Implementing the three-particle quantization condition: a progress report

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3-particle papers

<u>Max Hansen & SRS</u>:

"Relativistic, model-independent, three-particle quantization condition,"

arXiv:1408.5933 (PRD)

"Expressing the 3-particle finite-volume spectrum in terms of the 3-to-3 scattering amplitude,"

arXiv:1504.04028 (PRD)

"Perturbative results for 2- & 3-particle threshold energies in finite volume,"

arXiv:1509.07929 (PRD)

"Threshold expansion of the 3-particle quantization condition,"

arXiv:1602.00324 (PRD)

"Applying the relativistic quantization condition to a 3-particle bound state in a periodic box,"

arXiv: 1609.04317 (PRD)

"Lattice QCD and three-particle decays of Resonances,"

arXiv: 1901.00483 (to appear in Ann. Rev. Nucl. Part. Science)

Raúl Briceño, Max Hansen & SRS:

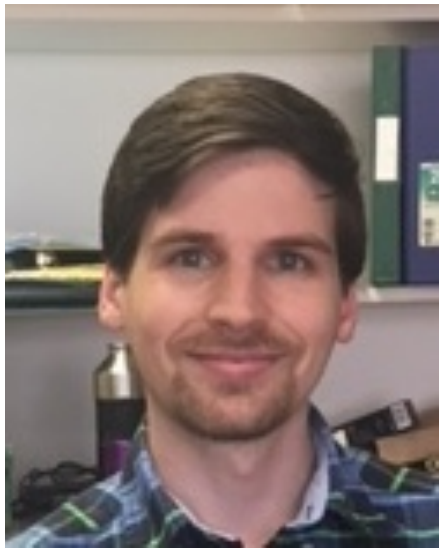
"Relating the finite-volume spectrum and the 2-and-3-particle S-matrix for relativistic systems of identical scalar particles," arXiv:1701.07465 (PRD)

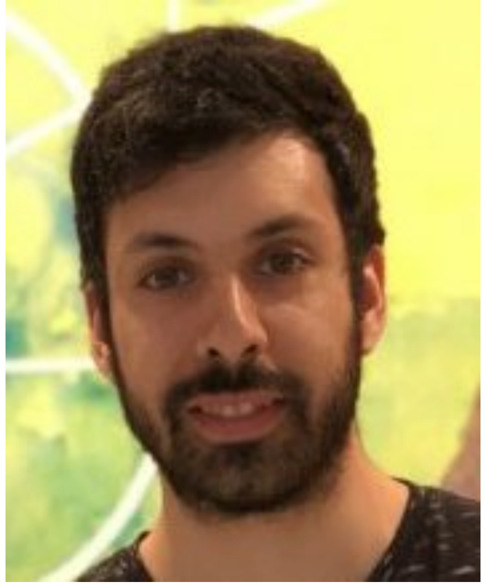
"Numerical study of the relativistic three-body quantization condition in the isotropic approximation," arXiv:1803.04169 (PRD)

"Three-particle systems with resonant sub-processes in a finite volume," arXiv:1810.01429 (PRD)

SRS

"Testing the threshold expansion for three-particle energies at fourth order in φ^4 theory," arXiv:1707.04279 (PRD)





Tyler Blanton, Fernando Romero-López & SRS:

"Implementing the three-particle quantization condition including higher partial waves," arXiv:1901.07095 (JHEP)

Tyler Blanton, Raúl Briceño, Max Hansen, Fernando Romero-López, SRS &Adam Szczepaniak:

works in progress

Outline

- Motivations for studying 3 (or more) particles
- Status of theoretical formalism for 2 and 3 particles
- Numerical implementation of 3-particle QC
  - Isotropic approximation
  - Including higher partial waves
  - Isotropic approx. v2: including two-particle bound states
- Outlook

Motivations for studying three (or more) particles using LQCD

Studying resonances

- Most resonances have 3 (or more) particle decay channels
    - $\omega(782, I^G J^{PC} = 0^- 1^{--}) \to 3\pi$ (no subchannel resonances)
    - $a_2(1320, I^G J^{PC} = 1^- 2^{++}) \to \rho\pi \to 3\pi$
    - Roper: $N(1440) \to \Delta\pi \to N\pi\pi$ (branching ratio 25-50%)
    - $X(3872) \to J/\Psi\pi\pi$
    - $Z_c(3900) \to \pi J/\psi, \pi\pi\eta_c, \bar{D}D^*$ (studied by HALQCD—talk by Ikeda)

- N.B. If a resonance has both 2- and 3-particle strong decays, then 2-particle methods fail—channels cannot be separated as they can in experiment

Weak decays

- Calculating weak decay amplitudes/form factors involving 3 particles, e.g. $K \to \pi\pi\pi$
- N.B. Can study weak $K \to 2\pi$ decays independently of $K \to 3\pi$, since strong interactions do not mix these final states (in isospin-symmetric limit)

