TKNN formula for general Hamiltonian

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arXiv:1903.11852

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Caution

Change of the topic

Previous speakers (Fukaya, Furuta)

Anomaly in four dimensions with boundary

This talk

Topological insulator in odd dimensions without boundary

(D=2+1, D=4+1)

1. Introduction

Topological insulator

- Interesting physics from non-trivial topology

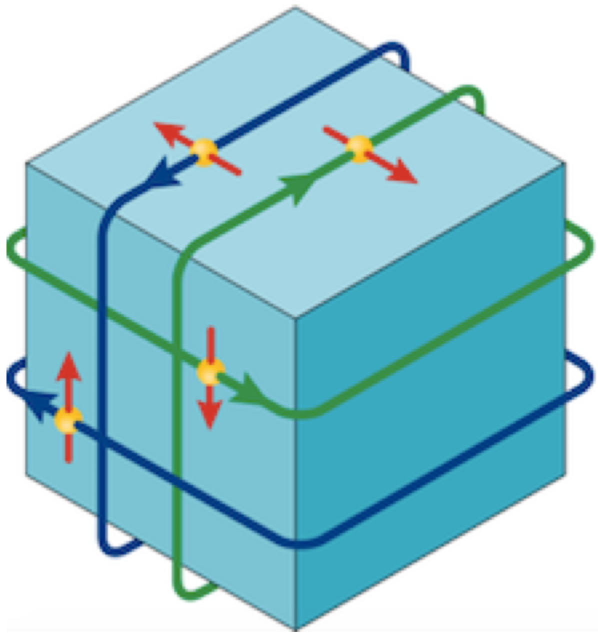


Figure from Tokura et al.
Nature Reviews Physics **vol 1**, 126 (2019)

Bulk: insulator
Surface: metal

Topology guarantees edge modes
(Bulk-Edge correspondence)

- Close relationship to domain-wall

New knowledge of topological matter
➔ new hints to lattice fermions by Domain-wall fermion

example: Gapped symmetric phase by 4-fermi interaction
(Talk by Kikukawa)

Two approaches to topology

- ## Microscopic approach:

  Study the wavefunction of the free electron Hamiltonian

  Applied to various different free systems (higher dim, higher symmetry)

  Classification of topology is highly developed

  Looks rather technical (at least to me)

  Applicable only to free fermion systems

  TKNN formula (D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs)

- ## Field theory approach:

  Introduce gauge field and study the effective action $S_{\text{eff}}(A)$

  Conceptually simple

  Applicable also to interacting fermion systems

What characterizes topology?

- **Microscopic approach:**

Topology of Berry connection of single particle wavefunctions

Top. # = $\sum_n \int d^2p \; c_1(\mathcal{A}^{(n)})$ $\qquad$ $\mathcal{A}^{(n)}_\mu \equiv -i\langle n,p|\frac{\partial}{\partial p_\mu}|n,p\rangle$

TKNN formula: Conductivity ←→ Top. #

- **Field theory approach:**

Top # = Chern-Simons level of the 3-dim effective gauge action

K. Ishikawa : Conductivity ← → Top. #

Question

Two topological characterizations are identical?

In some specific cases, yes.

How generally identical and why ?

We try to answer this question in this work.

Outline

✔ 1. Introduction
2. Review of TKNN formula
3. Review of field theory approach
4. Equivalence for general Hamiltonian
    1. Chern-Simons level → Winding number
    2. Winding number → TKNN formula
5. Summary

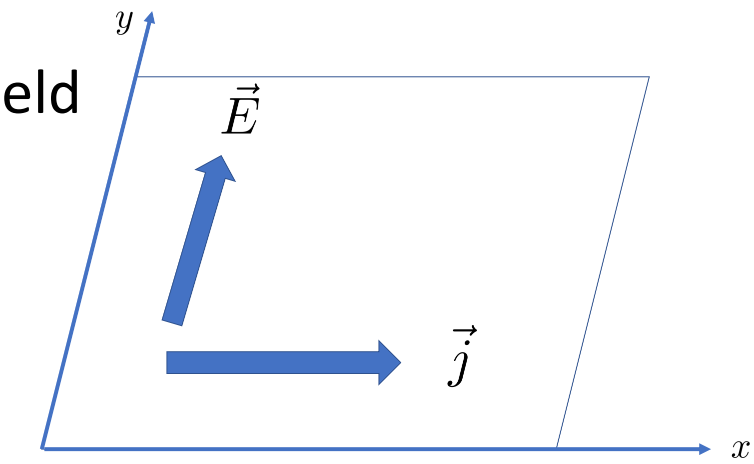
2. Review of TKNN formula

Anomalous Hall effect

2+1 dim system with Parity Violation

Hall current perpendicular to Electric field

$$\langle j_x \rangle_E = \sigma_{xy} E_y$$



Hall conductivity can be expressed by topological quantity using

1) Kubo formula from perturbation theory
2) Formulae in quantum mechanics

Electron states under electric field (perturbation theory)

$$|n\rangle_E = |n\rangle + \sum_{m\neq n} |m\rangle \frac{\langle m|eE_y y|n\rangle}{E_n - E_m}$$

$|n\rangle$: eigenstate in free theory

$|n\rangle_E$: perturbed state

Hall current under the electric field $\langle j_x\rangle_E \equiv \sum_{n,E_n<0} \langle n|_E \frac{-ev_x}{L^2} |n\rangle_E$

