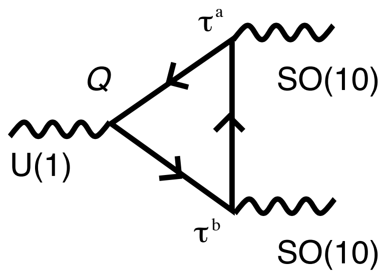
SU(3)xSU(2)xU(1) xU(1)$_{B-L}$

(<u>3</u> ,<u>2</u>) 1/6 **(<u>1</u>,<u>2</u>)** -1/2

(<u>3*</u>,<u>1</u>) -2/3 **(3*,1)** 1/3 **(<u>1</u>,<u>1</u>)** 1 **(<u>1</u>,<u>1</u>)** 0

SO(10)

<u>16</u>



Chiral Anomaly
=> Zero modes
=> 't Hooft vertex VEV
=> Fermion # violation

<u>16</u> x <u>16</u> x <u>16</u> x <u>16</u> => <u>10</u> x <u>10</u> => <u>1</u>

$$\mathrm{T}^a = \mathrm{C}\Gamma^a \quad \mathrm{T}^{aT} = \mathrm{T}^a$$

$$V_-^a(x) = \psi_-(x)^{\mathrm{T}} i\gamma_5 C_D \mathrm{T}^a \psi_-(x)$$

$$\bar{V}_-^a(x) = \bar{\psi}_-(x) i\gamma_5 C_D \mathrm{T}^{a\dagger} \bar{\psi}_-(x)^{\mathrm{T}}$$

$$\boxed{T_-(x) = \frac{1}{2} V_-^a(x) V_-^a(x)} \qquad \boxed{\bar{T}_-(x) = \frac{1}{2} \bar{V}_-^a(x) \bar{V}_-^a(x)}$$

Action & Path Integral measure for the <u>16</u>

[YK 2017]

Exact gauge-inv. & CP-inv.
Correct fermion # symm. breaking

$$\gamma_5 D + D\gamma_5 = 2aD\gamma_5 D$$

$$\hat{\gamma}_5 = \gamma_5(1 - 2aD) \qquad \hat{\gamma}_5{}^2 = \mathbb{I}$$

$$\hat{P}_\pm = \left(\frac{1 \pm \hat{\gamma}_5}{2}\right), \qquad P_\pm = \left(\frac{1 \pm \gamma_5}{2}\right)$$

$$\psi_-(x) = \hat{P}_- \psi(x) \qquad \bar{\psi}_-(x) = \bar{\psi}(x) P_+$$

$$S_W = a^4 \sum_x \bar{\underline{\psi}}(x)\, D\, \underline{\psi}(x)$$

$$\mathcal{D}[\underline{\psi}]\mathcal{D}[\bar{\underline{\psi}}] = \prod_x d\psi(x) d\bar{\psi}(x) \;\times$$

$$\prod_{x\in\Lambda} F(T_+(x)) \prod_{x\in\Lambda} F(\bar{T}_+(x))$$

SU(3)xSU(2)xU(1), <u>16</u> x3, Higgs & Yukawa int.

$$S_Y = \sum_x \Big[y_u\, \bar{q}_-^i(x)\tilde{\phi}(x)u_+^i(x) + y_u^*\, \bar{u}_+^i(x)\tilde{\phi}(x)^\dagger q_-^i(x)$$
$$+ y_d\, \bar{q}_-^i(x)\phi(x)d_+^i(x) + y_d^*\, \bar{d}_+^i(x)\phi(x)^\dagger q_-^i(x)$$
$$+ y_l\, \bar{l}_-(x)\phi(x)e_+(x) + y_l^*\, \bar{e}_+(x)\phi(x)^\dagger l_-(x) \Big]$$

The saturation of lattice fermion measures due to 't Hooft vertices

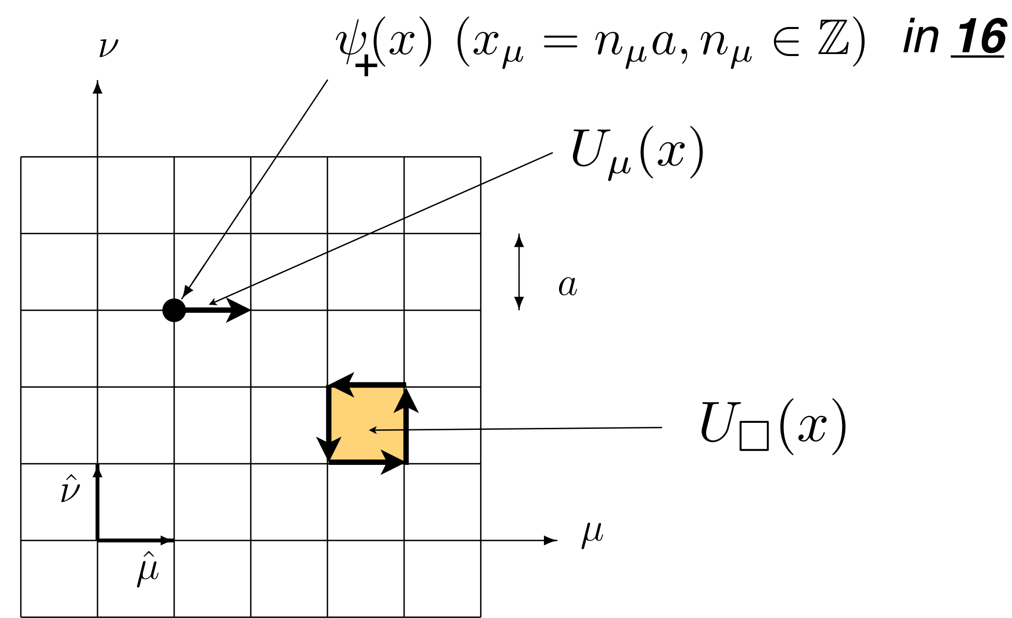
for the naive Weyl fermion with species doublers [*Eichten-Preskill(1986)*]

$\psi_+(x) = P_+\psi(x) \quad \bar\psi_+(x) = \bar\psi(x)P_-$ *in* ***16***

$$\int \prod_{x\in\Lambda}\prod_{\alpha=1}^{2}\prod_{s=1}^{16} d\psi_{\alpha s}(x) \prod_{x\in\Lambda}\frac{4!}{8!12!}\left\{\frac{1}{2}\,\psi(x)^{\mathrm{T}}P_+ i\gamma_5 C_D \mathrm{T}^a\psi(x) \quad \psi(x)^{\mathrm{T}}P_+ i\gamma_5 C_D \mathrm{T}^a\psi(x)\right\}^8 = 1$$

$$\int \prod_{x\in\Lambda}\prod_{\alpha=3}^{4}\prod_{s=1}^{16} d\bar\psi_{\alpha s}(x) \prod_{x\in\Lambda}\frac{4!}{8!12!}\left\{\frac{1}{2}\,\bar\psi(x)P_- i\gamma_5 C_D \mathrm{T}^a\bar\psi(x)^{\mathrm{T}}\,\bar\psi(x)P_- i\gamma_5 C_D \mathrm{T}^a\bar\psi(x)^{\mathrm{T}}\right\}^8 = 1$$

