Path optimization for the sign problem in low-dimensional QCD and QCD effective models at finite density

Akira Ohnishi [1], **Yuto Mori** [2], **Kouji Kashiwa** [3]

1. Yukawa Inst. for Theoretical Physics, Kyoto U.,
2. Dept. Phys., Kyoto U., 3. Fukuoka Inst. Tech.

YITP Molecule-type Workshop on
Frontiers in Lattice QCD and related topics
April 15-26, 2019, Kyoto, Japan.

Collaborators

Akira Ohnishi [1], Yuto Mori [2], Kouji Kashiwa [3]

1. Yukawa Inst. for Theoretical Physics, Kyoto U.,
2. Dept. Phys., Kyoto U., 3. Fukuoka Inst. Tech.

Y. Mori (PhD stu.)

K. Kashiwa

AO (11 yrs ago)

- *1D integral: Y. Mori, K. Kashiwa, AO, PRD 96 ('17), 111501(R) [arXiv:1705.05605]*
- *φ^4 w/ NN: Y. Mori, K. Kashiwa, AO, PTEP 2018 ('18), 023B04 [arXiv:1709.03208]*
- *Lat 2017: AO, Y. Mori, K. Kashiwa, EPJ Web Conf. 175 ('18), 07043 [arXiv:1712.01088]*
- *NJL thimble: Y. Mori, K. Kashiwa, AO, PLB 781('18),698 [arXiv:1705.03646]*
- *PNJL w/ NN: K. Kashiwa, Y. Mori, AO, PRD 99 ('19), 014033 [arXiv:1805.08940 [hep-ph]*
- *PNJL with vector int. using NN: K. Kashiwa, Y. Mori, AO, arXiv:1903.03679 [hep-lat]*
- *0+1D QCD: Y. Mori, K. Kashiwa, AO, in prep; AO, Y. Mori, K. Kashiwa, arXiv:1812.11506 (Lat2018 proc.)*

The Sign Problem

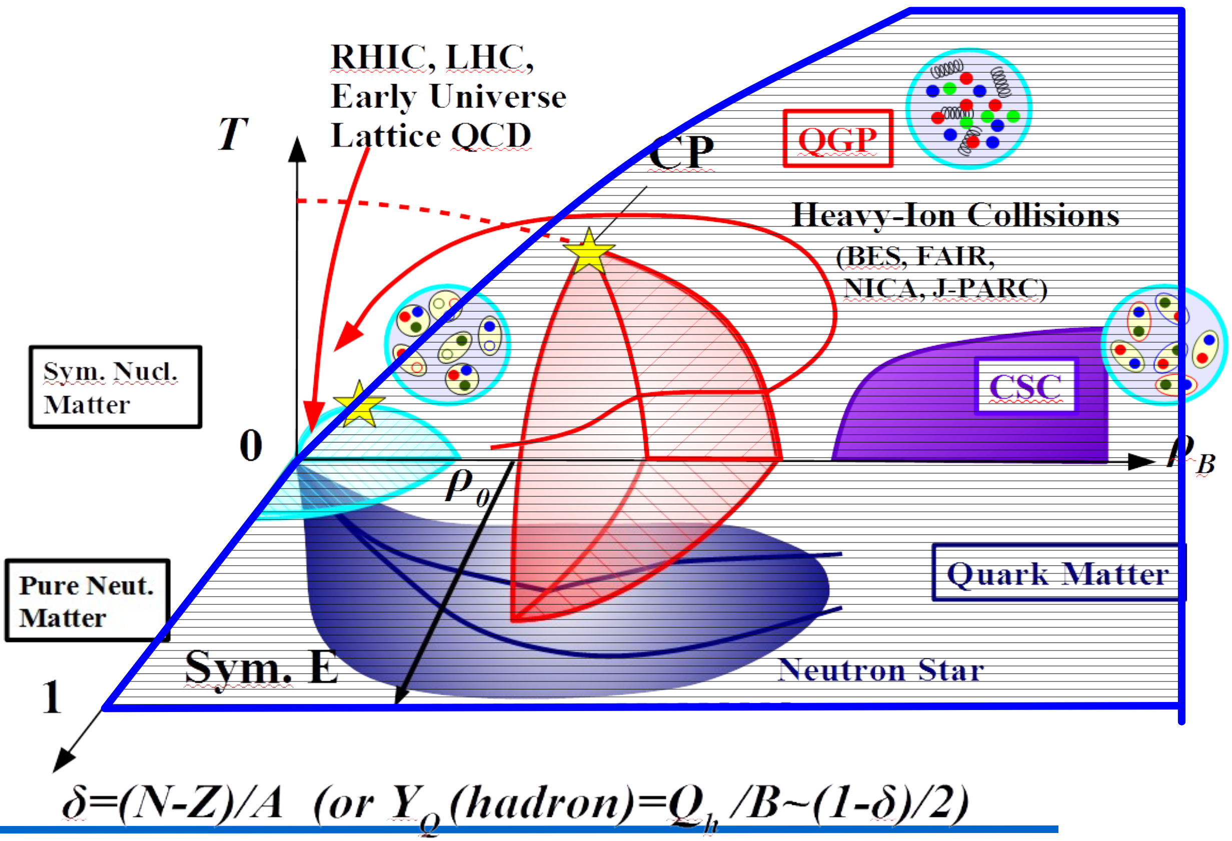
- **When the action is complex, strong cancellation occurs in the Boltzmann weight at large volume. = The Sign Problem**

$$\mathcal{Z} = \int \mathcal{D}x\, e^{-S(x)},\ |\mathcal{Z}| \ll \mathcal{Z}_{\rm pq} = \int \mathcal{D}x \left| e^{-S(x)} \right| \ \text{(at large } V\text{)}$$

- **Fermion det. is complex at finite density**

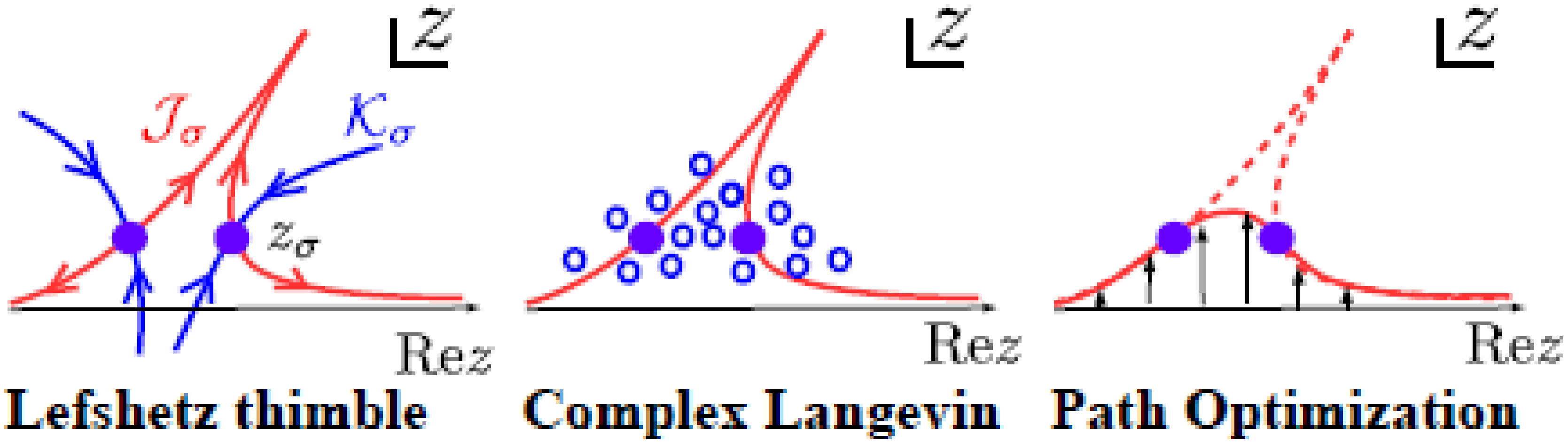
$$\det D(\mu) = (\det D(-\mu^*))^* \to S_{\rm eff} = S_{\rm boson} - \log\det D \in \mathbb{C}$$

- **Difficulty in studying finite density in LQCD**
  → Heavy-Ion Collisions, Neutron Star, Binary Neutron Star Mergers, Nuclei, …



Approaches to the Sign Problem

- **Standard approaches**
  - **Taylor exp., Imag. μ (Analytic cont. / Canonical), Strong coupling**
- **Integral in Complexified variable space**
  - **Lefschetz thimble method**
    Witten ('10), Cristoforetti+ (Aurora) ('12), Fujii+ ('13), Alexandru+ ('16).
  - **Complex Langevin method**
    Parisi ('83), Klauder ('83), Aarts+ ('11), Nagata+ ('16); Seiler+ ('13), Ito+ ('16).
  - **Path optimization method**
    Mori, Kashiwa, AO ('17,'18,'19); Kashiwa, Mori, AO ('18,19); AO, Mori, Kashiwa ('18,'19); Alexandru+('18), Bursa, Kroyter ('18)