Kubo formula

$$\sigma_{xy} = -\frac{ie^2}{L^2} \sum_{\vec{p}} \sum_{a} \sum_{b\neq a} \epsilon^{ij} \frac{\langle a,\vec{p}|v_i|b,\vec{p}\rangle\langle b,\vec{p}|v_j|a,\vec{p}\rangle}{(E_a(\vec{p}) - E_b(\vec{p}))^2}$$

where we have used

- Translational invariance: $n \Rightarrow (a, \vec{p})$
- Heisenberg equation: $[y, H] = iv_y$

a : band label,

$\vec{p}$: bloch momentum

Derivation of Useful formula from

$$v_i = \frac{\partial}{\partial p^i} H(\vec{p})$$
$$H(\vec{p})|a,\vec{p}\rangle = E_a(\vec{p})|a,\vec{p}\rangle$$
$$\langle a,\vec{p}|b,\vec{p}\rangle = 0 \qquad (a \neq b)$$

$$\langle a,\vec{p}|v_i|b,\vec{p}\rangle = (E_a(\vec{p}) - E_b(\vec{p})\langle a,\vec{p}|\frac{\partial}{\partial p^i}|b,\vec{p}\rangle \qquad (a \neq b)$$

Combining with Kubo formula and defining $\mathcal{A}_i^{(a)}(\vec{p}) \equiv -i\langle a,\vec{p}|\frac{\partial}{\partial p^i}|a,\vec{p}\rangle$

Berry connection

$$\sigma_{xy} = \frac{e^2}{L^2}\sum_{\vec{p}}\sum_a \epsilon^{ij}\frac{\partial}{\partial p^i}\mathcal{A}_j^{(a)}(\vec{p})$$
$$= \frac{e^2}{2\pi}\int\frac{d^2p}{2\pi}\sum_a \epsilon^{ij}\frac{\partial}{\partial p^i}\mathcal{A}_j^{(a)}(\vec{p})$$

Chern number c_1 !

TKNN formula

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Effective gauge action

Integrating out massive fermions in 3-dimensions

$$S_{\rm eff}(A) \equiv \ln \left[\int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-\int \bar{\psi}(D+m)\psi} \right]$$

Parity anomaly   S. Deser, R. Jackiw, S. Templeton 1982, N. Redlich 1984

$$S_{\rm eff}(A) = i c_{cs} S_{cs}(A) + \cdots$$

$$S_{\rm cs}(A) \equiv \int d^3x \; \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \qquad\qquad c_{cs} = -\frac{1}{8\pi}\frac{m}{|m|}$$

Parity violation of fermion induces Chern-Simons action

Anomalous Hall conductivity from Chern-Simons action

$$S_{\text{eff}}(A) = ic_{cs}\int d^3x\ \epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda$$

Hall conductivity is given by the Chern-Simons coupling c_cs

$$\begin{aligned}\langle j_i\rangle &\equiv \frac{\partial}{\partial A_i}S_{\text{eff}}(A)\\ &= 2c_{cs}\epsilon^{i\nu\lambda}\partial_\nu A_\lambda = 2c_{cs}\epsilon^{ij}E_j\end{aligned}$$

Winding number expression of Chern-Simons coupling

c_{cs} can be obtained from 2-point function with fermion 1-loop

Assuming multi-photon vertex does not contribute

True for continuum theory and Wilson fermion on the lattice

$$c_{cs} = \frac{\epsilon^{\alpha_0\beta_1\alpha_1}}{2\cdot 3!}\left(\frac{\partial}{\partial q_1}\right)_{\beta_1}\int \frac{d^3\ p}{(2\pi)^3} Tr\left[S(p)\Gamma^{(1)}_{\alpha_0}[q_1;p-q_1]S(p-q_1)\Gamma^{(1)}_{\alpha_1}(-q_1;p)\right]$$

$\Gamma^{(1)}_{\mu}[q;p]$: fermion-fermion-photon vertex

p: incoming fermion momentum,

q: incoming photon momentum

Assuming derivative of vertex function does not contribute

True for continuum theory and Wilson fermion on the lattice

$$c_{cs} = -\frac{\epsilon^{\alpha_0\beta_1\alpha_1}}{2\cdot 3!}\int \frac{d^3\ p}{(2\pi)^3} Tr\left[S(p)\Gamma^{(1)}_{\alpha_0}[0;p]\frac{\partial S(p)}{\partial p^{\beta_1}}\Gamma^{(1)}_{\alpha_1}(0;p)\right]$$

Using Ward-Takahashi identity $\Gamma_{\mu}^{(1)}[0;p] = -i \frac{\partial S^{-1}(p)}{\partial p^{\mu}}$

⬇

"Winding number" expression of Chern-Simons coupling

$$c_{cs} = -\frac{\epsilon^{\alpha_0 \beta_1 \alpha_1}}{2 \cdot 3!} \int \frac{d^3\ p}{(2\pi)^3} \mathrm{Tr} \left[S(p) \partial_{\alpha_0} S^{-1}(p) S(p) \partial_{\beta_1} S^{-1}(p) S(p) \partial_{\alpha_1} S^{-1}(p) \right]$$

K. Ishikawa 1984
Golterman, Jansen, Kaplan 1993

- Topological in S(p) has no singularity (true for gapped system)
- Winding number of a map $T^3 \rightarrow S^3$ for Wilson fermion