A more distant motivation

Observation of CP violation in charm decays

CERN-EP-2019-042
13 March 2019

LHCb collaboration†

Abstract

A search for charge-parity (CP) violation in $D^0 \to K^- K^+$ and $D^0 \to \pi^- \pi^+$ decays is reported, using pp collision data corresponding to an integrated luminosity of $6\,\mathrm{fb}^{-1}$ collected at a center-of-mass energy of $13\,\mathrm{TeV}$ with the LHCb detector. The flavor of the charm meson is inferred from the charge of the pion in $D^*(2010)^+ \to D^0\pi^+$ decays or from the charge of the muon in $\overline{B} \to D^0\mu^-\bar{\nu}_\mu X$ decays. The difference between the CP asymmetries in $D^0 \to K^- K^+$ and $D^0 \to \pi^- \pi^+$ decays is measured to be $\Delta A_{CP} = [-18.2 \pm 3.2\,(\text{stat.}) \pm 0.9\,(\text{syst.})] \times 10^{-4}$ for π-tagged and $\Delta A_{CP} = [-9 \pm 8\,(\text{stat.}) \pm 5\,(\text{syst.})] \times 10^{-4}$ for μ-tagged D^0 mesons. Combining these with previous LHCb results leads to

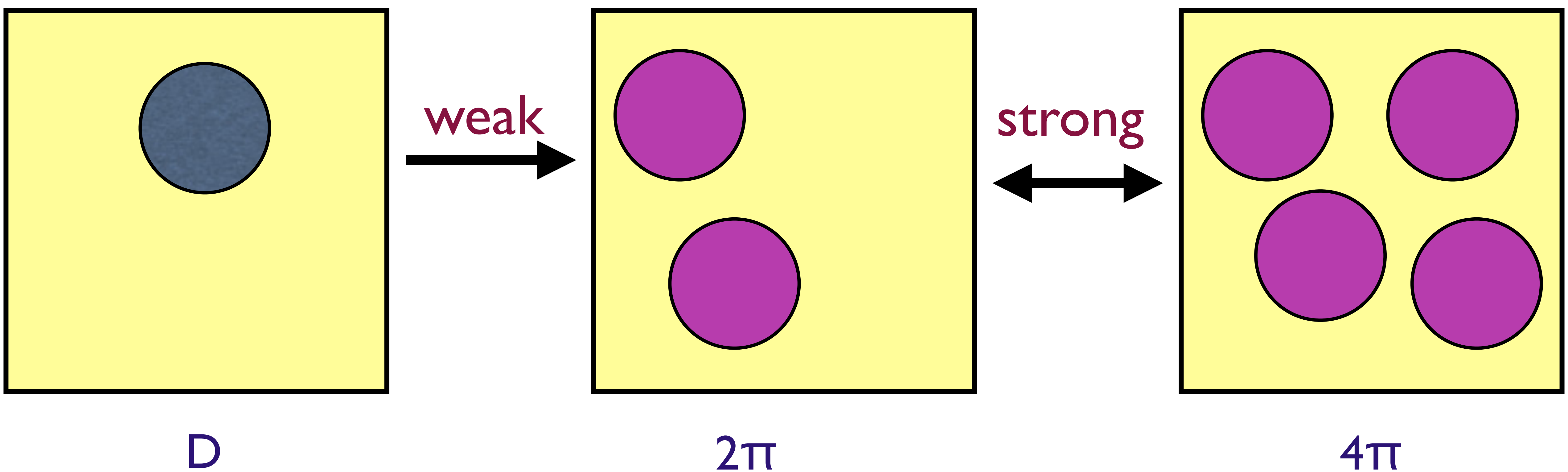
$$\Delta A_{CP} = (-15.4 \pm 2.9) \times 10^{-4},$$

where the uncertainty includes both statistical and systematic contributions. The measured value differs from zero by more than five standard deviations. This is the first observation of CP violation in the decay of charm hadrons.