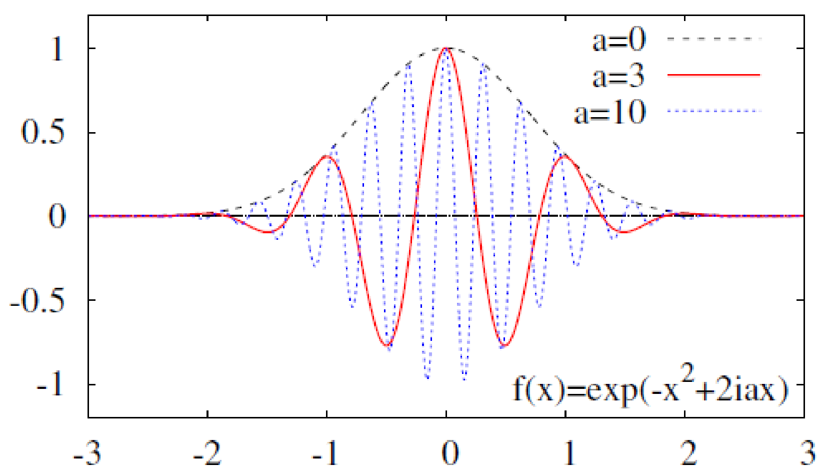


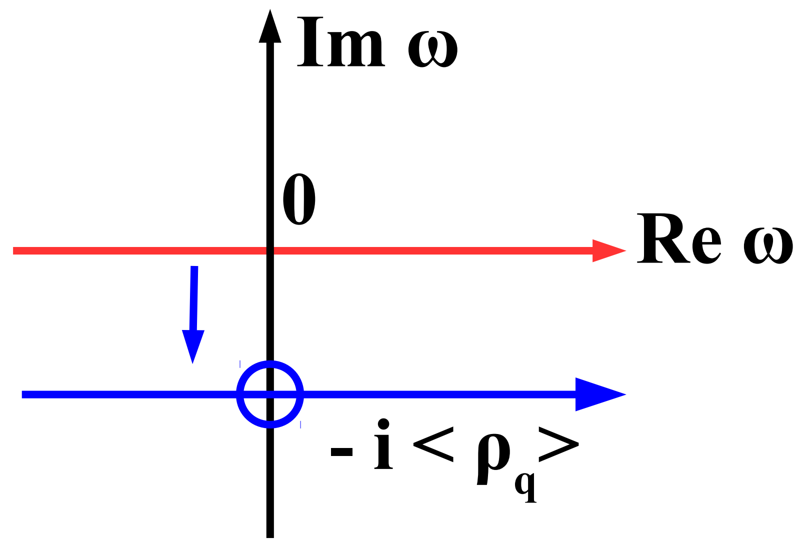
Integral in Complexified Variable Space

- **Simple Example: Gaussian integral (bosonized repulsive int.)**
  Mori, Kashiwa, AO ('18b)

$$\int_{\mathbb{R}} d\omega\, e^{-\omega^2/2 - i\omega\rho_q} = \int_{\mathbb{R}+i\rho_q} d\omega\, e^{-\underbrace{(\omega+i\rho_q)}_{z}{}^2/2-\rho_q^2/2}$$

$$= \exp(-\rho_q^2/2) \int_{\mathbb{R}} dz\, e^{-z^2/2}$$





Complexified variable methods
= Extension of the saddle point integral

Lefschetz thimble & Complex Langevin methods

- **Lefschetz thimble method**
  Witten ('10), Cristoforetti+ (Aurora) ('12), Fujii+ ('13), Alexandru+ ('16).
  - **Flow eq. from a fixed point σ → thimble (Im(S)=const.)**

$$\mathcal{J}_\sigma : \frac{dz_i(t)}{dt} = \overline{\left(\frac{\partial S}{\partial z_i}\right)} \to \frac{dS}{dt} = \sum_i \left|\frac{\partial S}{\partial z_i}\right|^2 \in \mathbb{R}, \quad \mathcal{C} = \sum_\sigma n_\sigma \mathcal{J}_\sigma$$

  - **Problems:Phase of Jacobian, Multimodal prb., Stokes phenomena, …**

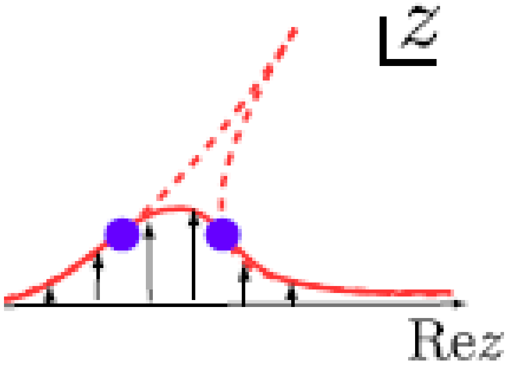
- **Complex Langevin method**
  Parisi ('83), Klauder ('83), Aarts+ ('11), Nagata+('16); Seiler+ ('13), Ito+ ('16).
  - **Complex Langevin eq.→ Configs.**

$$\frac{dz_i}{dt} = -\frac{\partial S}{\partial z_i} + \eta_i(t) (\eta_i : \text{White noize}), \ \langle \mathcal{O}(x) \rangle = \langle \mathcal{O}(z) \rangle$$

  - **Problems: Wrong conversion, Boundary terms, ...**

Path optimization method

Mori, Kashiwa, AO ('17,'18,'19); Kashiwa, Mori, AO ('18,19);
AO, Mori, Kashiwa ('18,'19); Alexandru+('18), Bursa, Kroyter ('18)



Path Optimization

- **Integration path is optimized**
  to evade the sign problem,
  i.e. to enhance the average phase factor.

$$\mathrm{APF} = \langle e^{i\theta} \rangle_{\mathrm{pq}} = \int dx e^{-S} / \int dx |e^{-S}| = \mathcal{Z} / \mathcal{Z}_{\mathrm{pq}}$$

Sign Problem → Optimization Problem

- **Cauchy(-Poincare) theorem: the partition fn. is invariant if**
  - **the Boltzmann weight W=exp(-S) is holomorphic (analytic), and the path does not go across the poles and cuts of W.**
- **S is singular but W is not singular when fermion det.=0.**

Application of POM to Field Theory

- **Cost function: a measure of the seriousness of the sign problem.**

$$\mathcal{F}[z(x)] = |\mathcal{Z}| \left(\left| \langle e^{i\theta} \rangle_{\mathrm{pq}} \right|^{-1} - 1 \right)$$

- **Optimization: Gradient Descent or Neural Network**
  - **Neural network = Combination of linear and non-linear transf.**

$$a_i = g(\underline{W_{ij}^{(1)}} x_j + \underline{b_i^{(1)}})$$

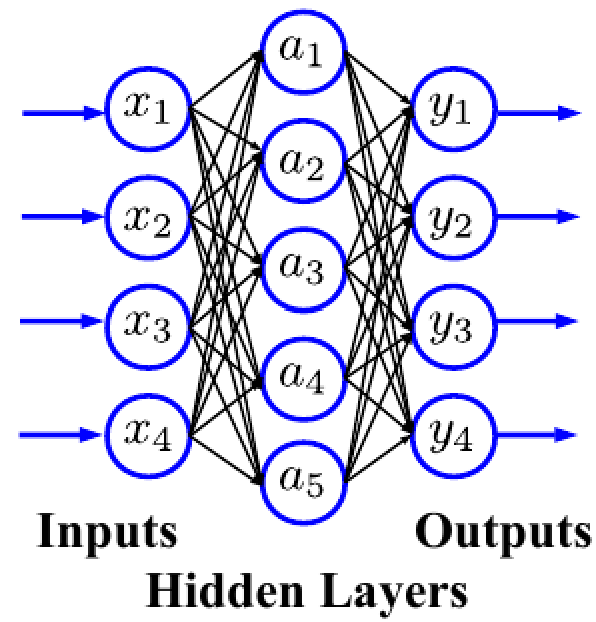
$$f_i = g(\underline{W_{ij}^{(2)}} a_j + \underline{b_i^{(2)}})$$

$$z_i = x_i + i(\underline{\alpha_i} f_i(x) + \underline{\beta_i})$$

$$g(x) = \tanh x \ \text{(activation fn.)}$$

variational parameters

  - **Universal approximation theorem**
    Any fn. can be reproduced
    at (hidden layer unit #) $\to \infty$
    G. Cybenko, MCSS 2 ('89) 303
    K. Hornik, Neural networks 4('91) 251



Optimization of many parameters

- **Stochastic Gradient Descent method, E.g. ADADELTA algorithm**
  M. D. Zeiler, arXiv:1212.5701

Grad. Desc. :

$$dc_i/dt = -\partial\mathcal{F}/\partial c_i$$

par. in (j+1)th step

Learning rate

$$c_i^{(j+1)} = c_i^{(j)} - \eta v_i^{(j+1)}$$

mean sq. ave. of v

$$v_i^{(j+1)} = \frac{\sqrt{s_i^{(j)} + \epsilon}}{\sqrt{r_i^{(j+1)} + \epsilon}} F_i^{(j)}$$

decay rate

mean sq. ave. of F

$$r_i^{(j+1)} = \gamma r_i^{(j)} + (1-\gamma)(F_i^{(j)})^2$$

$$s_i^{(j+1)} = \gamma s_i^{(j)} + (1-\gamma)(v_i^{(j+1)})^2$$

gradient evaluated in MC (batch training)

$$F_i = \partial\mathcal{F}/\partial c_i$$

Cost fn.