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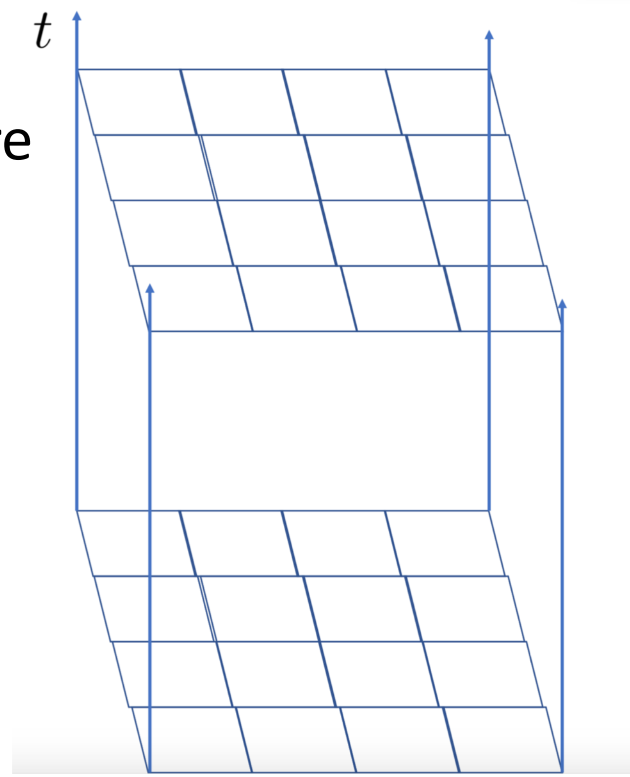
4. Equivalence for general Hamiltonian

Fukaya, T.O., Yamaguchi, Xi
arXiv:1903.11852

Gapped fermion system in D=2n+1 dimensions.

Fermions on 2n dim lattice with continuous time in Euclidean space

$$S_E = \int dt \sum_{\vec{r}} \psi^\dagger(t,\vec{r}) \left[\frac{\partial}{\partial t} + iA_0 + H(\vec{A}) \right] \psi(t,\vec{r})$$

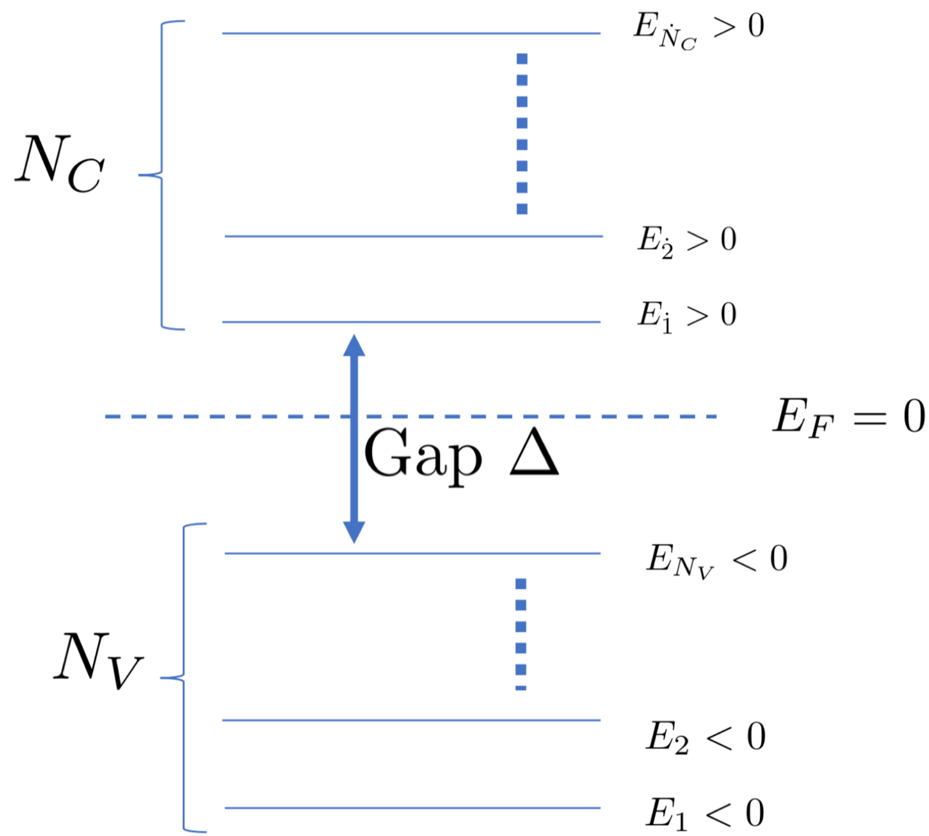


$H(\vec{A})\Big|_{\vec{A}=0}$: translational inv. ➔ band structure

$\psi, \psi^\dagger$: can have many internal DOF
➔ Many bands

No particular structure is assumed
such as relativistic fermion, or Wilson fermion, …..