32-components at a site !



for the **Overlap Weyl fermion** / the Ginsparg-Wilson relation it holds true, too !!

$$\psi_+(x) = \hat P_+\psi(x) \quad \bar\psi_+(x) = \bar\psi(x)P_-$$

$$\hat P_\pm = \left(\frac{1\pm\hat\gamma_5}{2}\right), \quad P_\pm = \left(\frac{1\pm\gamma_5}{2}\right)$$

A gauge invariant path-integral measure for the overlap Weyl fermions in 16 of SO(10)

YK (2017)

$$\mathcal{D}[\psi_-]\mathcal{D}[\bar\psi_-] \equiv \mathcal{D}[\psi]\mathcal{D}[\bar\psi] \prod_{x\in\Lambda} F(T_+(x)) \prod_{x\in\Lambda} F(\bar T_+(x))$$

$$\mathcal{D}[\psi]\mathcal{D}[\bar\psi] \equiv \prod_{x\in\Lambda}\prod_{\alpha=1}^{4}\prod_{s=1}^{16} d\psi_{\alpha s}(x) \prod_{x\in\Lambda}\prod_{\alpha=1}^{4}\prod_{s=1}^{16} d\bar\psi_{\alpha s}(x)$$

$$\psi_+(x) = \hat P_+ \psi(x) \quad \bar\psi(x)_+ = \bar\psi(x) P_-$$

$$T_+(x) = \frac{1}{2} V_+^a(x) V_+^a(x), \quad V_+^a(x) = \psi_+(x)^{\mathrm{T}} i\gamma_5 C_D \mathrm{T}^a \psi_+(x) \qquad \mathrm{T}^a = \mathrm{C}\Gamma^a$$

$$\bar T_+(x) = \frac{1}{2} \bar V_+^a(x) \bar V_+^a(x), \quad \bar V_+^a(x) = \bar\psi_+(x) i\gamma_5 C_D \mathrm{T}^a \bar\psi_+(x)^{\mathrm{T}} \qquad \mathrm{T}^{aT} = \mathrm{T}^a$$

cf. $\boxed{\hat P_+^T i\gamma_5 C_D P_+ \mathrm{T}^a E^a(x) \hat P_+ = (1-D)^T i\gamma_5 C_D P_+ \mathrm{T}^a E^a(x)(1-D)}$

$$F(w) \equiv 4!\,(z/2)^{-4} I_4(z)\Big|_{(z/2)^2=w} = 4! \sum_{k=0}^{\infty} \frac{w^k}{k!(k+4)!}$$

$$F(w)\Big|_{w=(1/2)u^a u^a} = (\pi^5/12)^{-1} \int \prod_{a=1}^{10} de^a \delta(\sqrt{e^b e^b} - 1)\, \mathrm{e}^{e^c u^c}$$

$$Z \equiv \int \mathcal{D}[U]\, \mathrm{e}^{-S_G[U]+\Gamma_W[U]}$$

$$\mathcal{D}[E] = \prod_{x\in\Lambda} (\pi^5/12)^{-1} \prod_{a=1}^{10} dE^a(x)\delta(\sqrt{E^b(x)E^b(x)}-1)$$

$$\mathcal{D}[\bar{E}] = \prod_{x\in\Lambda} (\pi^5/12)^{-1} \prod_{a=1}^{10} d\bar{E}^a(x)\delta(\sqrt{\bar{E}^b(x)\bar{E}^b(x)}-1)$$

$$\begin{aligned} \mathrm{e}^{\Gamma_W[U]} &\equiv \int \mathcal{D}[\psi_-]\mathcal{D}[\bar{\psi}_-]\, \mathrm{e}^{-S_W[\psi_-,\bar{\psi}_-]} \\ &= \int \mathcal{D}[\psi]\mathcal{D}[\bar{\psi}] \prod_{x\in\Lambda} F(T_+(x)) \prod_{x\in\Lambda} F(\bar{T}_+(x))\, \mathrm{e}^{-S_W[\psi_-,\bar{\psi}_-]} \\ &= \int \mathcal{D}[\psi]\mathcal{D}[\bar{\psi}]\, \mathcal{D}[E]\mathcal{D}[\bar{E}]\; \mathrm{e}^{-S_W[\psi_-,\bar{\psi}_-]+\sum_{x\in\Lambda}\{E^a(x)V_+^a(x)+\bar{E}^a(x)\bar{V}_+^a(x)\}[\psi_+,\bar{\psi}_+]} \end{aligned}$$

$$\frac{1}{2}\sum_{x\in\Lambda} \begin{pmatrix} \psi^T & \bar{\psi} \end{pmatrix}(x) \begin{pmatrix} -\hat{P}_+^T i\gamma_5 C_D \mathrm{T}^a E^a \hat{P}_+ & -\hat{P}_-^T D^T P_+^T \\ P_+ D \hat{P}_- & -P_- i\gamma_5 C_D \mathrm{T}^{a\dagger} \bar{E}^a P_-^T \end{pmatrix} \begin{pmatrix} \psi \\ \bar{\psi}^T \end{pmatrix}(x)$$

$$\psi_-(x) = \sum_j v_j(x) c_j, \quad \bar{\psi}_-(x) = \sum_k \bar{c}_k \bar{v}_k(x)$$

$$\psi_+(x) = \sum_j u_j(x) b_j, \quad \bar{\psi}_+(x) = \sum_k \bar{b}_k \bar{u}_k(x)$$

$$\sum_j (u_j, \delta_\eta u_j) + \sum_j (v_j, \delta_\eta v_j) = 0$$

$$\begin{aligned} &\mathrm{pf}\begin{pmatrix} -\hat{P}_+^T i\gamma_5 C_D \mathrm{T}^a E^a \hat{P}_+ & -\hat{P}_-^T D^T P_+^T \\ P_+ D \hat{P}_- & -P_- i\gamma_5 C_D \mathrm{T}^{a\dagger} \bar{E}^a P_-^T \end{pmatrix} \\ &= \mathrm{pf}\begin{pmatrix} -(u^T i\gamma_5 C_D \mathrm{T}^a E^a u) & 0 & 0 & 0 \\ 0 & 0 & 0 & -(v^T D^T \bar{v}^T) \\ 0 & 0 & -(\bar{u}\, i\gamma_5 C_D \mathrm{T}^{a\dagger} \bar{E}^a \bar{u}^T) & 0 \\ 0 & (\bar{v} D v) & 0 & 0 \end{pmatrix} \\ &= \det(\bar{v} D v) \times \int \mathcal{D}[\bar{E}]\, \mathrm{pf}(u^T i\gamma_5 C_D T^a E^a u) \int \mathcal{D}[\bar{E}]\, \mathrm{pf}(\bar{u}\, i\gamma_5 C_D \mathrm{T}^{a\dagger} \bar{E}^a \bar{u}^T) \end{aligned}$$