Machine learning
~ Educated algorithm to generic problems

Hybrid Monte-Carlo with Neural Network

Initial Config. on Real Axis

HMC $H(x,p) = \frac{p^2}{2} + \mathrm{Re}S(z(x))$

Jacobian
→ via Metropolis judge

Do k = 1, Nepoch
Do j = 1, Nconf/Nbatch

Mini-batch training of Neural Network

$$c_i^{(j+1)} = c_i^{(j)} - \eta v_i^{(j+1)}$$

Grad. wrt parameters (Nbatch configs.)

$$F_i = \frac{1}{N_{\mathrm{batch}}} \sum_n \partial \mathcal{F}(n)/\partial c_i$$

New Nbatch configs. by HMC

$$H(x,p) = \frac{p^2}{2} + \mathrm{Re}S(z(x))$$

Enddo
Enddo

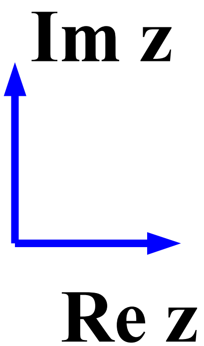
Nbatch ~ 10, Nconfig ~ 10,000, Nepoch ~ (10-20)

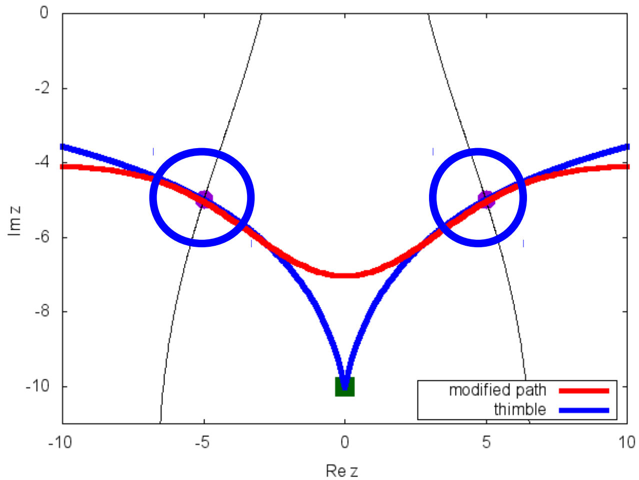
Benchmark test (1): 1 dim. integral

$$\mathcal{Z} = \int dx (x + i\alpha)^p \exp(-x^2/2)$$

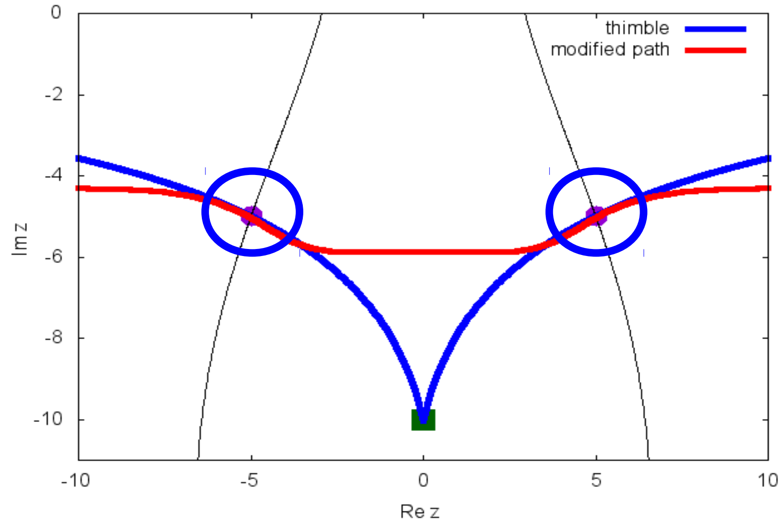
J. Nishimura, S. Shimasaki ('15)

Gaussian+Gradient Descent

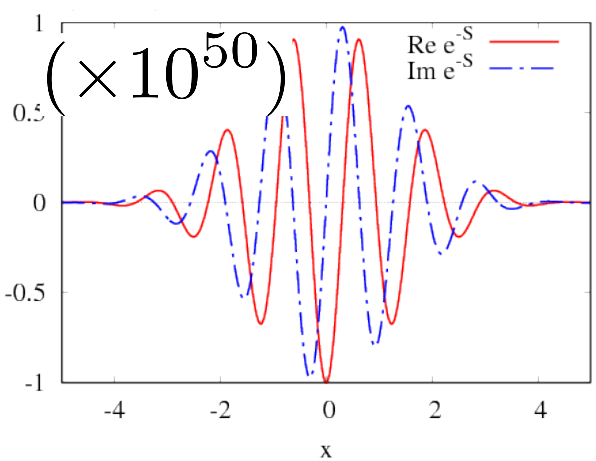




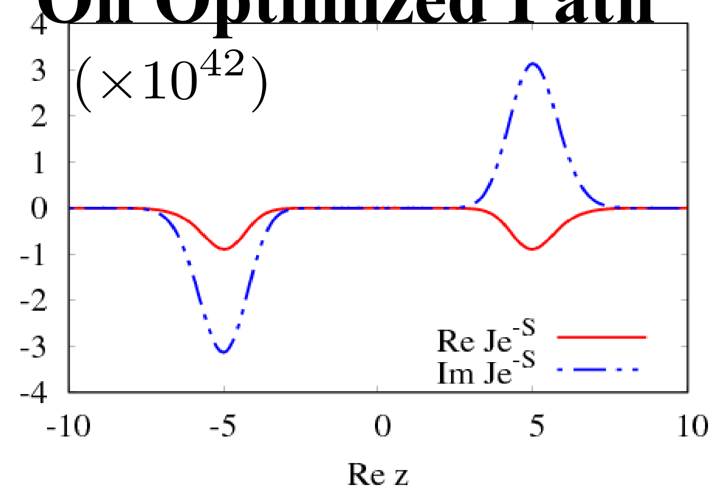
Neural Network



On Real Axis



On Optimized Path



Mori, Kashiwa, AO ('17); AO, Mori, Kashiwa (Lat 2017)

Benchmark test (2): Complex φ⁴ theory at finite μ

- **Complex Langevin & Lefschetz thimble work.**
  G. Aarts, PRL102('09)131601; H. Fujii, et al., JHEP 1310 (2013) 147

- **How about POM ?**
  - **1+1D Complex φ^4 theory**
    Y. Mori, K. Kashiwa, AO, PTEP 2018 ('18), 023B04 [arXiv:1709.03208]

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi - \lambda (\phi^* \phi)^2$$

$$S = \sum_x \Big[\frac{(4+m^2)}{2} \phi_{a,x}\phi_{a,x} + \frac{\lambda}{4} (\phi_{a,x}\phi_{a,x})^2 - \phi_{a,x}\phi_{a,x+\hat{1}} - \cosh\mu\, \phi_{a,x}\phi_{a,x+\hat{0}} + \underbrace{i\epsilon_{ab} \sinh\mu\, \phi_{a,x}\phi_{b,x+\hat{0}}}_{\text{complex}} \Big]$$

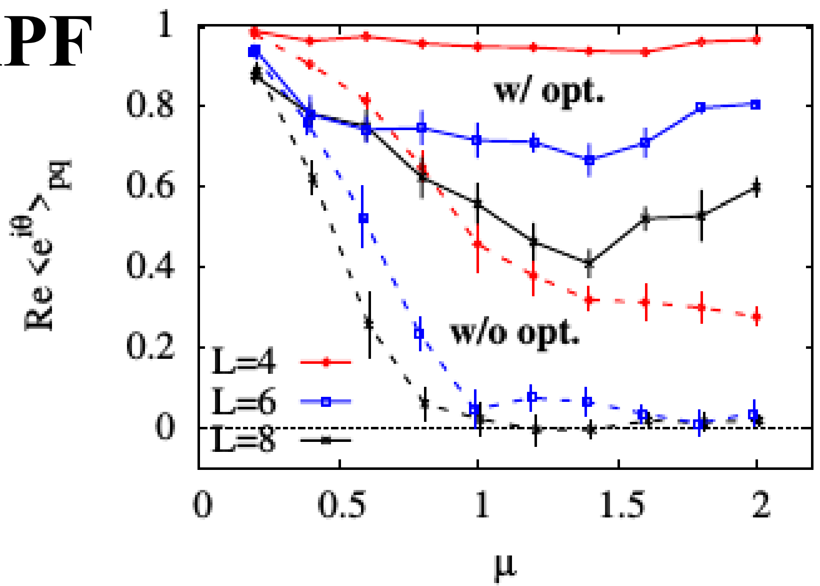
$$\left(\phi = \frac{1}{\sqrt{2}} (\underline{\phi_1} + i\underline{\phi_2}) \right)$$