Energy eigenstates for fixed $\vec{p}$



Nv Valence bands

Nc Conduction bands

Δ: Gap

Effective gauge action

We consider effective action after integrating out fermion

$$e^{S_{\mathrm{eff}}(A)} = \int \mathcal{D}\psi \mathcal{D}\psi^{\dagger} e^{-S_E}$$

Fermion system is gapped
➔ expanded as the sum of gauge inv. local actions

$$S_{\mathrm{eff}}(A) = \sum_k a_k S_k(A)$$

a_k : coefficients
S_k : Gauge inv actions

Chern-Simons action $S_{cs}(A)$

$$S_{\text{eff}}(A) = \cdots + ic_{cs}S_{cs}(A) + \cdots$$

$$S_{cs}(A) = \int d^{2n+1}x \epsilon_{\alpha_0\beta_1\alpha_1\cdots\beta_n\alpha_n} A_{\alpha_0}\partial_{\beta_1}A_{\alpha_1}\cdots\partial_{\beta_n}A_{\alpha_n}$$

Topological action ➔ defined for any geometry of the lattice

Coefficient is quantized due to gauge invariance

$$c_{cs} = \frac{k}{(2\pi)^n(n+1)!}, \quad k \in \mathbb{Z}$$

k = Chern-Simons level

We will see Chern-Simons level k is a toplogical invariant.

c_{cs} can be obtained by differentiating the effective action as

$$c_{cs} = \frac{(-i)^{n+1}\epsilon_{\alpha_0\beta_1\alpha_1\cdots\beta_n\alpha_n}}{(n+1)!(2n+1)!}\left(\frac{\partial}{\partial q_1}\right)_{\beta_1}\cdots\left(\frac{\partial}{\partial q_n}\right)_{\beta_n}$$
$$\times\prod_{i=1}^{n}\int d^{2n+1}x_i e^{iq_i x_i}\left.\frac{\delta^{n+1}S_{\text{eff}}(A)}{\delta A_{\alpha_0}(x_0)\delta A_{\alpha_1}(x_1)\cdots\delta A_{\alpha_n}(x_n)}\right|_{A=0,q_i=0}$$

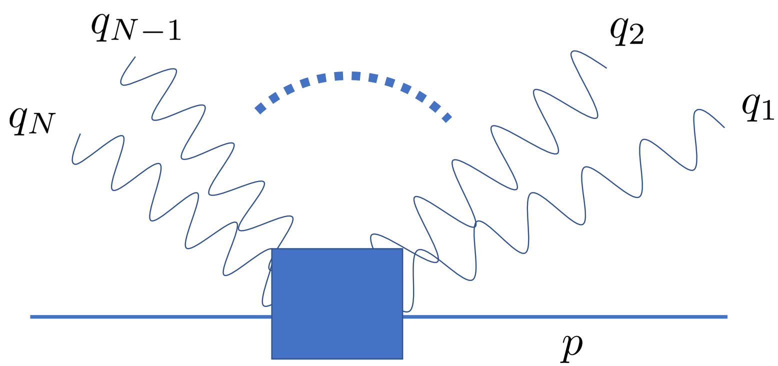
Fermion 1-loop diagram with n+1 external photons

4-1 Chern-Simons level → Winding number

For general Hamiltonian,

Feynman rule can have fermion-fermion-multiphoton vertices

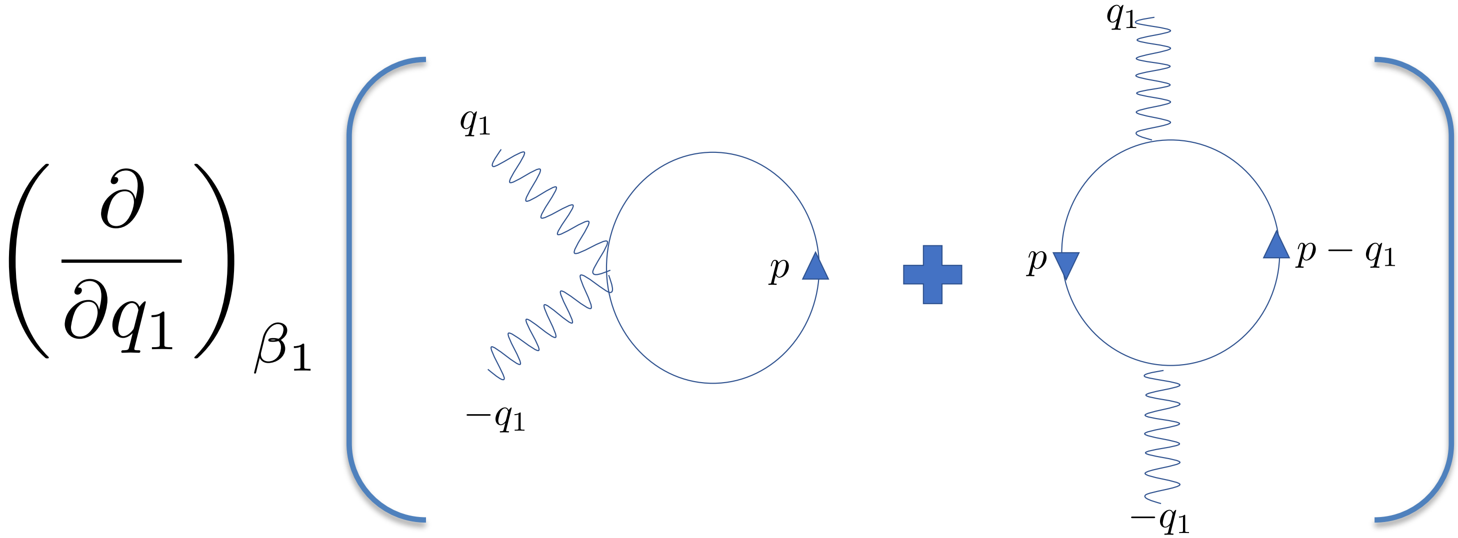
$$\Gamma^{(n)}[q_N, \alpha_N; \cdots ; q_1, \alpha_1; p]$$



➔ 1-loop n-point function from several diagrams in general.