5.3σ effect

A more distant motivation

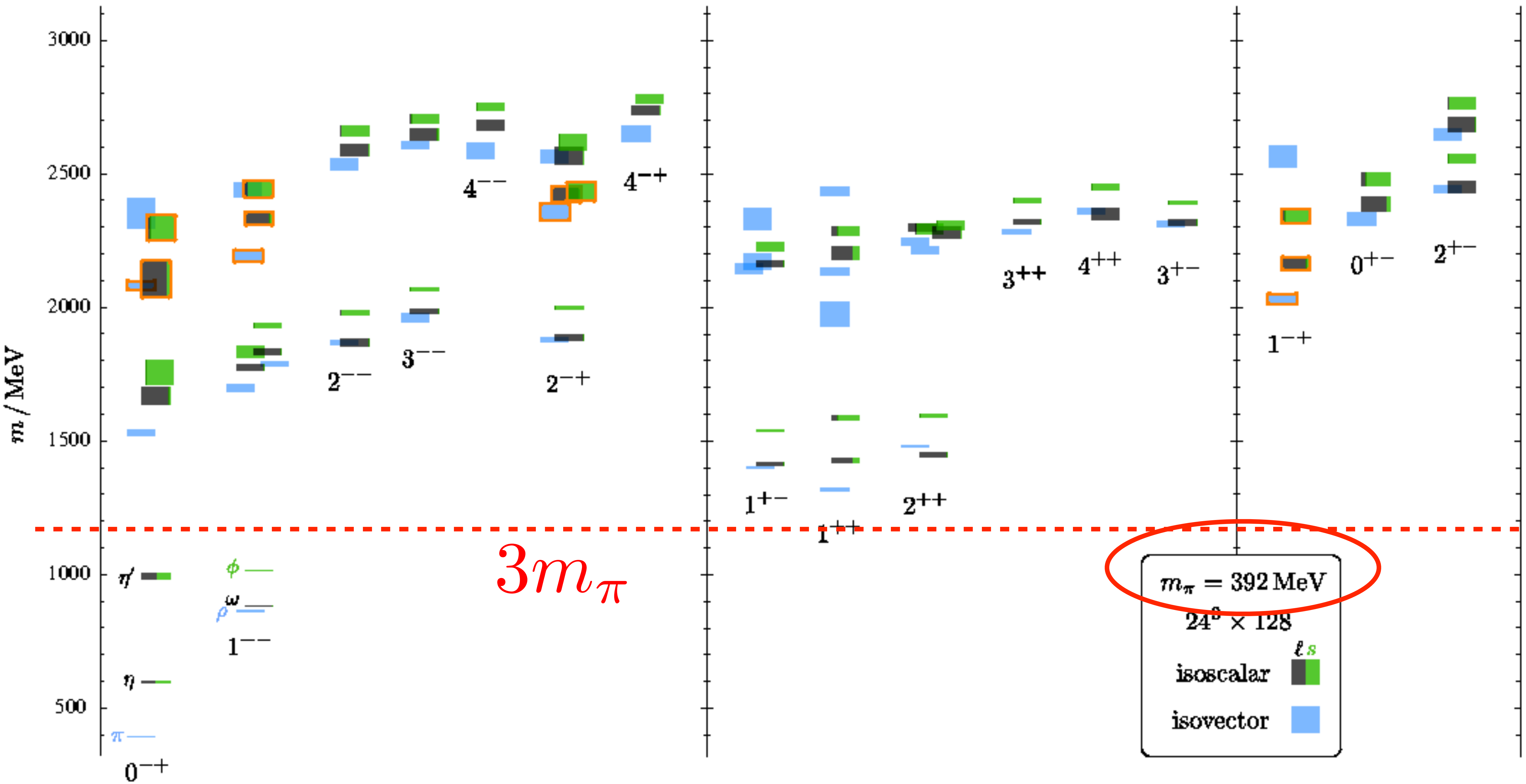
- Calculating CP-violation in $D \to \pi\pi, K\bar{K}$ in the Standard Model
- Finite-volume state is a mix of $2\pi, K\bar{K}, \eta\eta, 4\pi, 6\pi, \ldots$
- Need 4 (or more) particles in the box!



3-body interactions

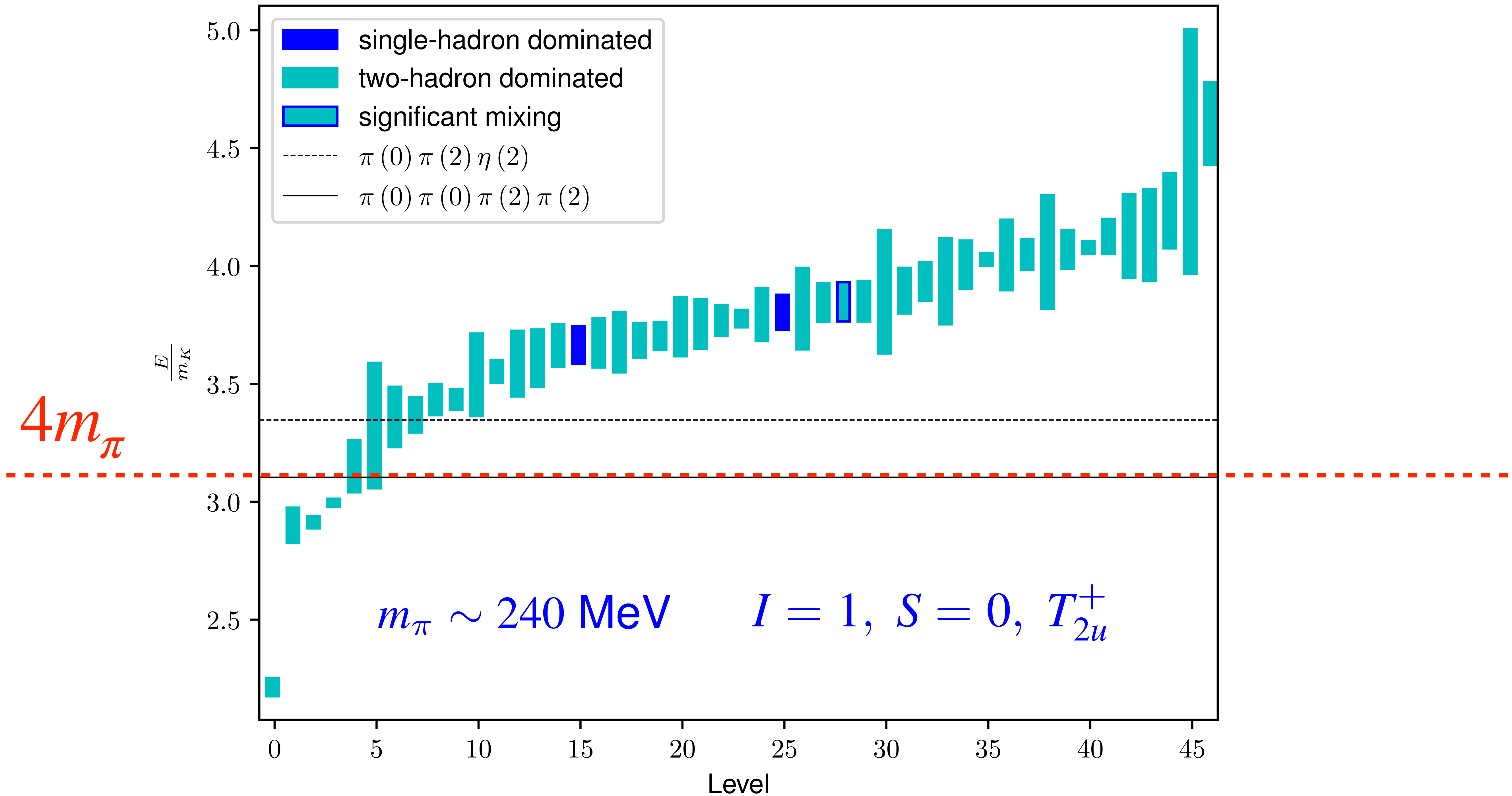
- Determining NNN interaction
  - Input for effective field theory treatments of larger nuclei & nuclear matter
- Similarly, $\pi\pi\pi$, $\pi K\bar{K}$, … interactions needed for study of pion/kaon condensation