Complexify

POM in 1+1D φ^4 theory

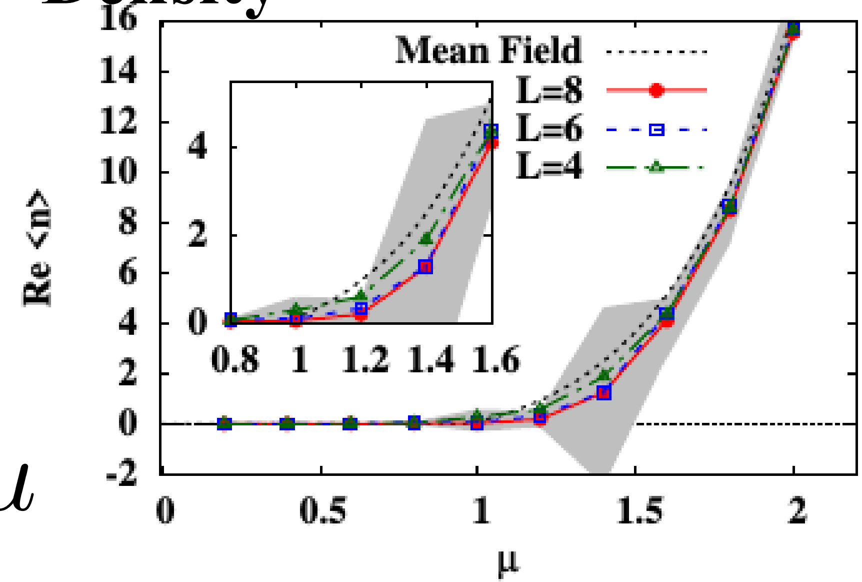
- **POM for 1+1D φ^4 theory**
  Y. Mori, K. Kashiwa, AO, PTEP 2018 ('18), 023B04 [arXiv:1709.03208]
  - **4^2, 6^2, 8^2 lattices, λ=m=1**
  - **$\mu_c \sim$ 0.96 in the mean field approximation**

APF



μ

Density



μ

> ***POM also works !***
> - ***Enhancement of the APF after optimization.***
> - ***Density is suppressed at μ < m. (Silver Blaze)***

Path Optimization Method w/ Neural Network seems to work in 1D integral and simple field theories.

How about gauge theory ?
What happens when phase transition occurs ?

Contents

- **Introduction of Path Optimization Method**

  Y. Mori, K. Kashiwa, AO, PRD 96 ('17), 111501(R) [1705.05605]
  Y. Mori, K. Kashiwa, AO, PTEP 2018 ('18), 023B04 [1709.03208]
  AO, Y. Mori, K. Kashiwa, EPJ Web Conf. 175 ('18), 07043 [1712.01088](Lat 2017)

- **Application to gauge theory: 1-dimensional QCD**

  Mori, K Kashiwa, AO, in prep.
  AO, Y. Mori, K. Kashiwa, PoS LATTICE2018 ('19), 023 (1-15) [1812.11506]

- **Application to QCD effective models**

  K. Kashiwa, Y. Mori, AO, PRD99('19)014033 [1805.08940]
  K. Kashiwa, Y. Mori, AO, arXiv:1903.03679 [hep-lat]

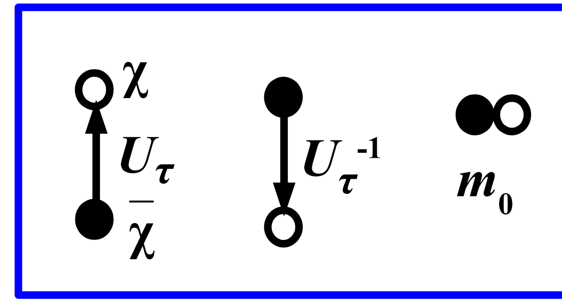
- **Discussions**
- **Summary**

Application to Gauge Theory: 1 dimensional QCD

0+1 dimensional QCD

- **0+1 dimensional QCD (1 dim. QCD)**
  with one species of staggered fermion on a 1xN_τ lattice

$$S = \frac{1}{2}\sum_\tau \left(\bar{\chi}_\tau e^{\mu} U_\tau \chi_{\tau+\hat{0}} - \bar{\chi}_{\tau+\hat{0}} e^{-\mu} U_\tau^{-1} \chi_\tau\right)$$
$$+ m\sum_\tau \bar{\chi}_\tau \chi_\tau = \frac{1}{2}\bar{\chi} D \chi$$



Bilic+('88), Ravagli+('07), Aarts+('10, CLM), Bloch+('13, subset), Schmidt+('16, LTM), Di Renzo+('17, LTM)

- **A toy model, but includes the actual source of 3+1D QCD sign prob.**
- **Reduces to a diagonalized one-link problem.**

$$\mathcal{Z} = \int dU e^{-S} = \int dx_1 dx_2 J \underbrace{\left[\frac{8}{3\pi^2}\prod_{a<b}\sin^2\left(\frac{z_a - z_b}{2}\right)\right]}_{\text{Haar measure}} \underbrace{\left[\prod_a (X_N + 2\cos(z_a - i\mu))\right]}_{\text{exp(-S)}}$$

→ Analytic results are known.

Fermion determinant in 1 dim. QCD

- **Fermion determinant (Temporal gauge) reduces to Nc x Nc det.**

$$
D = \begin{pmatrix} I_1 & e^{\mu} & 0 & & e^{-\mu}U^{-1} \\ -e^{-\mu} & I_2 & e^{\mu} & & \\ 0 & -e^{-\mu} & I_3 & e^{\mu} & \\ \vdots & & & \ddots & \\ -e^{\mu}U & & & -e^{-\mu} & I_N \end{pmatrix} \quad N_c \times N_\tau \qquad X = \begin{vmatrix} I_1 & e^{\mu} & 0 & & 0 \\ -e^{-\mu} & I_2 & e^{\mu} & & \\ 0 & -e^{-\mu} & I_3 & e^{\mu} & \\ \vdots & & & \ddots & \\ 0 & & & -e^{-\mu} & I_N \end{vmatrix}
$$

$$
\det D = \det\left[\underline{X \otimes 1_c} + \underline{(-1)^{N_\tau} e^{\mu/T} U} + \underline{e^{-\mu/T} U^{-1}} \right] \quad N_c \qquad I_k = 2m_q(\boldsymbol{x}, \tau_k)
$$

- **For constant σ, *X* is obtained as**

$$
\begin{aligned}
X &= 2\cosh(E/T) \\
E &= \mathrm{arcsinh}\, m_q \\
&\to \int dU \det D = \frac{\sinh[(N_c+1)E/T]}{\sinh(E/T)} + 2\cosh(N_c\mu/T)
\end{aligned}
$$

Faldt, Petersson ('86)

Partition Function in 1 dim. QCD

- **Partition Function**

$$D = X + e^{\mu/T} U + e^{-\mu/T} U^{-1}$$

$$\mathcal{Z} = \int dU \, \det D(U) = \frac{\sinh[(N_c+1)E/T]}{\sinh(E/T)} + 2\cosh(N_c\,\mu/T)$$

$$= X^3 - 2X + 2\cosh(N_c\,\mu/T) \quad (N_c = 3)$$

$$\det D = X^3 + N_c X (N_c \overline{P}_U P_U - 1) + 2\cosh(N_c \mu/T)$$
$$+ N_c X^2 (e^{\mu/T} P_U + e^{-\mu/T} \overline{P}_U) + N_c X (e^{2\mu/T} \overline{P}_U + e^{-2\mu/T} P_U)$$
$$+ N_c e^{\mu/T} (N_c \overline{P}_U^2 - 2P_U) + N_c e^{-\mu/T} (N_c P_U^2 - 2\overline{P}_U)$$
$$(N_c = 3, P_U = \mathrm{tr} U/N_c, \int dU \overline{P}_U P_U = 1/N_c^2)$$