D=2+1 case

$$c_{\rm cs} = -\frac{(-i)^2 \epsilon_{\alpha_0\beta_1\alpha_1}}{2!3!}\int \frac{d^3p}{(2\pi)^3}\left(\frac{\partial}{\partial q_1}\right)_{\beta_1}$$
$$\left\{ {\rm Tr}\left[S_F(p)\Gamma^{(2)}[-q_1,\alpha_0;q_1,\alpha_1;p]\right]\right.$$
$$\left.\left. + \ {\rm Tr}\left[S_F(p-q_1)\Gamma^{(1)}[-q_1,\alpha_0;p]S_F(p)\Gamma^{(1)}[q_1,\alpha_1;p-q_1]\right]\right\}\right|_{q_1=0}$$



Two new contributions in general case

1. Multi-photon vertex contribution → non-zero
2. Momentum derivative of the vertex function → non-zero

These new contributions can give corrections to the winding number expression.

However, one can show that they cancel due to new Ward-Takahashi identities

New Ward-Takahashi identities

Gauge invariant lattice action can be formally expanded by infinite series of covariant derivatives.

Example: $\psi^\dagger(t,\vec{x}) e^{i\int_{\vec{x}}^{\vec{x}+a\vec{\mu}} d\vec{r}'\cdot\vec{A}(\vec{r}')}\psi(t,\vec{x}+a\vec{\mu}) = \psi^\dagger(t,\vec{x})\sum_{n=0}^{\infty}\frac{a^n}{n!}\left(D_\mu^n\psi\right)(t,\vec{x})$

Therefore, formally action can be expressed as

$$S = \int dt \sum_{\vec{x}} \sum_{n=0}^{\infty} \psi^\dagger(t,\vec{x}) M_{\mu_1\cdots\mu_n}(D_{\mu_1}\cdots D_{\mu_n}\psi)(t,\vec{x})$$

Same coefficient M appear in propagator and vertices

Formal expansions of propagator and vertices

Using the coefficients M,

$$S_F^{-1}(p) = \sum_{n=0}^{\infty} M_{\mu_1\cdots\mu_n} \prod_{i=1}^{n} (ip_{\mu_i})$$

$$\Gamma^{(1)}[k,\mu;p] = -i \sum_{n=1}^{\infty} \sum_{a=1}^{n} M_{\mu_1\cdots\mu_{a-1}\mu\mu_{a+1}\cdots\mu_n} \prod_{i=1}^{a-1} (i(p+k)_{\mu_i}) \prod_{i=a+1}^{n} (ip_{\mu_i})$$

$$\begin{aligned}
&\Gamma^{(2)}[k,\mu;l,\nu;p] \\
&= -i^2 \sum_{n=1}^{\infty} \sum_{\substack{a,b=1 \\ a<b}}^{n} M_{\mu_1\cdots\mu_{a-1}\mu\mu_{a+1}\cdots\mu_{b-1}\nu\mu_{b+1}\cdots\mu_n} \prod_{i=1}^{a-1} (i(p+k+l)_{\mu_i}) \prod_{i=a+1}^{b-1} (i(p+l)_{\mu_i}) \prod_{i=b+1}^{n} (ip_{\mu_i}) \\
&- i^2 \sum_{n=1}^{\infty} \sum_{\substack{a,b=1 \\ a<b}}^{n} M_{\mu_1\cdots\mu_{a-1}\nu\mu_{a+1}\cdots\mu_{b-1}\mu\mu_{b+1}\cdots\mu_n} \prod_{i=1}^{a-1} (i(p+k+l)_{\mu_i}) \prod_{i=a+1}^{b-1} (i(p+k)_{\mu_i}) \prod_{i=b+1}^{n} (ip_{\mu_i})
\end{aligned}$$

New Ward-Takahashi identity

The formal expression reproduces usual Ward-Takahashi identities.

In addition, one also obtains the following 2^{nd} order W-T identity

$$\left.\frac{\partial^2\Gamma^{(1)}[k,\mu;p]}{\partial k_\nu \partial p_\lambda}\right|_{k=0} = \left.\frac{\partial\Gamma^{(2)}[k,\mu;0,\lambda;p]}{\partial k_\nu}\right|_{k=0} = \left.\frac{\partial\Gamma^{(2)}[0,\lambda;l,\mu;p]}{\partial l_\nu}\right|_{l=0}$$

1st derivative of the two-photon vertex with respect to momentum is related to 2^{nd} derivative of the single photon vertex.

↓

Correction terms to 1-loop expression is shown to be total derivatives and vanish.

Therefore, for general Hamiltonian we obtain

$$c_{\mathrm{cs}} = \frac{(-i)^2 \epsilon_{\alpha_0 \beta_1 \alpha_1}}{2!3!} \int \frac{dp_0}{2\pi} \int_{\mathrm{BZ}} \frac{d^2 p}{(2\pi)^2}$$
$$\times \operatorname{Tr}\left[S_F(p) \frac{\partial S_F^{-1}(p)}{\partial p_{\alpha_0}} S_F(p) \frac{\partial S_F^{-1}(p)}{\partial p_{\beta_1}} S_F(p) \frac{\partial S_F^{-1}(p)}{\partial p_{\alpha_1}} \right]$$

D=4+1 case

Chern-Simons coupling is obtained from 3-point 1-loop diagrams.