LQCD spectrum already includes 3+ particle states



Dudek, Edwards, Guo & C.Thomas [HadSpec], arXiv:1309.2608

LQCD spectrum already includes 3+ particle states

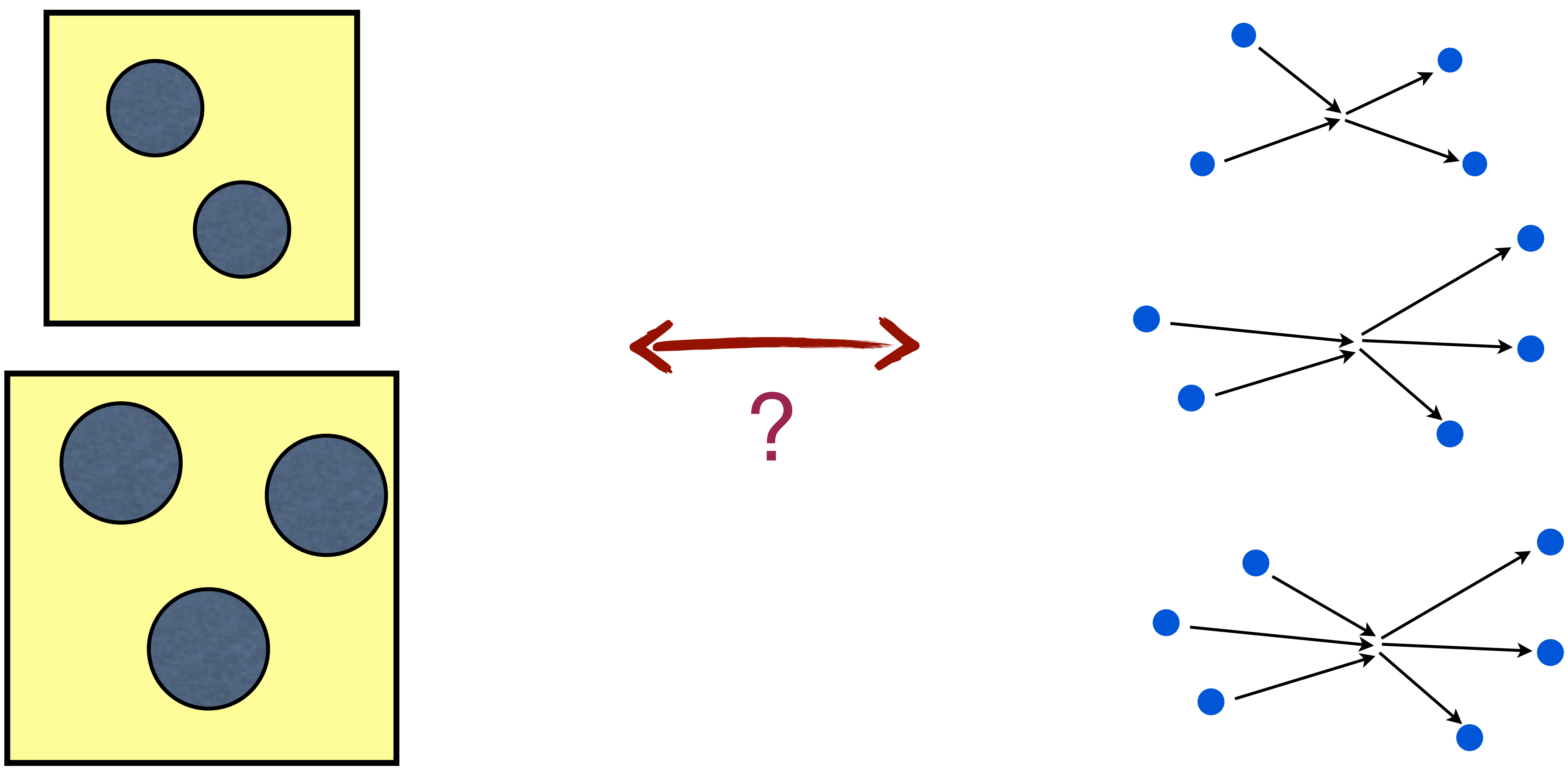


Slide from seminar by Colin Morningstar, Munich, 10/18