- **Chiral condensate, Quark number density, Polyakov loop**

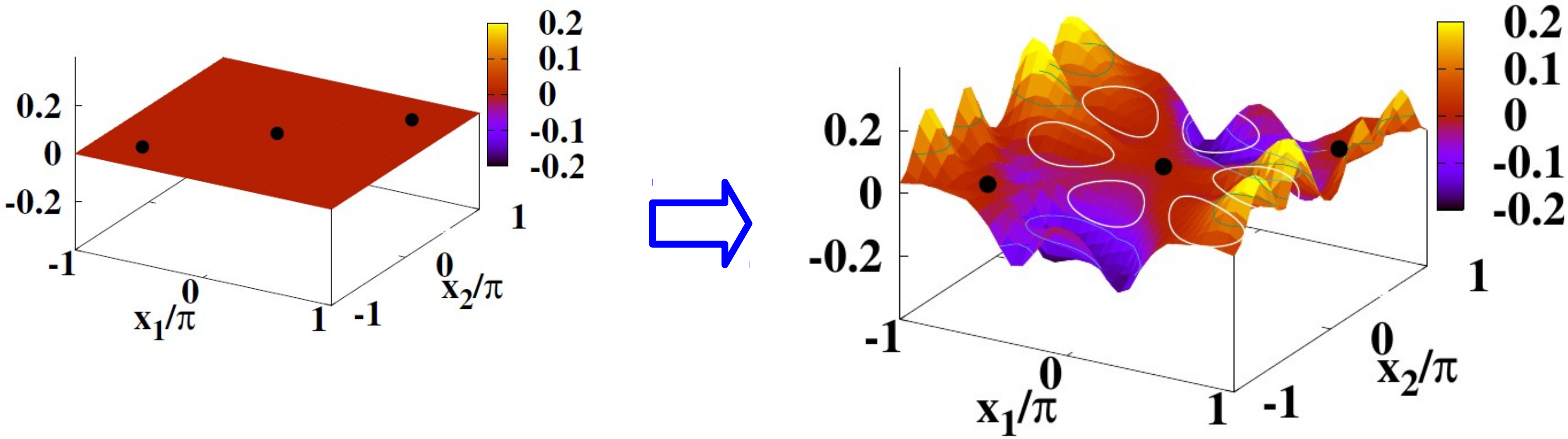
$$\sigma = \frac{T}{V}\frac{\partial}{\partial m}\log Z, \quad n_q = -\frac{T}{V}\frac{\partial}{\partial \mu}\log Z$$

$$P = \left\langle \frac{1}{N_c} \mathrm{Tr}\, U \right\rangle = \frac{1}{N_c} \frac{(X^2-1)e^{-\mu/T} + X e^{2\mu/T}}{X^3 - 2X + 2\cosh(3\mu/T)}$$

Faldt, Petersson ('86)
Bilic, Demeterfi ('88)

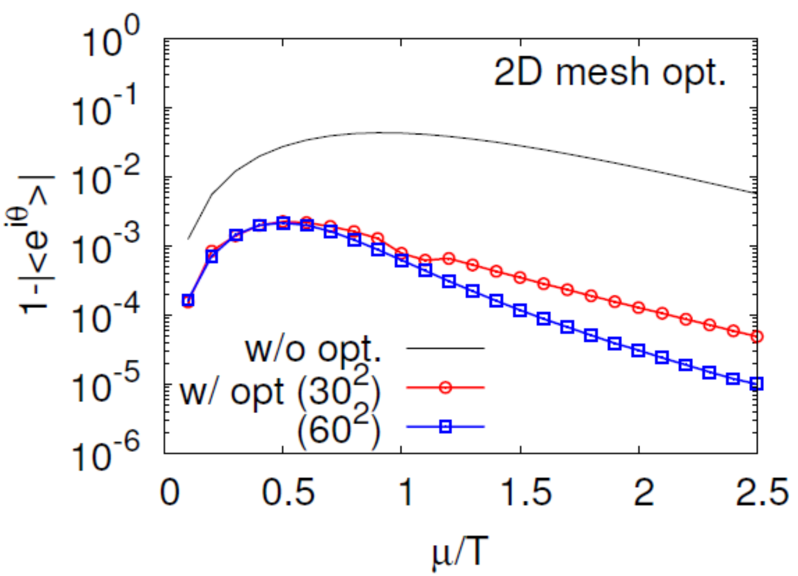
1 dim. QCD in diagonalized gauge (1)

- 2 variable problem → 2D mesh point integral
  → $y_{1,2}(x_1,x_2)$ are variational parameters by themselves.



- Average phase factor > 0.997
  - (Normal) gradient descent
  - Good enough for small lattice in 3+1D.

$$(\mathrm{APF})_{0+1}^{L^3} \simeq 0.21 (L=8)$$



Mori, Kashiwa, AO, in prep.

1 dim. QCD in diagonalized gauge (2)

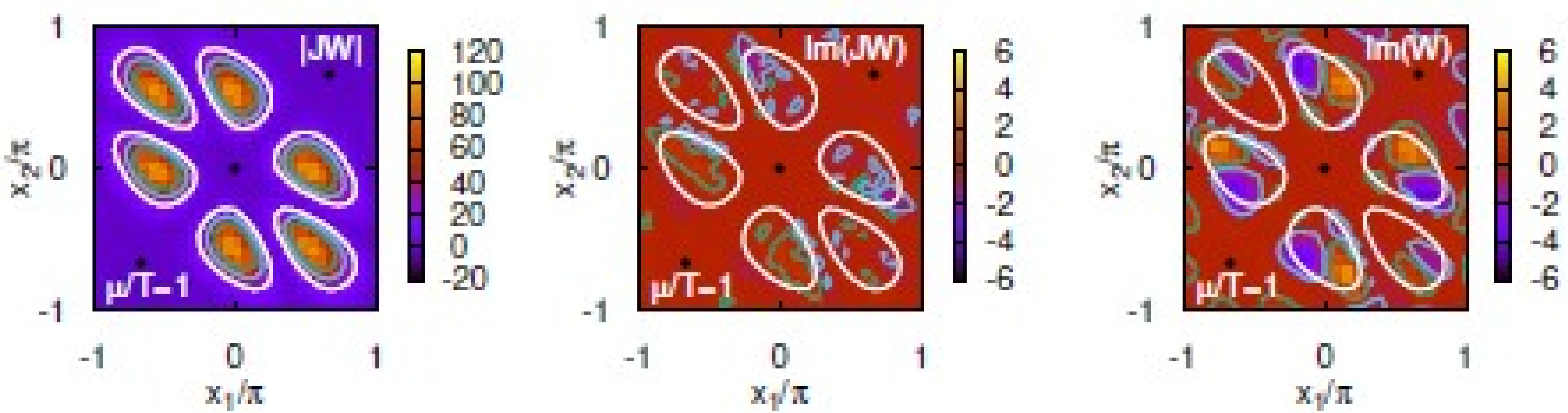
- **Jacobian is also important !**

$$|\mathrm{Im}(JW)| \ll |\mathrm{Im}W| \quad (W = H\exp(-S))$$

- **There are six regions with large stat. weight | *JW* |.**

$$\text{Symmetry}: \quad S(-z) = (S(z^*))^*, z_i \leftrightarrow z_j (i, j = 1, 2, 3)$$

→ Problematic in sampling in Hybrid MC



Mori, Kashiwa, AO, in prep.

1 dim. QCD w/o diagonalized gauge fixing (1)

- **Complexification of link variable**

$$U \to \mathcal{U}(U) = U \prod_{a=1}^{N_c^2-1} e^{-y_i \lambda_i}$$

$$= U \boxed{e^{-y_1\lambda_1} e^{-y_2\lambda_2} \cdots e^{-y_8\lambda_8}} \in \mathrm{SL}(3)$$

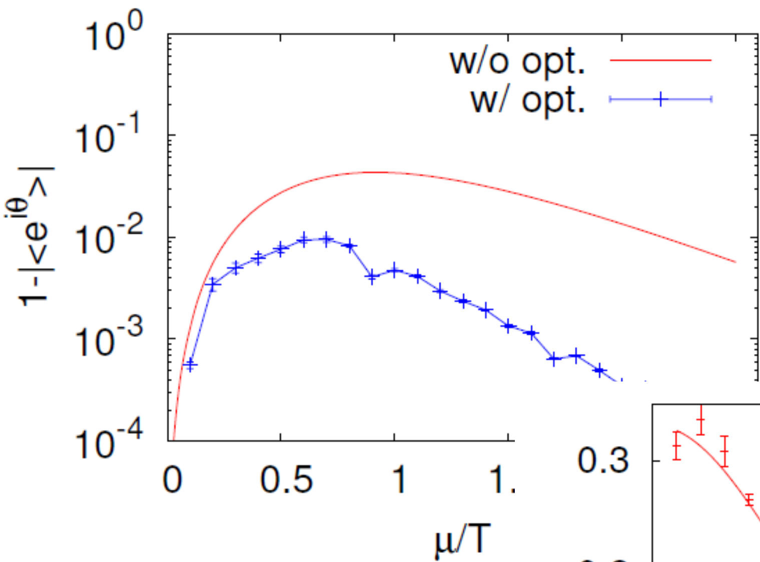
  - **Derivative wrt y's is easy. Parametrization deps. is taken care by J.**