For general Hamiltonian, multi-photon couplings can contribute and momentum derivative of vertex functions do not vanish.

However, one can derive new 3rd order WT-identity from formal expansion of the action in terms of covariant derivatives as

$$\left.\frac{\partial^2\Gamma^{(3)}[q,\mu;r,\nu;s,\lambda;p]}{\partial q_\alpha \partial r_\beta}\right|_{q,r,s=0} = \left.\frac{\partial^3\Gamma^{(2)}[q,\mu;r,\nu;p]}{\partial q_\alpha \partial r_\beta \partial p_\lambda}\right|_{q,r=0}$$

Using previous 2nd order WT-identity and new 3rd order WT-identity

One can show correction terms cancel and c_cs is given by winding number expressions as

$$c_{\rm cs} = -\frac{(-i)^3 \cdot 2}{3!5!} \int \frac{d^5 p}{(2\pi)^5} \epsilon_{\alpha_0 \beta_1 \alpha_1 \beta_2 \alpha_2}$$
$$\times \,{\rm Tr}\left[S_F(p) \frac{\partial S_F^{-1}(p)}{\partial p_{\alpha_0}} S_F(p) \frac{\partial S_F^{-1}(p)}{\partial p_{\beta_1}} S_F(p) \frac{\partial S_F^{-1}(p)}{\partial p_{\alpha_1}} S_F(p) \frac{\partial S_F^{-1}(p)}{\partial p_{\beta_2}} S_F(p) \frac{\partial S_F^{-1}(p)}{\partial p_{\alpha_2}} \right].$$

4-2 Winding number → TKNN formula

This part was essentially already given by

Qi, Hughes, Zhang , Phys. Rev. B78, 195424, 2008

Idea : Evaluate the winding number expression as follows

1. Rewrite the fermion propagator using eigenstates

$$S(p) = \sum_{\alpha} |\alpha, \vec{p}\rangle \frac{1}{ip^0 + E_\alpha(\vec{p})} \langle \alpha, \vec{p}|$$

2. Continuously deform only the eigenvalues to degenerate flat band

$$E_\alpha(\vec{p})(< 0) \longrightarrow E_v = \text{constant}$$
$$E_\alpha(\vec{p})(> 0) \longrightarrow E_c = \text{constant}$$

3. Carry out momentum integral over p^0

Step 1

Winding number expression can be rewritten as

$$c_{\rm cs} = -\frac{n! \cdot (2n+1)(-i)^{n+1}\epsilon^{i_1 i_2 \cdots i_{2n}}}{(n+1)!(2n+1)!}\int \frac{d^{2n}p}{(2\pi)^{2n}}\int \frac{dp^0}{2\pi}{\rm Tr}\left[\frac{1}{ip^0+H}\ i\ \prod_{k=1}^{2n}\left(\frac{1}{ip^0+H}(\partial_{i_k}H)\right)\right]$$

Inserting complete set of energies eigenstates, one obtains

$$c_{\rm cs} = \frac{n!(-i)^{n+2}}{(n+1)!(2n)!}\int \frac{d^{2n}p}{(2\pi)^{2n}} J$$

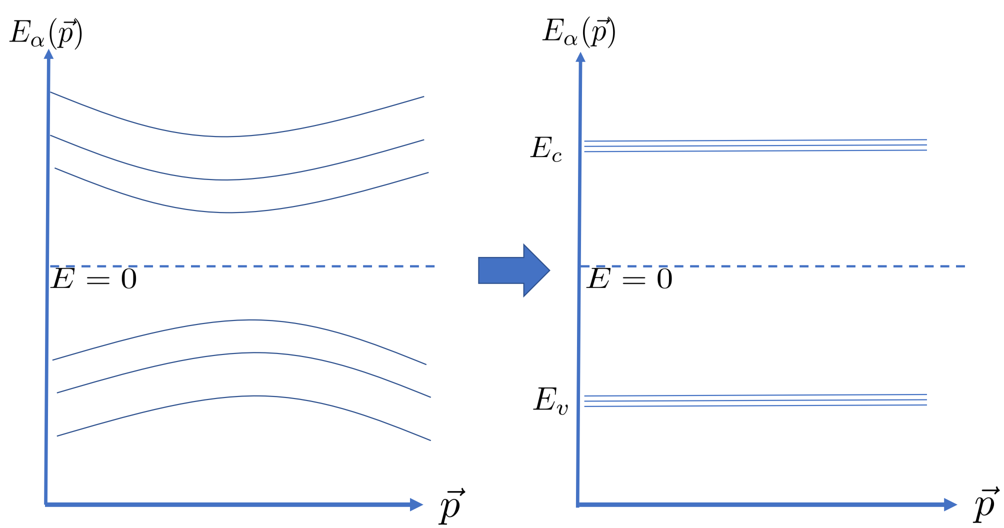
$$J = \sum_{\alpha_1,\cdots,\alpha_{2n}} \epsilon^{i_1 i_2 \cdots i_{2n}} \int \frac{dp^0}{2\pi}\frac{\langle \alpha_1 \partial_{i_1} H|\alpha_2\rangle\langle\alpha_2|\partial_{i_2}H|\alpha_3\rangle\cdots\langle\alpha_{2n}|\partial_l H|\alpha_1\rangle}{(ip^0+E_{\alpha_1})^2(ip^0+E_{\alpha_2})\cdots(ip^0+E_{\alpha_{2n}})}$$