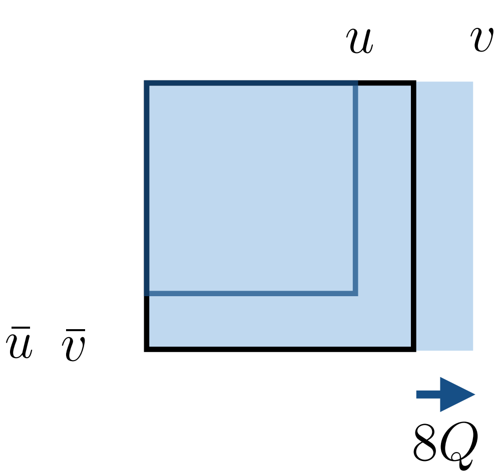
Chiral determinant and 't Hooft vertex pfaffians

$$\mathrm{e}^{\Gamma_W[U]} = \det(\bar v D v) \times \int \mathcal{D}[\bar E]\, \mathrm{pf}(u^T i\gamma_5 C_D T^a E^a u) \int \mathcal{D}[\bar E]\, \mathrm{pf}(\bar u\, i\gamma_5 C_D \mathrm{T}^{a\dagger} \bar E^a \bar u^T)$$

$(\bar v D v)_{ki}$ $\quad (k = 1, \cdots, n/2; i = 1, \cdots, n/2 + 8Q)$ **[variable, rectangular]**

$\left(u^{\mathrm{T}} i\gamma_5 C_D \mathrm{T}^a E^a u\right)_{ij}$ $\quad (i, j = 1, \cdots, n/2 - 8Q)$ **[variable, square]**

$\left(\bar u i\gamma_5 C_D \mathrm{T}^{a\dagger} \bar E^a \bar u^{\mathrm{T}}\right)_{kl}$ $\quad (k, l = 1, \cdots, n/2)$ **[fixed, square]**



—> Left-handed parts: correct behavior of `chiral determinant'

- overlap formula for the chiral determinant

  Zero modes *Narayanan-Neuberger(1997)*

  VEV of 't Hooft vertex >> Fermion # non-conservation

- gauge-invarinat formulation *Luscher (1999)*

  measure term (local counter terms) $-i\mathfrak{L}_\eta = \sum_j (v_j, \delta_\eta v_j)$

—> Right-handed parts : non-vanishing in all topological sec.

"Saturation of the right-handed part of fermion measure"

$$\sum_j (u_j, \delta_\eta u_j) + \sum_j (v_j, \delta_\eta v_j) = 0$$

Locality issue ?   the right-handed sector is a gapped system !

- **Schwinger-Dyson eq. for link field:** local operator insertions!

$$\Big\langle \Big[-\delta_\eta S_G[U] - \sum_{x\in\Lambda} \bar\psi(x) P_+ \delta_\eta D \psi(x) + 2 \sum_{x\in\Lambda} \psi^{\mathrm T} \hat P_+^T i\gamma_5 C_D \mathrm{T}^a E^a \delta_\eta \hat P_+ \psi(x) \Big] \Big\rangle = 0$$

- **Fermion two-point correlation functions:** short-range in the right-handed sector!

$$\big\langle \psi_-(x)\,\bar\psi_-(y) \big\rangle_F = \hat P_- D^{-1} P_+(x,y) \langle 1 \rangle_F, \qquad \mathbf{16_-} \quad \mathbf{16_-^*}$$

$$\Big\langle \psi_+(y) \left[\psi_+^{\mathrm T} i\gamma_5 C_D \mathrm{T}^a E^a \hat P_+(x) \right] \Big\rangle_F = -\frac{1}{2} \hat P_+(y,x) \langle 1 \rangle_F, \qquad \mathbf{16_+}$$

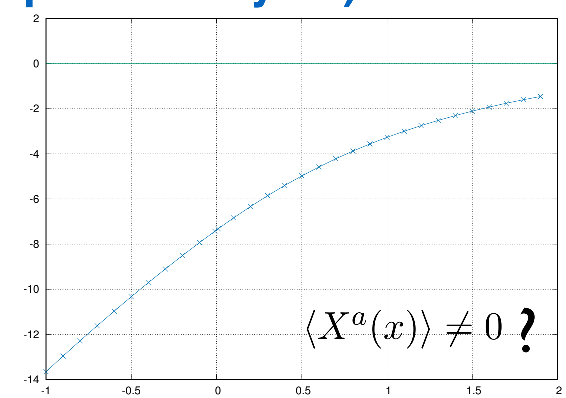
$$\Big\langle \left[P_- i\gamma_5 C_D \mathrm{T}^{a\dagger} \bar E^a \bar\psi_+^{\mathrm T}(x) \right] \bar\psi_+(y) \Big\rangle_F = -\frac{1}{2} P_- \delta_{xy} \langle 1 \rangle_F, \qquad \mathbf{16_+^*}$$

- **SO(10)-vector spin field dynamics:** disordered! (in a saddle point analysis)

$$\langle 1 \rangle_E = \int \mathcal{D}[E]\, \mathrm{pf}\big(u^{\mathrm T}\, i\gamma_5 C_D \mathrm{T}^a E^a u\big)$$

$$= \int \mathcal{D}[X]\mathcal{D}[\lambda]\, \mathrm{pf}\big(u^{\mathrm T}\, i\gamma_5 C_D \mathrm{T}^a X^a u\big)\, \mathrm{e}^{i \sum_x \lambda(x)(X^a(x)X^a(x)-1)}$$

$$f(m_0) \equiv 1 - \frac{9}{32} \frac{1}{V} \sum_{k\neq 0} \frac{4}{-\tilde D(k) + 2} \qquad \leq \ 0 \ \text{ for } \ m_0 \ < \ 2$$



- **Link-field dependence of Effective action:** should be local in the right-handed sector

$$\delta_\eta \Gamma_W[U] = \mathrm{Tr}\{P_+ \delta_\eta D D^{-1}\} - i\mathfrak{T}_\eta$$

cf. measure term
(local counter terms)

$$-i\mathfrak{T}_\eta \equiv -\mathrm{Tr}\big\{\delta_\eta \hat P_+ \big\langle \psi_+ \big[\psi_+^{\mathrm T} i\gamma_5 C_D \mathrm{T}^a E^a\big] \big\rangle_F \big\} / \langle 1 \rangle_F$$

$$-i\mathfrak{L}_\eta = \sum_j (v_j, \delta_\eta v_j)$$

- Eichten-Preskill model (species doublers)
- Mirror Fermion model (mirror modes)

realized by the Overalp Fermions/GW rel.