Status of theoretical formalism for 2 & 3 particles

The fundamental issue

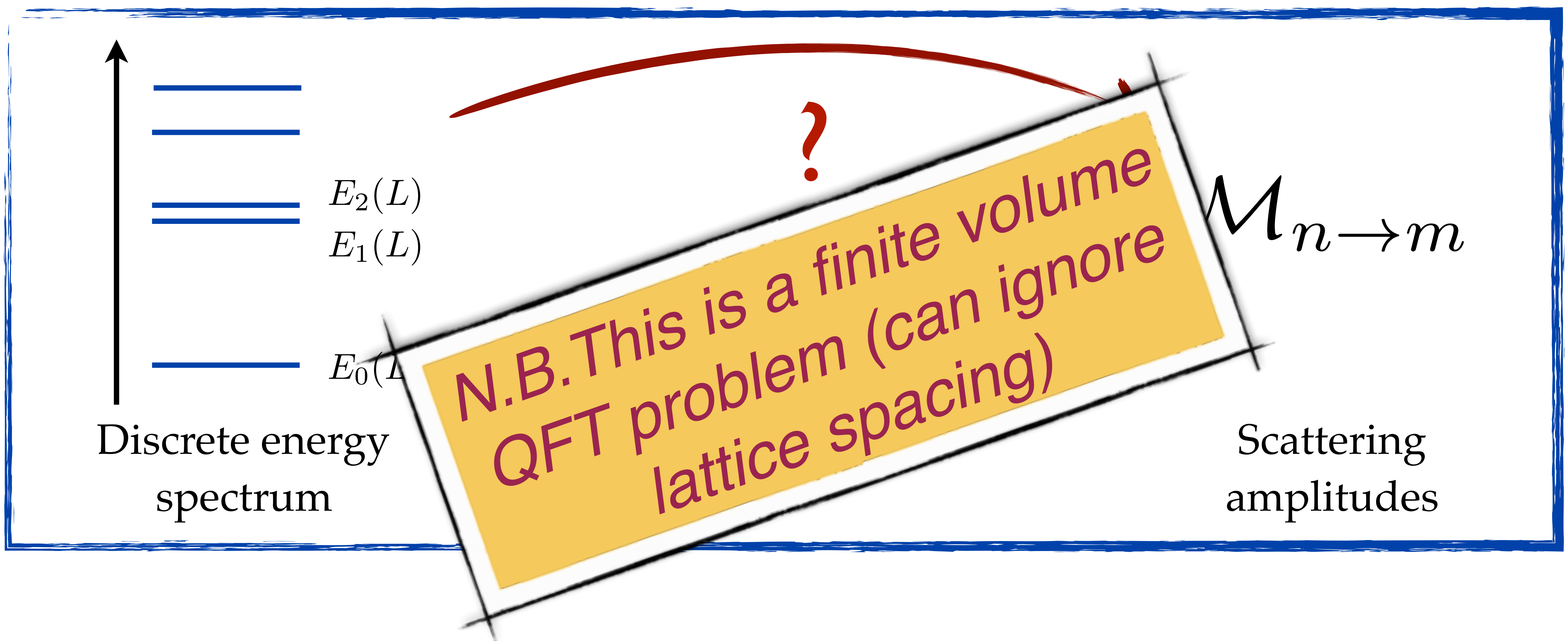
- Lattice simulations are done in finite volumes; experiments are not



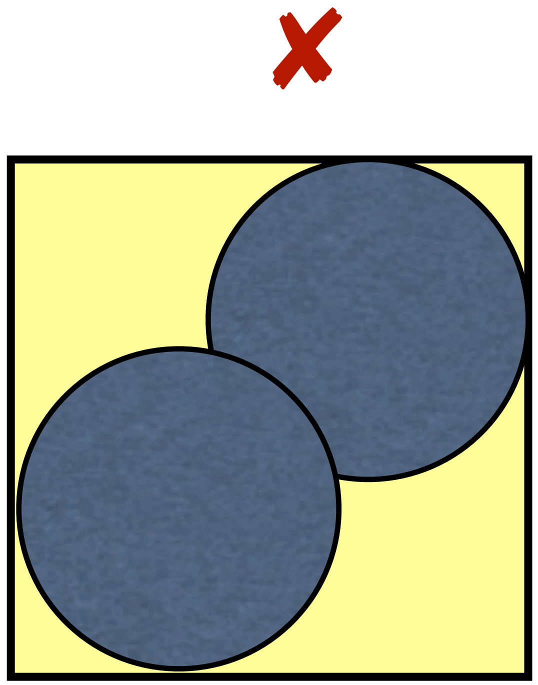
How do we connect these?