- **Hybrid Monte-Carlo in 1 dim. QCD**

$$H = \frac{P^2}{2} + \mathrm{Re}(S(\mathcal{U}(U)))$$

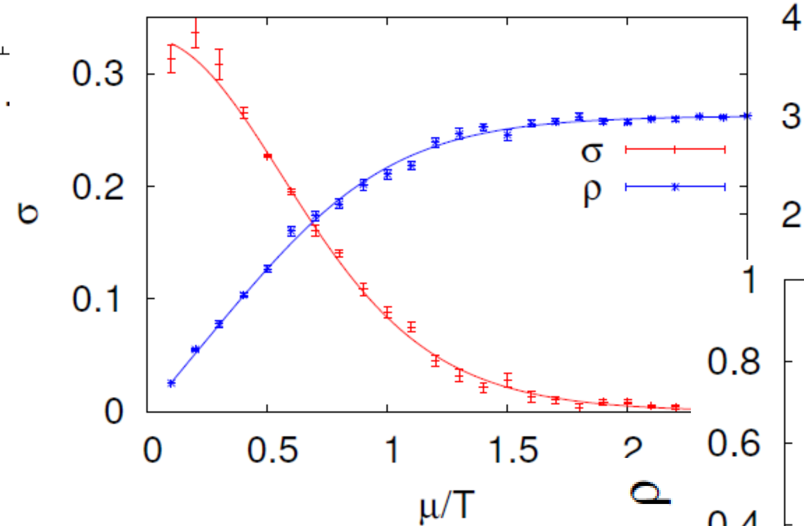
- **8 variables → path optimization using Neural Network**

1 dim. QCD w/o diagonalized gauge fixing (2)

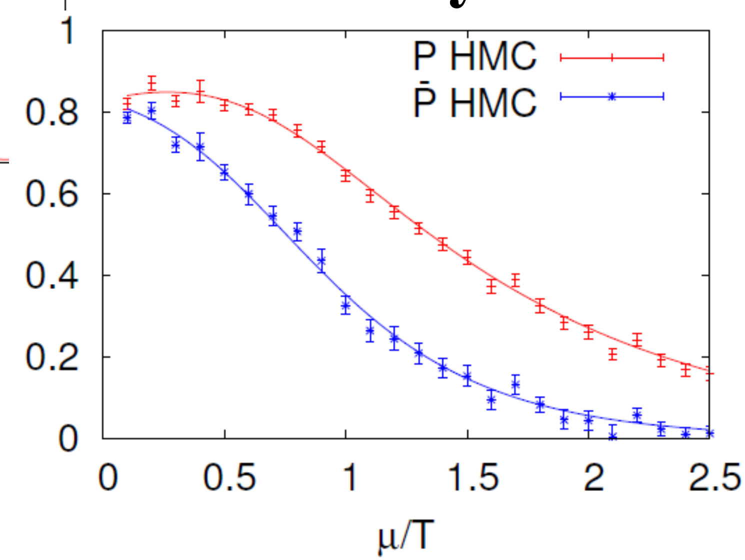


Average phase factor

Chiral condensate
& Quark number density



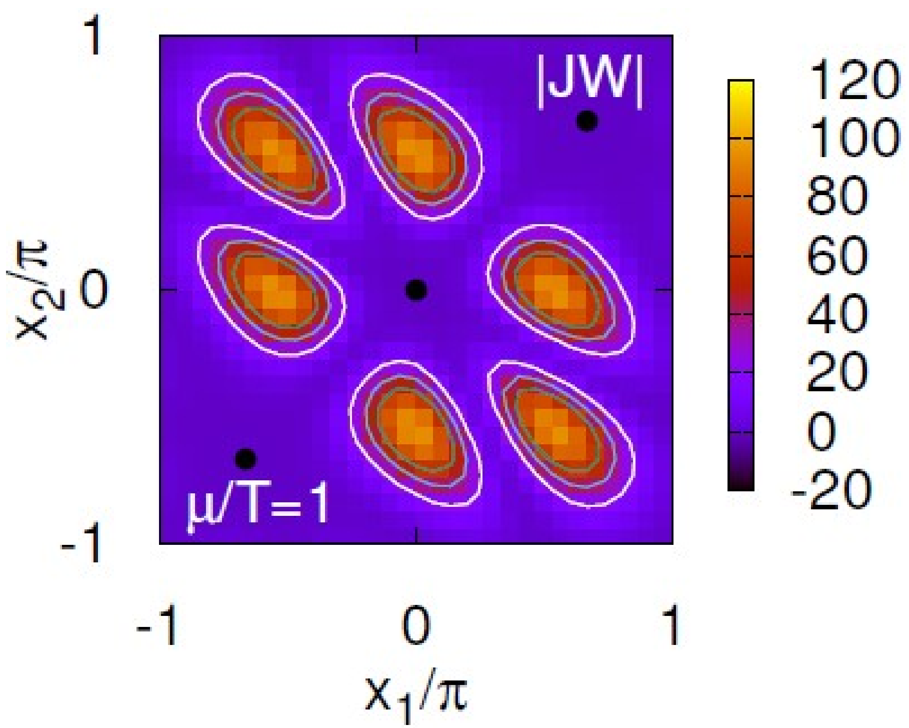
Polyakov loop

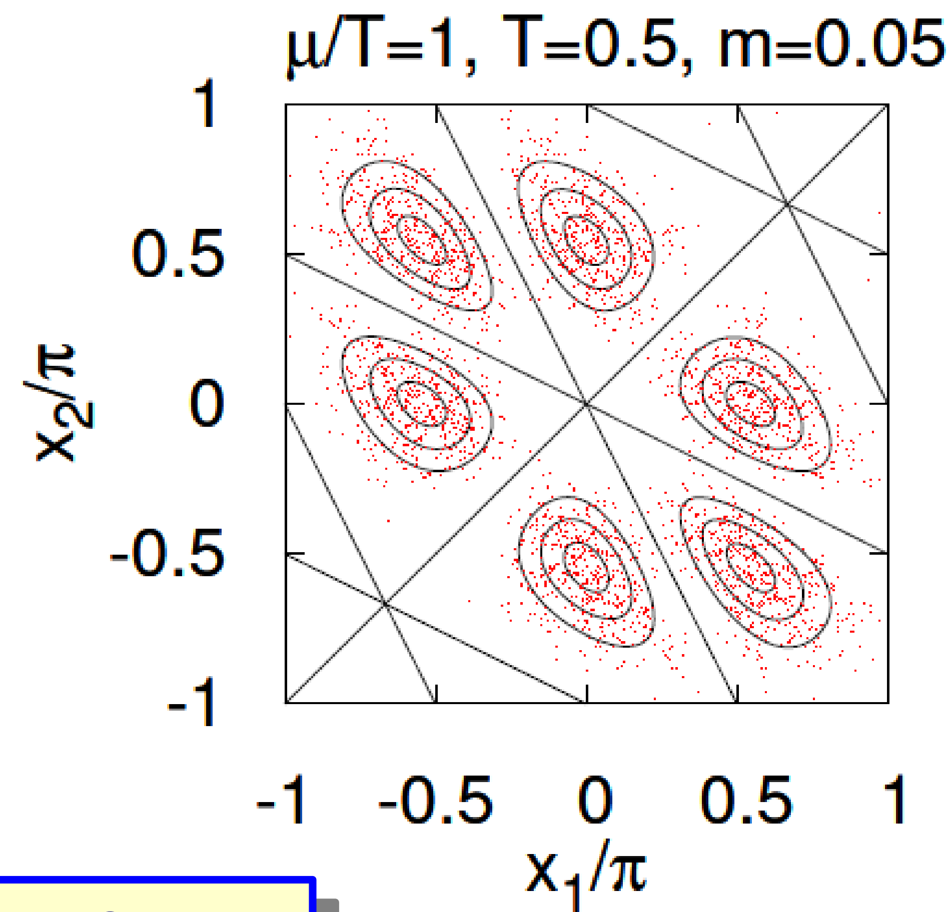


Mori, Kashiwa, AO, in prep.

1 dim. QCD w/o diagonalized gauge fixing (3)

- **Statistical weight distribution in diagonalized gauge**
  ~ Config. dist. in Hybrid MC w/o diag. gauge fixing





It would be possible to apply POM in more realistic cases !

Mori, Kashiwa, AO, in prep.

Application to QCD effective models

Polyakov-loop-extended NJL (PNJL) model

- **Sign problem is more severe around the phase boundary.**
  e.g. S. Tsutsui et al., 1811.07647; Y. Ito et al., 1811.12688.
  → Let us discuss QCD effecitve models !