[*Eichten-Preskill(1986)*]

[*Montvay(1987)*]

cf. [*Poppitz et al (2006)*]

$$S_{\mathrm{Ov/Mi}}[\psi,\bar\psi,X^a,\bar X^a] = \sum_{x\in\Lambda}\{\bar\psi_-(x)D\psi_-(x) + z_+\bar\psi_+(x)D\psi_+(x)\}$$
$$- \sum_{x\in\Lambda}\{y\,X^a(x)\psi_+^{\mathrm{T}}(x)i\gamma_5 C_D \mathrm{T}^a\psi_+(x) + \bar y\,\bar X^a(x)\bar\psi_+(x)i\gamma_5 C_D \mathrm{T}^{a\dagger}\bar\psi_+(x)^T\}$$
$$+ S_X[X^a]$$

cf. $\boxed{\hat P_+^T i\gamma_5 C_D P_+ \mathrm{T}^a E^a(x)\hat P_+ = (1-D)^T i\gamma_5 C_D P_+ \mathrm{T}^a E^a(x)(1-D)}$

Decoupling limit of the mirror (right-handed) Overlap Weyl fermions

$$y=\bar y, \quad \frac{z_+}{\sqrt{y\bar y}} \to 0,$$
$$v=\bar v = 1, \quad \lambda' = \bar\lambda' \to \infty$$
$$\kappa = \bar\kappa \to 0.$$

(PMS phase)

$$S_{\mathrm{Ov}}[\psi,\bar\psi,E^a,\bar E^a] = \sum_{x\in\Lambda}\bar\psi_-(x)D\psi_-(x)$$
$$- \sum_{x\in\Lambda}\{E^a(x)\psi_+^{\mathrm{T}}(x)i\gamma_5 C_D \mathrm{T}^a\psi_+(x) + \bar E^a(x)\bar\psi_+(x)i\gamma_5 C_D \mathrm{T}^{a\dagger}\bar\psi_+(x)^T\}$$

$$E^a(x)E^a(x) = 1 \qquad K_{\mathrm{hop}} = 0$$

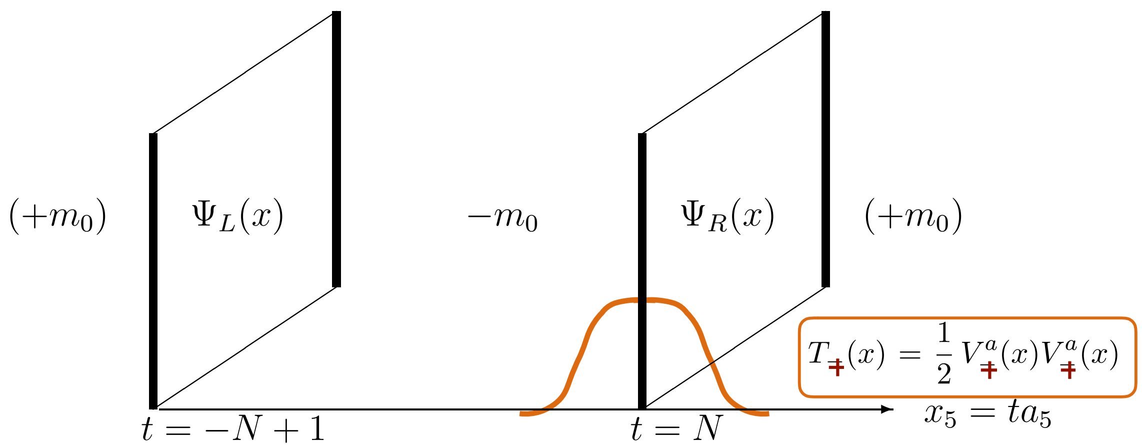
[*Kaplan(1992), Creutz et al (1997)*]

- 4+1 dim. DWF w/ boundary Eichten-Preskill term
- 4dim. TI/TSC w/ gapped boundary phase

[*Wen(2013), You-BenTov-Xu(2014), You-Xu (2015)*]

—> Low energy effective ***local lattice model*** through ***appropriate*** boundary terms

$$\boxed{\mathrm{e}^{\Gamma_W[U]} = \det(\bar{v}Dv) \times \int \mathcal{D}[\bar{E}]\,\mathrm{pf}(u^T i\gamma_5 C_D T^a E^a u) \int \mathcal{D}[\bar{E}]\,\mathrm{pf}(\bar{u}\, i\gamma_5 C_D \mathrm{T}^{a\dagger} \bar{E}^a \bar{u}^T)}$$



- 't Hooft vertices around the "right-handed wall"
  4+1dim. DWF class AII [Z] —> class DIII(BdG) [0]
- kinetic term z_+ —> 0 ($-m_0$ —> 0, 0 —> $+m_0$)
- $\pi_d(S^9) = 0$ (d=0, …,9)
  No topological obstructions/singularity
  No massless excitations around topol. singularity

[*Wen(2013), Furusaki et al (2015)*]

- 1dim. Majorana chain x 8
  Refinement of free fermion classification of TI/TSC due to interactions: $Z \to Z_8, Z_{16}$

[*Fidkowski-Kitaev (2010)*]

- Dai-Freed anomaly
  $\Omega^{spin}_5(BSpin(10)) = 0$,
  $\Omega_5(Spin(5)\times Spin(10)/Z_2) = Z_2$

[*Garcia-Etxebarria&Montero, Wang-Wen-Witten (2018)*]

The SM / SO(10) chiral lattice gauge theory with 16s in the framework of overlap fermion/the Ginsparg-Wilson rel.

- manifestly gauge-invariant by using full Dirac-field measure, but **saturating the right-handed part with 't Hooft vertices completely !**

- **all possible topological sectors**
- **zero modes, 't Hooft vertex VEV, fermion number non-conservation**
- **CP invariance** $\Gamma_W[U^{\mathrm{CP}}] = \Gamma_W[U]$
- **locality/smoothness** Issues

  Testable: To see if it works, examine $\langle \psi_+(y) \, [\psi_+^{\mathrm{T}} i\gamma_5 C_D \mathrm{T}^a E^a \hat{P}_-(x)] \, \rangle_F$

  MC studies in weak gauge-coupling limit feasible without sign problem

  Analytic studies desirable

- >>> SU(5), SU(4) x $SU(2)_L$ x $SU(2)_R$, **$SU(3)_c$ x $SU(2)_L$ x $U(1)_Y$ (+ ν_R)**
- Making the 't Hooft vertex terms **well-defined in large coupling limit**,

  Established the relations with GW Mirror-fermion model

  DW fermion with boundary EP terms

  4D TI/TSC with Gapped boundary phase explicitly

Eichten-Preskill model revisited

[*Eichten-Preskill(1986)*]

$$
\begin{aligned}
S_{\mathrm{EP}} \;=\; & \sum_x \Big\{ \bar{\psi}(x)\gamma_\mu P_- \frac{1}{2}(\nabla_\mu - \nabla_\mu^\dagger)\psi(x) \\
& -\frac{\lambda}{24}\left[\psi_-(x)^T i\gamma_5 C_D \mathrm{T}^a \psi_-(x)\right]^2 - \frac{\lambda}{24}\left[\bar{\psi}_-(x) i\gamma_5 C_D \mathrm{T}^a \bar{\psi}_-(x)^T\right]^2 \\
& -\frac{\lambda}{48}\triangle\left[\psi_-(x)^T i\gamma_5 C_D \mathrm{T}^a \psi_-(x)\right]^2 - \frac{\lambda}{48}\triangle\left[\bar{\psi}_-(x) i\gamma_5 C_D \mathrm{T}^a \bar{\psi}_-(x)^T\right]^2 \Big\}
\end{aligned}
$$

Explicit breaking of chiral symmetry by ’t Hooft vertices

generalized Wilson-term

$$
\begin{aligned}
& \Delta\{A(x)B(x)C(x)D(x)\} \\
& \quad \equiv +\frac{1}{2}\sum_\mu \Big\{ \big(\nabla_\mu\nabla_\mu^\dagger A(x)\big)B(x)C(x)D(x) + A(x)\big(\nabla_\mu\nabla_\mu^\dagger B(x)\big)C(x)D(x) \\
& \qquad\quad + A(x)B(x)\big(\nabla_\mu\nabla_\mu^\dagger C(x)\big)D(x) + A(x)B(x)C(x)\big(\nabla_\mu\nabla_\mu^\dagger D(x)\big) \Big\}.
\end{aligned}
$$