The fundamental issue

- Lattice QCD can calculate energy levels of multiparticle systems in a box
- How are these related to infinite-volume scattering amplitudes (which determine resonance properties)?



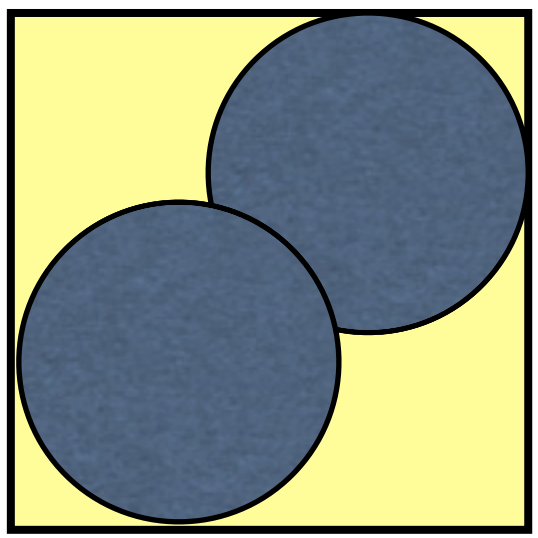
When is the spectrum related to scattering amplitudes?



L<2R
No "outside" region.
Spectrum NOT related to scatt. amps.
Depends on finite-density properties

When is the spectrum related to scattering amplitudes?

✘



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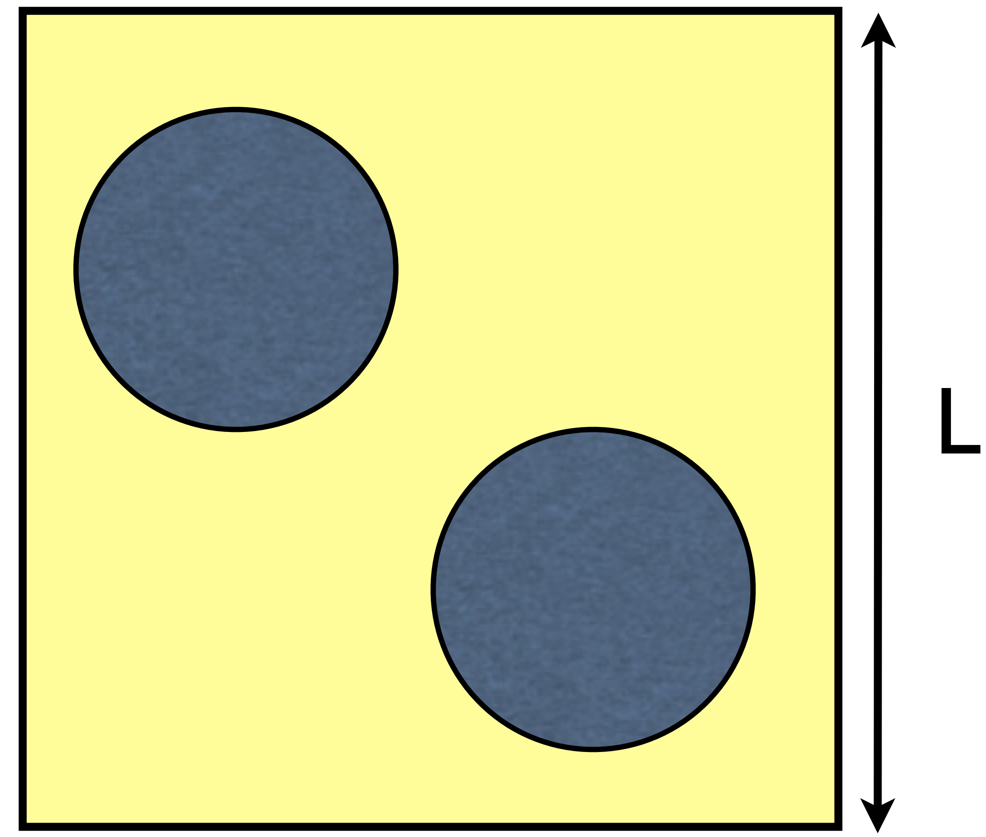
✔