- **Polyakov-loop-extended Nambu-Jona-Lasinio (PNJL) model with vector coupling**

$$
\begin{aligned}
\mathcal{L}_E =& \bar{q}(\not{D} + m_0)q - G\left[(\bar{q}q)^2 + (\bar{q}i\gamma_5\boldsymbol{\tau}q)^2\right] \\
& + \underbrace{\mathcal{V}_g(\Phi,\bar{\Phi})}_{\text{Polyakov}} + \underbrace{G_v(\bar{q}\gamma_\mu q)^2}_{\text{Vector}}
\end{aligned}
$$

- **Bosonizaiton & Truncation to homogeneous aux. field**

$$
\begin{aligned}
\mathcal{V} =& -2N_f \int^\Lambda \frac{d^3p}{(2\pi)^3}\left[N_c E_p + T\log(f^- f^+)\right] \\
& + G(\sigma^2 + \boldsymbol{\pi}^2) + G_v\omega^2 + \mathcal{V}_g
\end{aligned}
$$

$$
f^- = 1 + 3\Phi e^{-\beta E_p^-} + 3\bar{\Phi}e^{-2\beta E_p^-} + e^{-3\beta E_p^-}
$$

$$
E_p = \sqrt{p^2 + (m_0 - 2G\sigma)^2},\ E_p^{\mp} = E_p \mp \tilde{\mu},\ \tilde{\mu} = \mu - 2iG_v\omega
$$

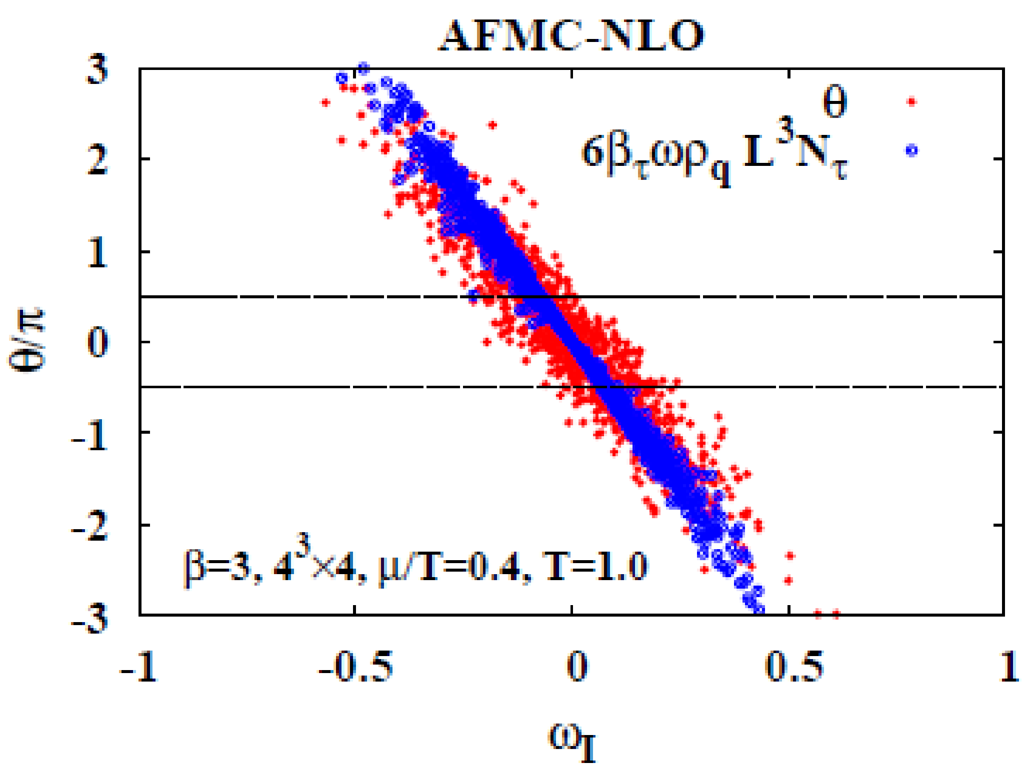
Φ, $\bar{\Phi}$, ω cause the sign prb.

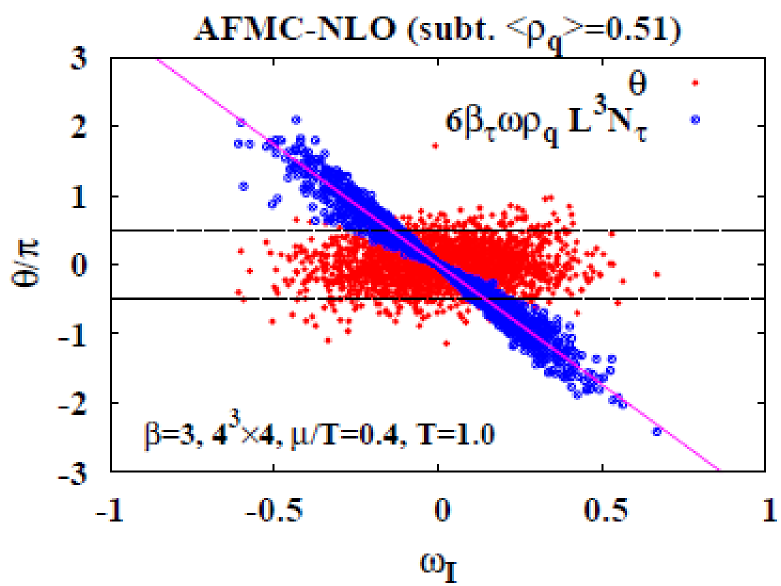
Repulsive interaction causes the sign problem

- **Hubbard-Stratonovich transformation of repulsive interaction**

$$\exp[-\alpha(\bar{q}\Gamma q)^2] = \int d\omega \exp[-\alpha\omega^2 + i\alpha\omega\bar{q}\Gamma q]$$

- **This "model" sign problem causes trouble in Shell Model Monte-Carlo, Strong-coupling LQCD, …**





Complexification & Shift

AO, Ichihara (Lattice2015)

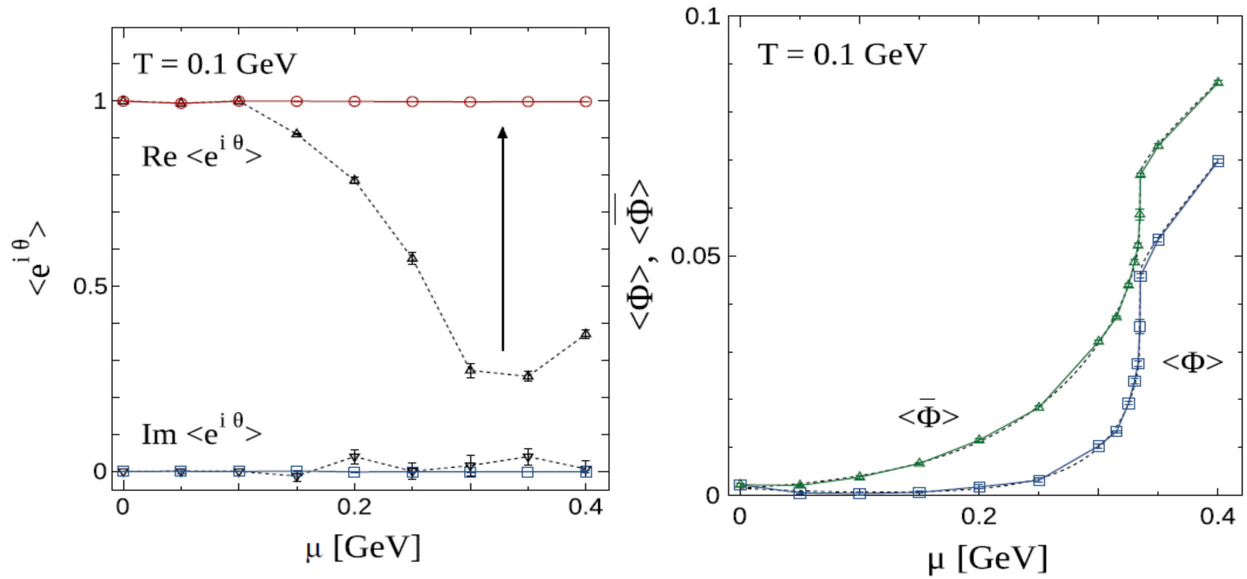
POM for PNJL (w/o vector coupling)

- **Partition function**

$$\mathcal{Z} = \int dX \exp(-\beta\mathcal{V}/V) = \int dX \exp(-k\mathcal{V}/T^4) \ (dX = dA_3 dA_8 d\sigma d\omega \ldots)$$

Homogenous field ansatz & const. k approx.
→ Results converge to mean field results at large k
Cristoforetti, Hell, Klein, Weise ('10) (MC-NJL)

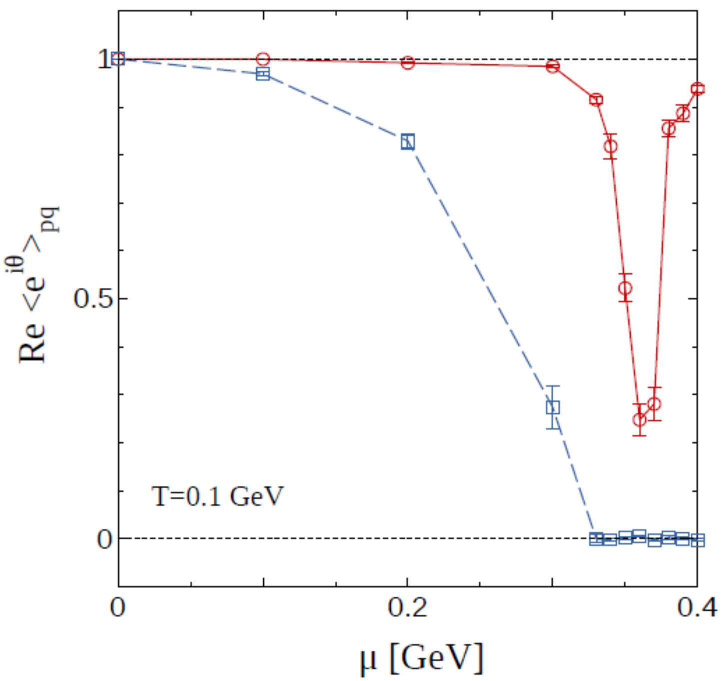
- **POM works in PNJL !**
  - **Average phase factor ~ 1, Pol. loop converges to MF results.**