- resolve the degenerated physical and species-doubling modes
  { $(\underline{16})$- + $(\underline{16})$+ } x 8 —> light $(\underline{16})$- + heavy { $(\underline{16})$- x 7 + $(\underline{16})$+ x 8}
- fine-tune to the massless limit within a SO(10)-symmetric phase

Wilson-Yukawa-type model

[*Golterman-Petcher-Rivas(1986)*]

$$
\begin{aligned}
S_{\mathrm{EP/WY}} \;=\; \sum_x \Big\{ & \bar\psi(x)\gamma_\mu P_- \frac{1}{2}(\nabla_\mu - \nabla_\mu^\dagger)\psi(x) \\
& - y\big[\psi_-(x)^T i\gamma_5 C_D \mathrm{T}^a \psi_-(x) + \bar\psi_-(x) i\gamma_5 C_D \mathrm{T}^a \bar\psi_-(x)^T\big] E^a(x) \\
& - w\triangle\big[\psi_-(x)^T i\gamma_5 C_D \mathrm{T}^a \psi_-(x) + \bar\psi_-(x) i\gamma_5 C_D \mathrm{T}^a \bar\psi_-(x)^T\big] E^a(x) \Big\}
\end{aligned}
$$

Mirror-Fermion-type model

$$
\begin{aligned}
S_{\mathrm{EP/Mi}} \;=\; \sum_x \Big\{ & \bar\psi(x)\big[\gamma_\mu(z_- P_- + z_+ P_+)\frac{1}{2}(\nabla_\mu - \nabla_\mu^\dagger)\psi(x) + \frac{w}{2}\nabla_\mu\nabla_\mu^\dagger\big]\psi(x) \\
& - \lambda\big[\psi_+(x)^T i\gamma_5 C_D \mathrm{T}^a \psi_+(x) + \bar\psi_+(x) i\gamma_5 C_D \mathrm{T}^a \bar\psi_+(x)^T\big] E^a(x) \Big\}
\end{aligned}
$$

Integrate out the right-handed Mirror modes

$$
\begin{aligned}
S'_{\mathrm{EP/Mi}} \;=\; \sum_x \Big\{ & z_-\,\bar\psi(x)\gamma_\mu P_- \frac{1}{2}(\nabla_\mu - \nabla_\mu^\dagger)\psi(x) \\
& - \frac{w^2}{\lambda}\big[\triangle\psi_-(x)^T i\gamma_5 C_D \mathrm{T}^a \triangle\psi_-(x) + \triangle\bar\psi_-(x) i\gamma_5 C_D \mathrm{T}^a \triangle\bar\psi_-(x)^T\big] E^a(x) \Big\}
\end{aligned}
$$

$E^a(x)E^a(x) = 1 \qquad K_{\mathrm{hop}} = 0$

(not explored by *GPV*)

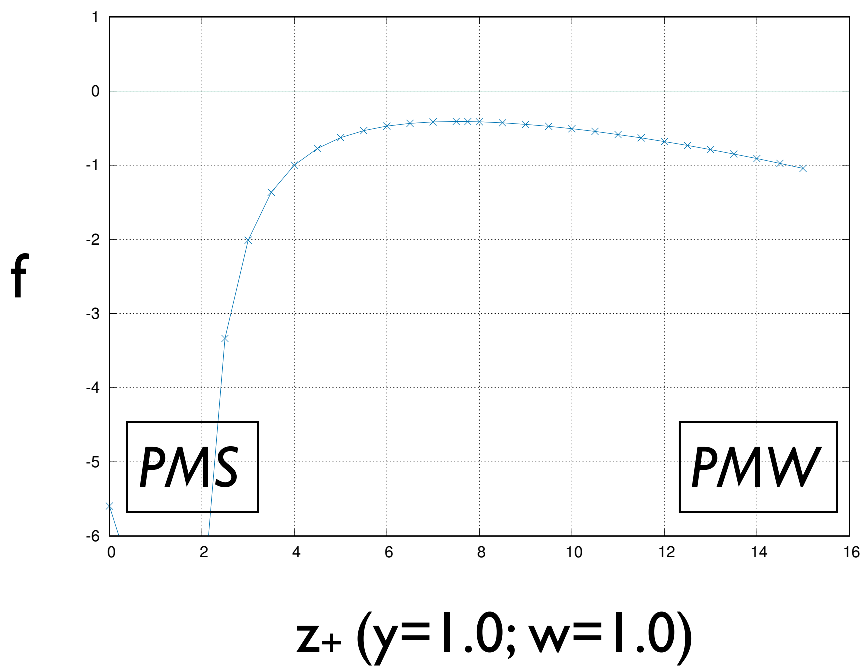
- **No SO(10) symmetry breaking !** cf. [*Golterman-Petcher-Rivas(1986)*]

$$E^a(x)E^a(x) = 1 \qquad K_{\rm hop} = 0$$

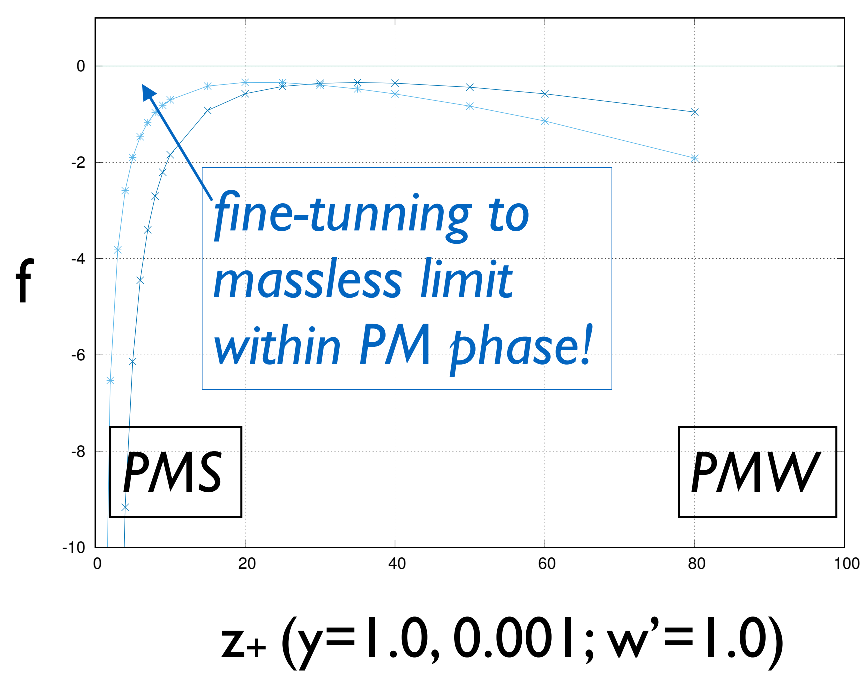
O(N)-vector model / saddle point analysis

$$\langle X^a(x)\rangle \neq 0 \quad \Rightarrow \quad f(m_0, z_+, y, w) \equiv 1 - \frac{9}{32}\frac{1}{V}\sum_{k\neq 0}\frac{4}{-\tilde{D}'(k\,;X_0) + \tilde{D}'(k^0\,;X_0)}\Big|_{X_0^c X_0^c = 1} > 0$$