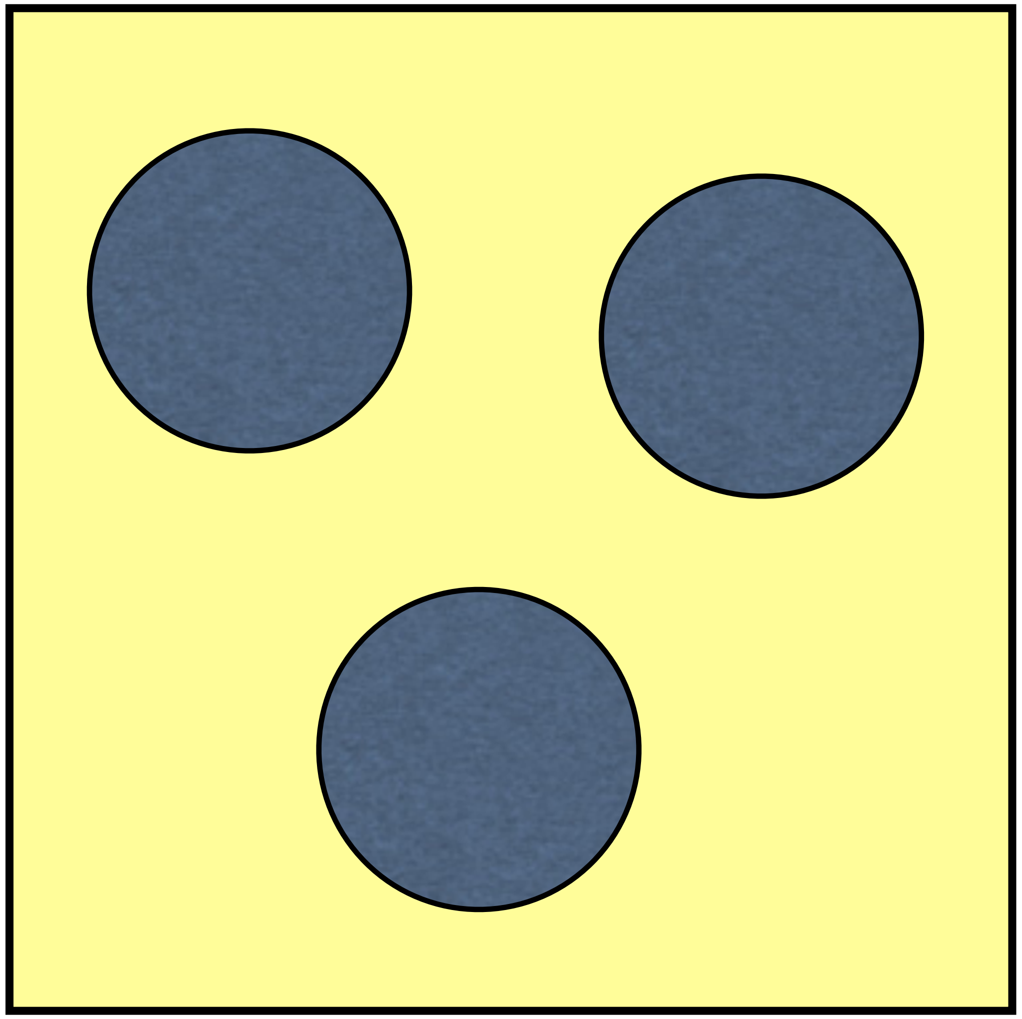


L>2R
There is an "outside" region.
Spectrum IS related to scatt. amps.
up to corrections proportional to $e^{-M_\pi L}$

[Lüscher]

Theoretically understood;
numerical implementations mature.

…and for 3 particles?



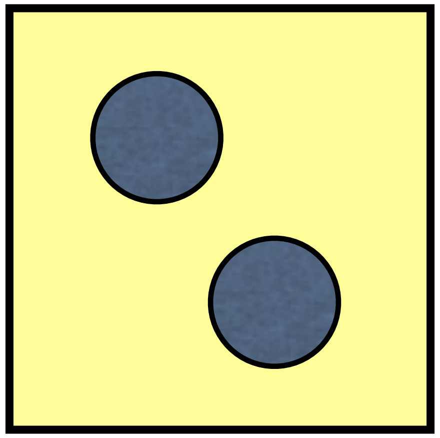
- Spectrum IS related to 2→2, 2→3 & 3→3 scattering amplitudes up to corrections proportional to e^{-ML} [Polejaeva & Rusetsky]

- Formalism developed in a generic relativistic EFT [Hansen & SRS, Briceño, Hansen & SRS]

- Alternative approaches based on NREFT [Hammer, Pang & Rusetsky] and on ``finite-volume unitarity" [Döring & Mai] under development (reviewed in [Hansen & SRS])

- HALQCD approach can be extended to 3 particles in NR domain [Aoki et al.]