Step 2

Continuously deform Hamiltonian by changing only the eigenvalues keeping the gap to degenerate flat band.

$$H(\vec{p}) \equiv \sum_{a=1}^{N_v} E_a(\vec{p})|a(\vec{p})\rangle\langle a(\vec{p})| + \sum_{\dot{b}=1}^{N_c} E_{\dot{b}}(\vec{p})|\dot{b}(\vec{p})\rangle\langle \dot{b}(\vec{p})|$$

$$H_{\rm new}(\vec{p}) = E_v \sum_{a=1}^{N_v} |a(\vec{p})\rangle\langle a(\vec{p})| + E_c \sum_{\dot{b}=1}^{N_c} |\dot{b}(\vec{p})\rangle\langle \dot{b}(\vec{p})|$$

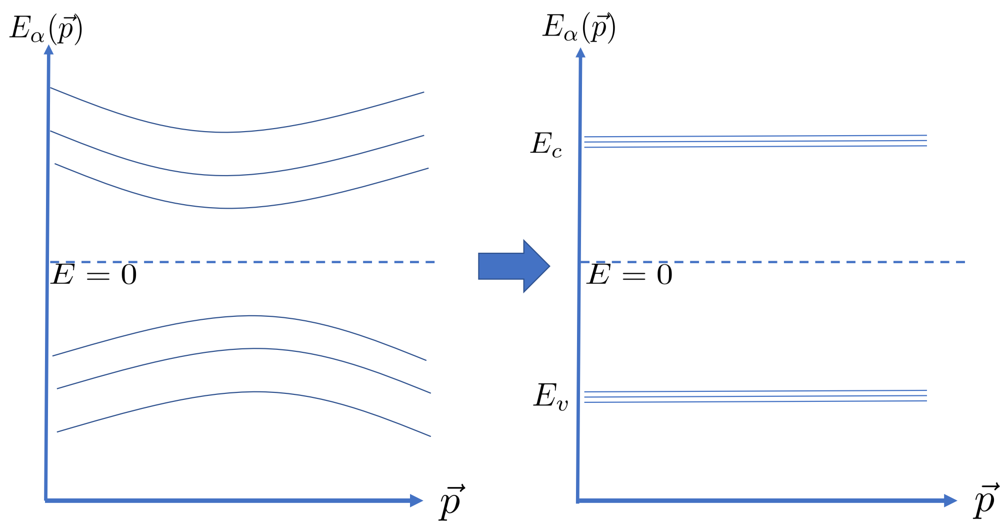


Step 2

Continuously deform Hamiltonian by changing only the eigenvalues keeping the gap to degenerate flat band.

$$H(\vec{p}) \equiv \sum_{a=1}^{N_v} E_a(\vec{p}) |a(\vec{p})\rangle\langle a(\vec{p})| + \sum_{\dot{b}=1}^{N_c} E_{\dot{b}}(\vec{p}) |\dot{b}(\vec{p})\rangle\langle \dot{b}(\vec{p})|$$

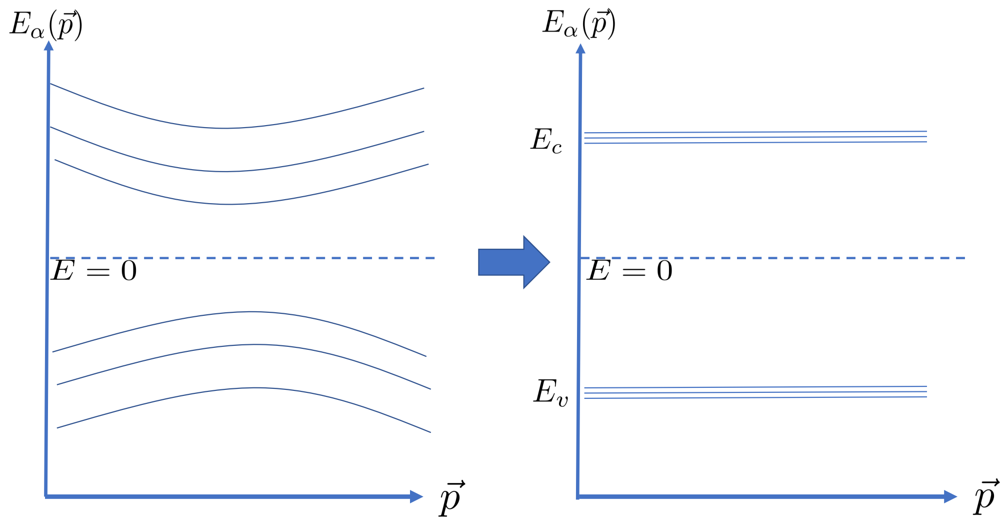
$$H_{\mathrm{new}}(\vec{p}) = E_v \sum_{a=1}^{N_v} |a(\vec{p})\rangle\langle a(\vec{p})| + E_c \sum_{\dot{b}=1}^{N_c} |\dot{b}(\vec{p})\rangle\langle \dot{b}(\vec{p})|$$



Continuously deform Hamiltonian by changing only the eigenvalues keeping the gap to degenerate flat band.