Wilson-Yukawa-type model



Mirror-Fermion-type model



- **Correct # of zero (light) modes**

$$\det C\big|_{\rm EP/Mi} = \int \mathcal{D}[E]\,{\rm pf}\begin{pmatrix} -2w'^2\sigma_2\triangle\cdot T^a E^a\cdot\triangle & z_-\sigma_\mu^T\bar{\nabla}_\mu \\ z_-\sigma_\mu\bar{\nabla}_\mu & -2w'^2\sigma_2\triangle\cdot T^{a\dagger}E^a\cdot\triangle \end{pmatrix} \underset{(z_-\to 0)}{\simeq} \det(w'^2\triangle^2)^{32}$$

applications of lattice Standard Model/ SO(10) CGTs

1) Phase transitions, Phase structures in EW theory & GUT theories
2) Realizations of gauge and flavors symmetries in EW theory & GUT theories
3) Baryon & Lepton numbers generations
   a. B symmetry violation/chiral anomaly, CP violation, non-equilibrium process
   b. Chern# diffusion process, Spharelon process
4) Phase transitions in the early Universe, Dynamics of Inflation

and so on

- Schwinger-Keldysh formalism for lattice gauge theories

  real-time, non-equilibrium dynamics / finite-temperature・density

- Lefschetz-Thimble methods

  sign problem

  generalized method(GLTM), tempered method(tLTM)

Schwinger-Keldysh formalism for LFT

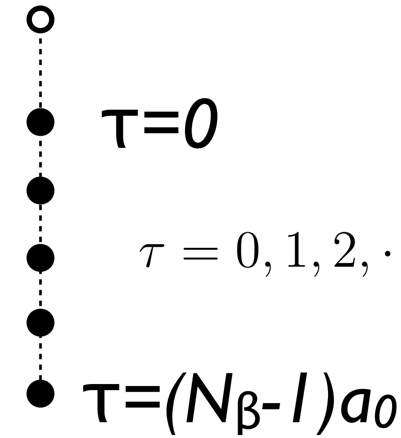
$$\langle \hat{O}(\tau) \rangle_{\beta,\mu} \equiv \mathrm{Tr}\left[\{\hat{T}_{+1}\}^{\beta-\tau} \hat{O} \{\hat{T}_{+1}\}^{\tau} \right] / Z(\beta,\mu)$$

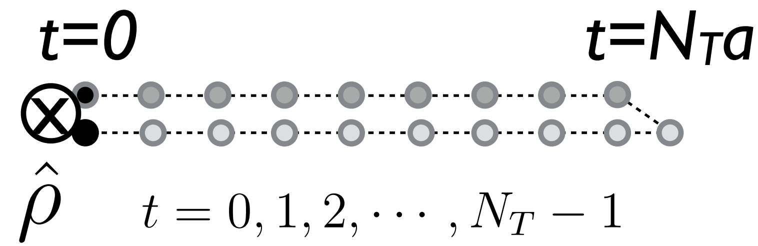
$$\hat{T}_{+1} = \mathrm{e}^{-a_0 \hat{H}} \quad \left(= \mathrm{e}^{-a_0 \hat{V}/2} \mathrm{e}^{-a_0 \hat{\Pi}^2/2} \mathrm{e}^{-a_0 \hat{V}/2} \right)$$

$$\Longrightarrow \quad \hat{T}_{\pm i} = \mathrm{e}^{\mp i a_0 \hat{H}}$$

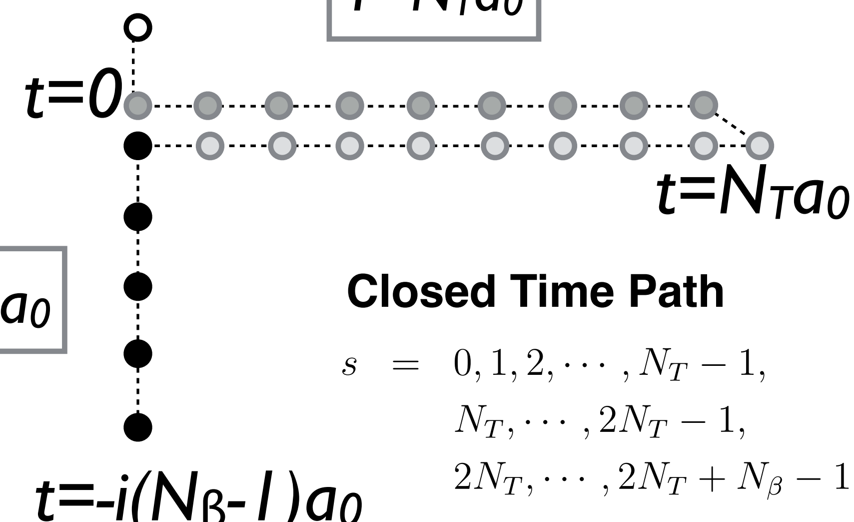
$$\langle \hat{O}(t) \rangle \equiv \mathrm{Tr}\left[\hat{\rho} \{\hat{T}_{-i}\}^{T} \{\hat{T}_{+i}\}^{T-t} \hat{O} \{\hat{T}_{+i}\}^{t} \right] / Z$$

$$\langle \hat{O}(t) \rangle \equiv \mathrm{Tr}\left[\{\hat{T}_{+1}\}^{\beta} \{\hat{T}_{-i}\}^{T} \{\hat{T}_{+i}\}^{T-t} \hat{O} \{\hat{T}_{+i}\}^{t} \right] / Z$$





$T = N_T a_0$



$\beta = N_\beta a_0$

Closed Time Path

$$\begin{aligned} s \;=\; & 0, 1, 2, \cdots, N_T - 1, \\ & N_T, \cdots, 2N_T - 1, \\ & 2N_T, \cdots, 2N_T + N_\beta - 1 \end{aligned}$$

$$\Delta_s = \begin{cases} +i + \epsilon \\ -i + \epsilon \\ +1 \end{cases}$$

Construct Transfer matrixes for Scalar, Link and Wilson fermion fields so that

$$\hat{T}_{-i}\hat{T}_{+i} = \boldsymbol{I}, \quad \hat{T}_{\pm i} = \hat{A}\, \hat{U}_{\pm i}\, \hat{A}^{-1}$$
$$\hat{T}_{+1}\hat{T}_{\pm i} - \hat{T}_{\pm i}\hat{T}_{+1} \neq 0$$

* to recover in the continuum limit

[*Alexandru et al.(2016)*] [*Fujii, Hoshina & YK*]

Schwinger-Keldysh formalism for Lattice Gauge Theory(QCD)