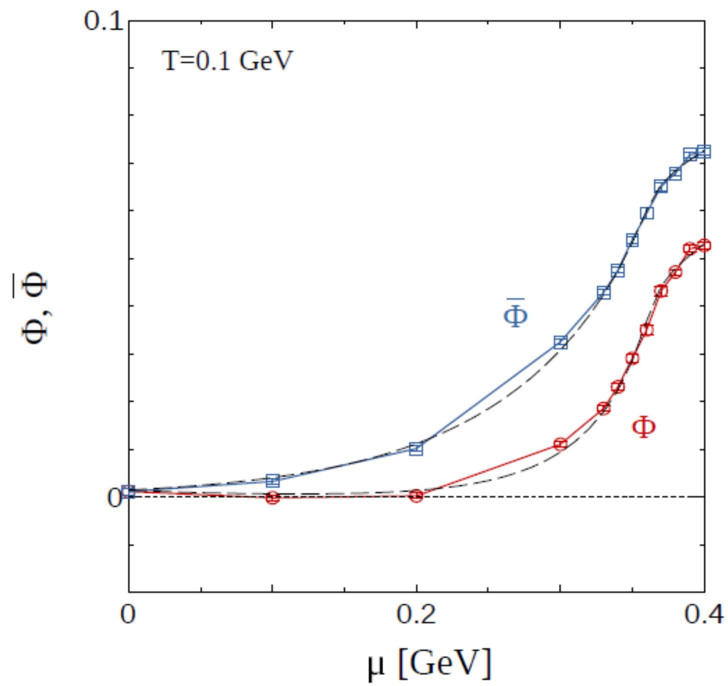


K. Kashiwa, Y. Mori, AO, PRD 99 ('19)

POM for PNJL with vector coupling

- **POM works in PNJLv**
  - **Pol. loop converges to MF results.**
  - **Average phase factor is enhanced significantly, but we still find the region, APF < 1 and we need special care for the initial config. distribution. (Optimization is not automatic in this case.)**

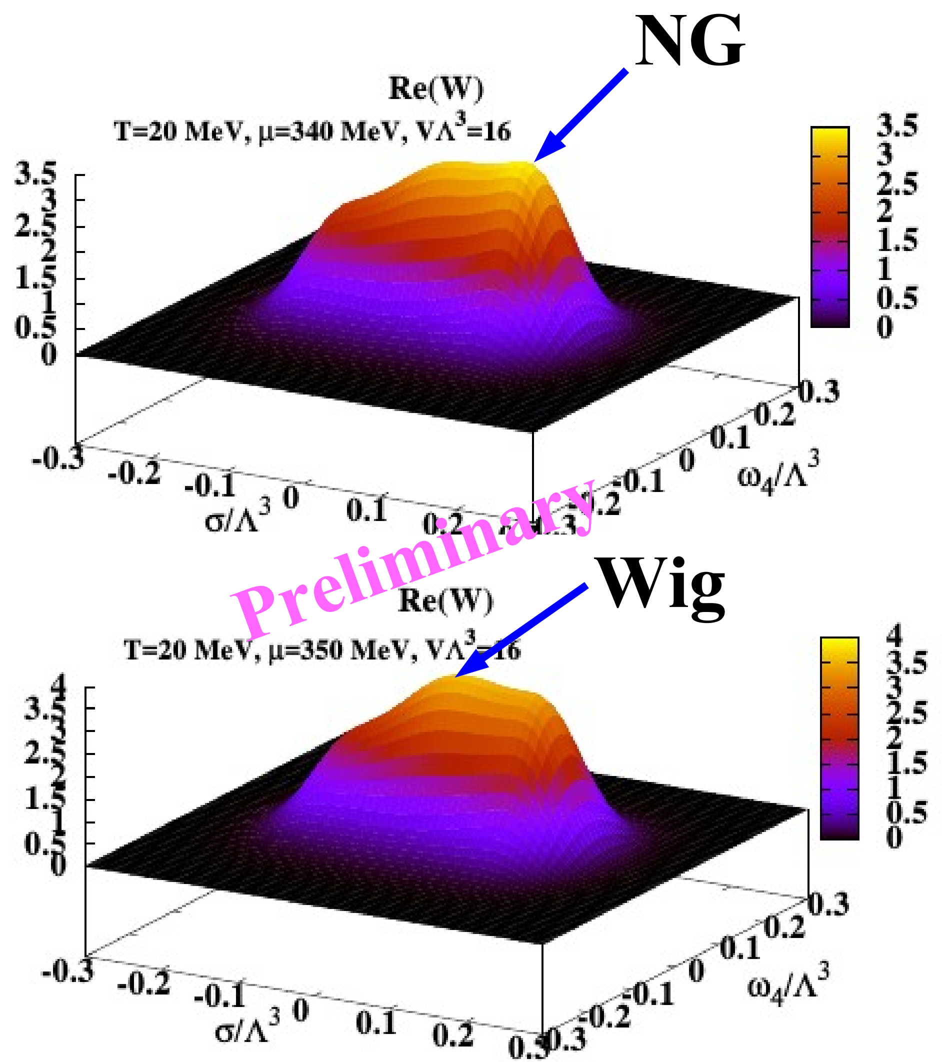




K. Kashiwa, Y. Mori, AO, arXiv:1903.03679

Do we describe multi thimbles ?

- **It seems yes.**
- **Statistical weight in NJLv on (σ, ω_4) plain after optimization on the 2D mesh.**
  $\rightarrow$ Three peaks (or shoulders)
- **Transition from the Nambu-Goldstone phase to the Wigner phase occurs at μ=(340-350) MeV**
- **In the V $\rightarrow \infty$ limit, this corresponds to the phase transition. (We may need exchange MC or different tempering.)**

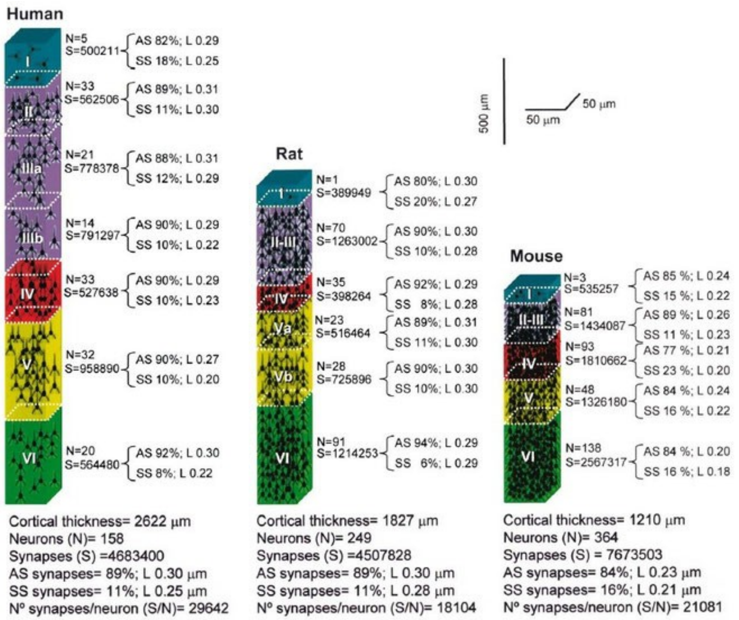


Summary

- Complexified variable methods (LTM, CLM, POM) are promising tools to tackle the sign problem. [c.f. Talk by Fukuma]
- Path optimization method has been demonstrated to work in
  1 dim. integral, 1+1 dim. scalar theory at finite density,
  0+1 dim. QCD, and PNJL model w/ and w/o vector coupling.
  - POM does not suffer from zero point of fermion det., since it is not a singular point of the Boltzmann weight.
  - Complex phase from Jacobian and the Boltzmann weight cancels with each other, and the residual sign problem is evaded.
  - In 1 dim. QCD, an apparent multimodal problem in the diag. gauge can be avoided by calc. w/o diag. gauge fixing.
  - Sometimes, the average phase factor does not easily grow during the optimization. Improving the opt. method and/or knowledge of preferred path would be necessary (not yet *ab initio*).
- To do: 1+1 D QCD, Reducing cost $O(V^3)$, ...

Prospect

- **Path optimization in 3+1 D field theories would require reduction of numerical cost.**
  Imaginary part
  = f (real parts of same point and nearest neighbor points)
  may be a good guess.
- **Deep learning (# of hidden layers > 3) may be helpful to explore complex paths, which human beings (~ 7 layers) cannot imagine, while "Understanding" the results of machine learning need to be done by human beings (at present).**



Defelipe 2011a (Review). The evolution of the brain, the human nature of cortical circuits, and intellectual creativity. Front Neuroanat 5, 29.

Thank you for your attention !