$$H(\vec{p}) \equiv \sum_{a=1}^{N_v} E_a(\vec{p})|a(\vec{p})\rangle\langle a(\vec{p})| + \sum_{\dot{b}=1}^{N_c} E_{\dot{b}}(\vec{p})|\dot{b}(\vec{p})\rangle\langle \dot{b}(\vec{p})|$$

$$H_{\text{new}}(\vec{p}) = E_v \sum_{a=1}^{N_v} |a(\vec{p})\rangle\langle a(\vec{p})| + E_c \sum_{\dot{b}=1}^{N_c} |\dot{b}(\vec{p})\rangle\langle \dot{b}(\vec{p})|$$



Useful formulae

$$\langle a(\vec{p})|\partial_\mu H(\vec{p})|b(\vec{p})\rangle = 0, \qquad \langle \dot{a}(\vec{p})|\partial_\mu H(\vec{p})|\dot{b}(\vec{p})\rangle = 0,$$
$$\langle a(\vec{p})|\partial_\mu H(\vec{p})|\dot{b}(\vec{p})\rangle = (E_c - E_v)\langle a|\partial_\mu \dot{b}\rangle,$$
$$\langle \dot{a}(\vec{p})|\partial_\mu H(\vec{p})|b(\vec{p})\rangle = -(E_c - E_v)\langle \dot{a}|\partial_\mu b\rangle,$$
$$(a, b = 1, \cdots, N_v, \quad \dot{a}, \dot{b} = 1, \cdots, N_c).$$

shows that inserted states should be valence and conduction band appearing alternately.

$$J = \sum_{a_1,\cdots,a_n=1}^{N_v} \sum_{\dot{a}_1,\cdots,\dot{a}_n=1}^{N_c} \epsilon^{i_1 j_1 \cdots i_{2n} j_{2n}}$$
$$\Big[\int \frac{dp^0}{2\pi} \frac{1}{(ip^0+E_v)^{n+1}(ip^0+E_c)^n} \langle a_1|\partial_{i_1} H|\dot{a}_1\rangle\langle \dot{a}_1|\partial_{j_1} H|a_2\rangle \times \cdots \times \langle a_n|\partial_{i_n} H|\dot{a}_n\rangle\langle \dot{a}_n|\partial_{j_n} H|a_1\rangle$$
$$+ \int \frac{dp^0}{2\pi} \frac{1}{(ip^0+E_c)^{n+1}(ip^0+E_v)^n} \langle \dot{a}_1|\partial_{i_1} H|a_1\rangle\langle a_1|\partial_{j_1} H|\dot{a}_2\rangle \times \cdots \times \langle \dot{a}_n|\partial_{i_n} H|a_n\rangle\langle a_n|\partial_{j_n} H|\dot{a}_1\rangle\Big]$$

Step 3

p^0 integration can be easily carried out by Cauchy integral

$$J = \sum_{a_1,\cdots,a_n=1}^{N_v} \sum_{\dot{a}_1,\cdots,\dot{a}_n=1}^{N_c} \epsilon^{i_1 j_1 \cdots i_{2n} j_{2n}} (-1)^{n+1} \frac{(2n)!}{(n!)^2}$$
$$\times \langle a_1 | \partial_{i_1} \dot{a}_1 \rangle \langle \dot{a}_1 | \partial_{j_1} a_2 \rangle \times \cdots \times \langle a_n | \partial_{i_n} \dot{a}_n \rangle \langle \dot{a}_n | \partial_{j_n} a_1 \rangle.$$

Inserting this expression into c_cs using J, and using the definition of the Berry curvature (skipping detail) one obtains

$$c_{\rm cs} \equiv \frac{k}{(n+1)!(2\pi)^n} = \frac{(-1)^n}{(n+1)!(2\pi)^n} \int_{BZ} {\rm ch}_n(\mathcal{A}),$$
$${\rm ch}_n(\mathcal{A}) = \frac{1}{n!} \frac{1}{(2\pi)^n} {\rm tr}(\mathcal{F}^n)$$

This result shows that

Chern-Simons level in field theory approach

and

Chern number in microscopic approach (TKNN)

are identical for general Hamiltonian bilinear in fermion for D=2+1, 4+1 dimensions.

Outline

- ✔ 1. Introduction
- ✔ 2. Review of TKNN formula
- ✔ 3. Review of field theory approach
- ✔ 4. Equivalence for general Hamiltonian
    1. Chern-Simons level → Winding number
    2. Winding number → TKNN formula
- 5. Summary

5. Summary

- We have shown microscopic approach (TKNN) and field theory approach give identical topological number for general Hamiltonian bilinear in fermion.
- A series of Ward-Takahashi identities are crucial to show the equivalence.
- One should note no other details beyond gauge symmetry (such as existence of relativistic field theory at low energy) is needed.

- In 4+1 dimensions, there are two independent Chern numbers. However, only a particular Chern number appeared.

- This means that topological classification in microscopic approach may be finer, or those detailed structure may not be robust.

- It would be interesting to see similar equivalence holds or not for other cases such as systems with higher symmetry or systems with interacting fermions.

Back-up

1. Introduction

Topological insulators & Domain-wall fermion in D=2n+1 dimension

Very closely related

- Characterized by topology
- Mass Gap in the bulk
- Bulk-Edge Correspondence ⇔ Gauge symmetry