[Fujii, Hoshina & YK]

$$U_0(x,s)=1\,(s=0,\cdots,2N_T+N_\beta-2),\;\; U_0(x,2N_T+N_\beta-1)=P(x)$$

$$S_G^{[\Delta]}=\sum_{s,x}\left\{\sum_k K_{\Delta_s}\left[U_k(s){U_k}^\dagger(s-1)\right]+\sum_{kl}\frac{\Delta_s+\Delta_{s-1}}{2g^2}\mathrm{ReTr}(1-U_{kl})(s)\right\}$$

$$\mathrm{e}^{-K_\Delta[UU'^\dagger]}=\begin{cases}\mathrm{e}^{-\frac{1}{g^2}\mathrm{Tr}\left\{2-UU'^\dagger-U'U^\dagger\right\}} & (\Delta=+1)\\ \mathrm{e}^{-\frac{2N_c}{g^2\Delta}}\sum_r d_r\left\{L_r\left(2N_c/g^2\right)\right\}^\Delta \mathrm{Tr}_r\{UU'^\dagger\} & (\Delta=\pm i)\end{cases}$$

$$\begin{aligned}S_W^{[\Delta]} &= \sum_{s,s',x}\bar\psi(s,x)\left\{-\left(\frac{1-\gamma_0}{2}\right)\delta_{s+1,s'}\,\mathrm{e}^{-\mu}-\left(\frac{1+\gamma_0}{2}\right)\delta_{s,s'+1}\,\mathrm{e}^{+\mu}+\delta_{ss'}\right.\\ &\left.\qquad +a_0\left[\gamma_k\frac{1}{2}\left(\nabla_k-\nabla_k^\dagger\right)+\frac{1}{2}\nabla_k\nabla_k^\dagger+m\right]V_{s,s'}\right\}\psi(s',x)\end{aligned}$$

$$\begin{aligned}V_{s,s'} &= \frac{\Delta_s-1}{2}\left(\frac{1-\gamma_0}{2}\right)\delta_{s,s'-1}+\frac{\Delta_{s-1}-1}{2}\left(\frac{1+\gamma_0}{2}\right)\delta_{s,s'+1}\\ &\quad+\left[\frac{\Delta_s+1}{2}\left(\frac{1+\gamma_0}{2}\right)+\frac{\Delta_{s+1}-1}{2}\left(\frac{1-\gamma_0}{2}\right)\right]\delta_{s,s'}\end{aligned}$$

Closed Time Path

$$\begin{aligned}s &= 0,1,2,\cdots,N_T-1,\\ &\quad N_T,\cdots,2N_T-1,\\ &\quad 2N_T,\cdots,2N_T+N_\beta-1\end{aligned}$$

$$\Delta_s=\begin{cases}+i+\epsilon\\ -i+\epsilon\\ +1\end{cases}$$

- **Spectral function** $p_0 \in \left[-\frac{\pi}{a_0}, +\frac{\pi}{a_0}\right]$ $T \to \infty$

$$\boxed{\rho(p,\beta)_T \equiv \sum_{t=-T/2}^{T/2} \sum_{x} \mathrm{e}^{ip_0 t - i\boldsymbol{p}\boldsymbol{x}} \left\langle \left[\phi(t+T/2,\boldsymbol{x}), \phi(T/2,\boldsymbol{0})\right] \right\rangle_\beta}$$

- **Responce to time-dependent external source**

ex. EM field (U(1) link field) $V_\mu(t,\boldsymbol{x}) = \mathrm{e}^{ie_0 A_\mu(t,\boldsymbol{x})} \Longleftrightarrow J_\mu(t,\boldsymbol{x})$

conserved U(1) current of the lattice fermions

$$\langle \bar{J}_k(t,\boldsymbol{x})\rangle_\beta \,[V_0]$$

$$\langle \bar{J}_k(t,\boldsymbol{x})\rangle_\beta \,[V_0] \simeq \sum_{t'\boldsymbol{x}'} G_R(t,\boldsymbol{x};t',\boldsymbol{x}') A_0(t',\boldsymbol{x}')$$

$$\boxed{G_R(x,x') = i\theta(t-t') \left\langle \left[\bar{J}_k(t,\boldsymbol{x}), \bar{J}_0(t',\boldsymbol{x}')\right] \right\rangle_\beta}$$

- **Conductivity & Kubo's responce function**

$$\boxed{\sigma = \frac{1}{d} \sum_{t=t_0'}^{T} \sum_{s'=t_0'}^{-N_\beta - t_0'} \sum_{\boldsymbol{x}} \left\langle \left[\bar{J}_k(t,\boldsymbol{x}), \bar{J}_k(s',\boldsymbol{x}')\right] \right\rangle_\beta}$$

$$G^K_{kl}(x,x') = \frac{1}{N_\beta} \sum_{s'=t'}^{-N_\beta - t'} \sum_{\boldsymbol{x}} \left\langle \left[\bar{J}_k(t,\boldsymbol{x}), \bar{J}_l(s',\boldsymbol{x}')\right] \right\rangle_\beta$$

- **DWF**

$$D_{5\mathrm{w}}^{[\Delta]} = \left\{\gamma_0 \frac{1}{2}(\nabla_0 - \nabla_0^\dagger) + \frac{1}{2}\nabla_0^\dagger \nabla_0\right\} V^{-1} + D_{3w} - m_0 - \left(\frac{1-\gamma_5}{2}\right)\delta_{t_5+1,t_5'} - \left(\frac{1+\gamma_5}{2}\right)\delta_{t_5,t_5'+1}$$

- **Overlap Fermion**

$$D_{\mathrm{ov}}^{[\Delta]} = \frac{1}{2}\left(1 + X\frac{1}{\sqrt{\gamma_5 X \gamma_5 X}}\right)$$

$$X = \left\{\gamma_0 \frac{1}{2}(\nabla_0 - \nabla_0^\dagger) + \frac{1}{2}\nabla_0^\dagger \nabla_0\right\} V^{-1} + D_{3w} - m_0$$

- **Overlap Weyl Fermion**
  Schwinger-Kelydish formalism for the SM / SO(10)

Path-integral on Lefschetz thimbles

F. Pham (1983);
E. Witten, arXiv:1001.2933;
L. Scorzato et al.
Phys. Rev. D 86, 074506 (2012),
arXiv:1205.3996

$$Z = \int_{\mathcal{C}_{\mathbb{R}}} \mathcal{D}[x] \exp\{-S[x]\} = \int_{\mathcal{C}} \mathcal{D}[z] \exp\{-S[z]\}$$

$$x \in \mathcal{C}_{\mathbb{R}} \, (\subseteq \mathbb{R}^n) \longrightarrow x + iy = z \in \mathbb{C}^n$$

$$S[x] \to S[x+iy] = S[z] \qquad \left(\mathcal{D}[x] = d^n x \right)$$

the contour of path-integration is selected based on the result of Morse theory [*F. Pham (1983)*]

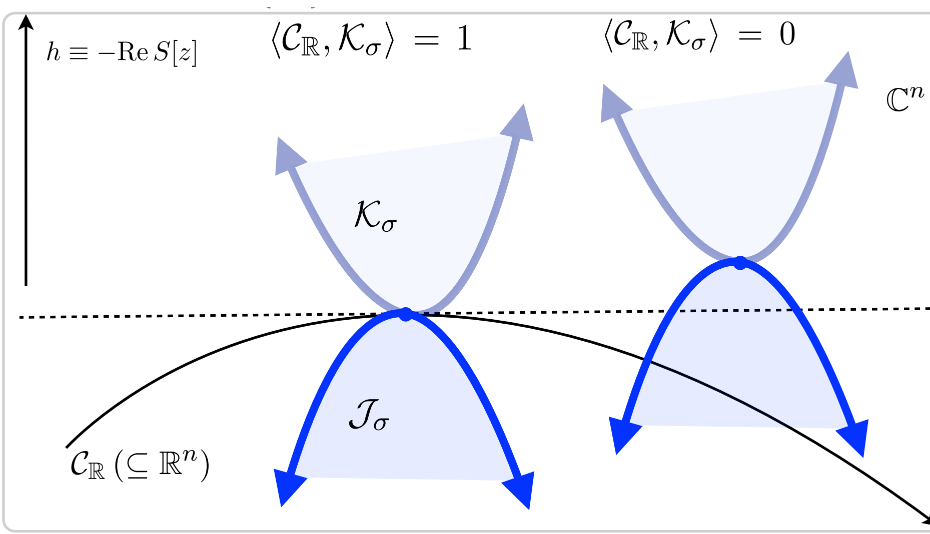
$$\mathcal{C}_{\mathbb{R}} = \sum_{\sigma \in \Sigma} n_\sigma \mathcal{J}_\sigma, \qquad n_\sigma = \langle \mathcal{C}_{\mathbb{R}}, \mathcal{K}_\sigma \rangle$$

$$h \equiv -\mathrm{Re}\, S[z]$$

$$\frac{d}{dt} z(t) = \frac{\partial \bar{S}[\bar{z}]}{\partial \bar{z}}, \qquad \frac{d}{dt} \bar{z}(t) = \frac{\partial S[z]}{\partial z}, \qquad t \in \mathbb{R}$$

critical points $\boldsymbol{z_\sigma}$ **:** $\left. \dfrac{\partial S[z]}{\partial z} \right|_{z=z_\sigma} = 0$

Lefschetz thimble $\mathcal{J}_\sigma$ $(\mathcal{K}_\sigma)$ *(n-dim. real mfd.)*
= the union of all down(up)ward flows which trace back to z_σ in the limit t goes to $-\infty$



$\langle \mathcal{J}_\sigma, \mathcal{K}_\tau \rangle = \delta_{\sigma\tau}$ (intersection numbers)

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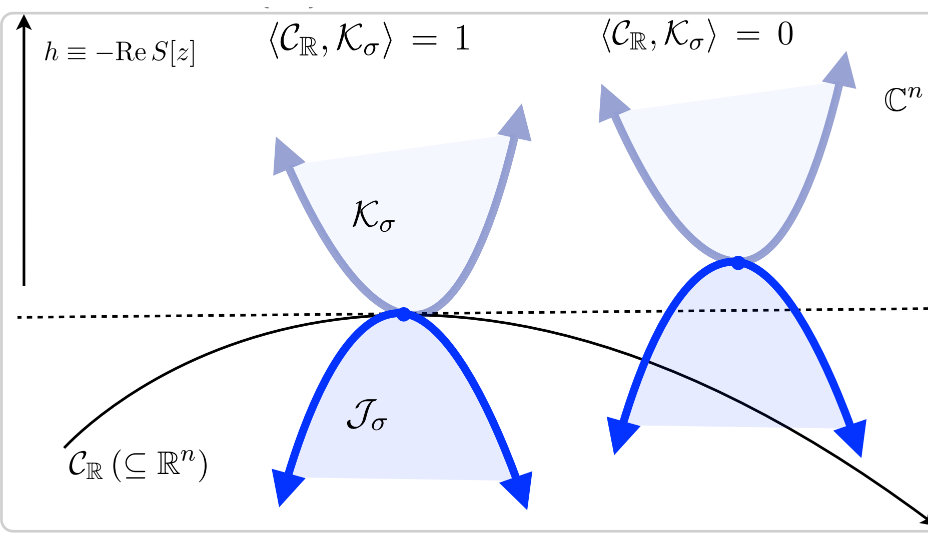
$$\frac{d}{dt} z(t) = \frac{\partial \bar{S}[\bar{z}]}{\partial \bar{z}}, \qquad \frac{d}{dt} \bar{z}(t) = \frac{\partial S[z]}{\partial z}, \qquad t \in \mathbb{R}$$

$$\frac{d}{dt} h = -\frac{1}{2}\left\{ \frac{\partial S[z]}{\partial z} \cdot \frac{d}{dt} z(t) + \frac{\partial \bar{S}[\bar{z}]}{\partial \bar{z}} \cdot \frac{d}{dt} \bar{z}(t) \right\} = -\left| \frac{\partial S[z]}{\partial z} \right|^2 \leq 0$$

$$\frac{d}{dt} \mathrm{Im}\, S[z] = \frac{1}{2i}\left\{ \frac{\partial S[z]}{\partial z} \cdot \frac{d}{dt} z(t) - \frac{\partial \bar{S}[\bar{z}]}{\partial \bar{z}} \cdot \frac{d}{dt} \bar{z}(t) \right\} = 0 \quad !$$



$\langle \mathcal{J}_\sigma, \mathcal{K}_\tau \rangle = \delta_{\sigma\tau}$ (intersection numbers)

Partition function

$$Z = \sum_{\sigma \in \Sigma} n_\sigma \exp\{-S[z_\sigma]\} \, Z_\sigma , \qquad n_\sigma = \langle \mathcal{C}_\mathbb{R}, \mathcal{K}_\sigma \rangle$$

$$Z_\sigma = \int_{\mathcal{J}_\sigma} \mathcal{D}[z] \exp\{-\mathrm{Re}\big(S[z] - S[z_\sigma]\big)\}$$

Observables

$$\langle O[z] \rangle = \frac{1}{Z} \sum_{\sigma \in \Sigma} n_\sigma \exp\{-S[z_\sigma]\} \, Z_\sigma \, \langle O[z] \rangle_{\mathcal{J}_\sigma}$$

$$\langle O[z] \rangle_{\mathcal{J}_\sigma} = \frac{1}{Z_\sigma} \int_{\mathcal{J}_\sigma} \mathcal{D}[z] \exp\{-\mathrm{Re}\big(S[z] - S[z_\sigma]\big)\} \, O[z]$$

Monte Carlo on Lefschetz Thimbles:

- no \`local' sign-problem, but huge numerical cost
- multiple Thimbles may constribute, then \`global' sign-problem may remains
- simple Thimble structure in SK formalism

generalized LTM:

- **GLTM(contraction algo.)**
- **tLTM (parallel tempering)**

[*Alexandru et al.(2016)*]

[*Fukuma & Umeda(2017)*]

one-site Hubbard, 0,1,2+1 massive Thirring, 1+1 massive Schwinger model

0,1,3+1 $\lambda\varphi^4{}_\mu$ model, 1+1 massless Schwinger model

Summary

- the Standard Model / SO(10) chiral gauge theory on the lattice
- Schwinger-Keldysh formalism for lattice gauge theories

  real-time, non-equilibrium dynamics / finite-temperature・density
- Lefschetz-Thimble methods !?

  to overcome the sign problem

  generalized method(GLTM), tempered method(tLTM)

What is the sound of one hand clapping?

両手の鳴る音は知る。
片手の鳴る音はいかに？
― 禅の公案 ―