Reminder of 2-particle quantization condition



Single-channel 2-particle quantization condition

[Lüscher 86 & 91; Rummukainen & Gottlieb 85; Kim, Sachrajda & SRS 05; …]

- Two particles (say pions) in cubic box of size L with PBC and total momentum **P**
- Below inelastic threshold (4 pions), the finite-volume spectrum E_1, E_2, ... is given by solutions to a equation in partial-wave (*l,m)* space (up to exponentially suppressed corrections)

$$\boxed{\det\left[(F_{\widetilde{\mathrm{PV}}})^{-1}+\mathcal{K}_2\right]=0}$$

- $\mathcal{K}_2 \sim \tan\delta/q$ is the K-matrix, which is diagonal in *l,m*
- F_{PV} is a known kinematical "zeta-function", depending on the box shape & E; It is off-diagonal in *l,m*, since the box violates rotation symmetry

Single-channel 2-particle quantization condition

$$\det\left[(F_{\rm PV})^{-1} + \mathcal{K}_2\right] = 0$$

- Infinite-dimensional determinant must be truncated to be practical; truncate by assuming that $\mathcal{K}_2$ vanishes above l_{max}
- If l_{max}=0, obtain one-to-one relation between energy levels and $\mathcal{K}_2$

$$E_n^* = \sqrt{E_n^2 - \vec{P}^2}$$

CM energy

$$\mathcal{K}_{2,s}(E_n^*) = -\frac{1}{F_{\rm PV;00;00}(E_n, \vec{P}, L)}$$

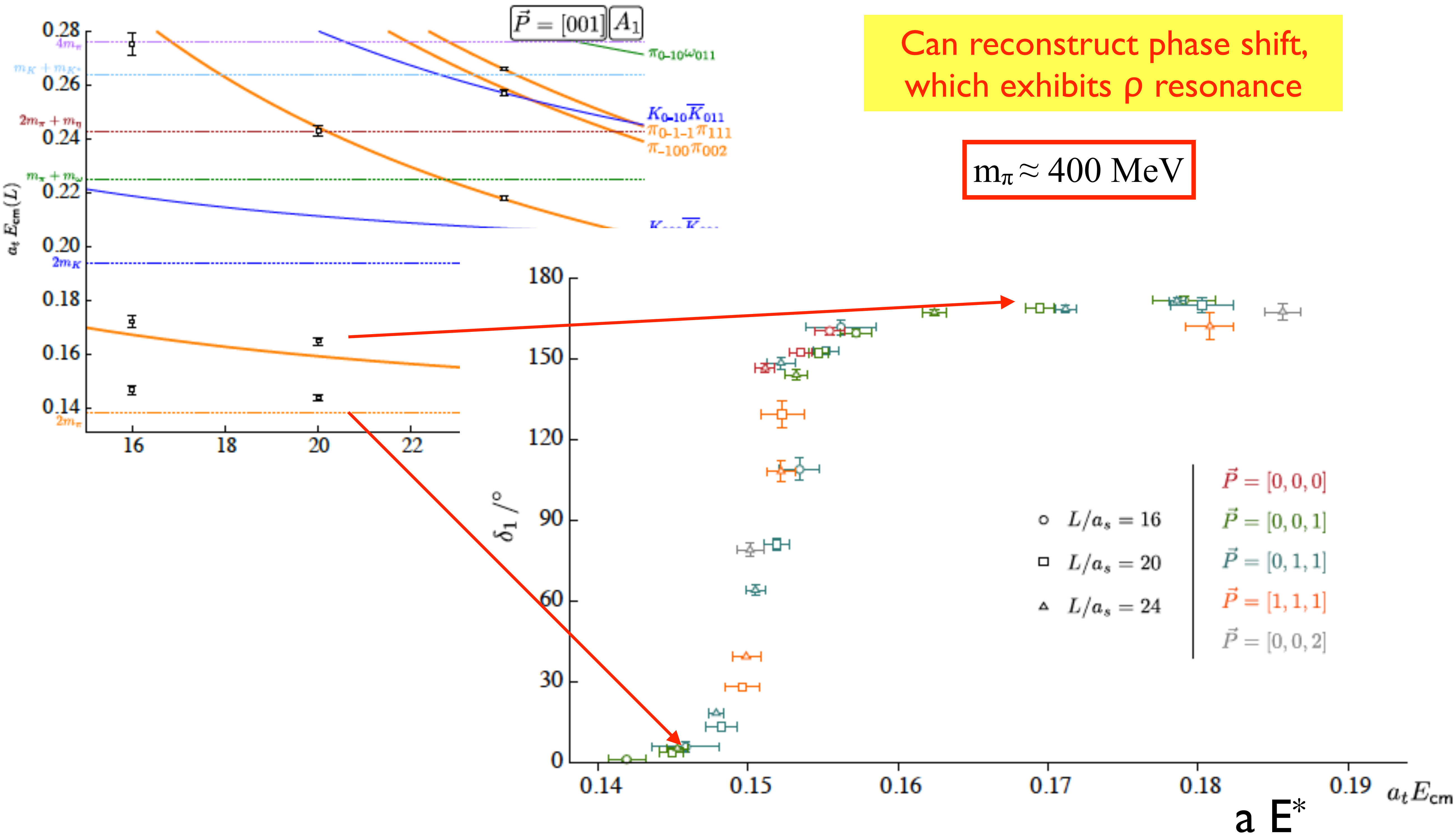
"measured" energy-level

Application to ρ meson

[Dudek, Edwards & Thomas, 1212.0830]

Can reconstruct phase shift, which exhibits ρ resonance

$m_\pi \approx 400$ MeV

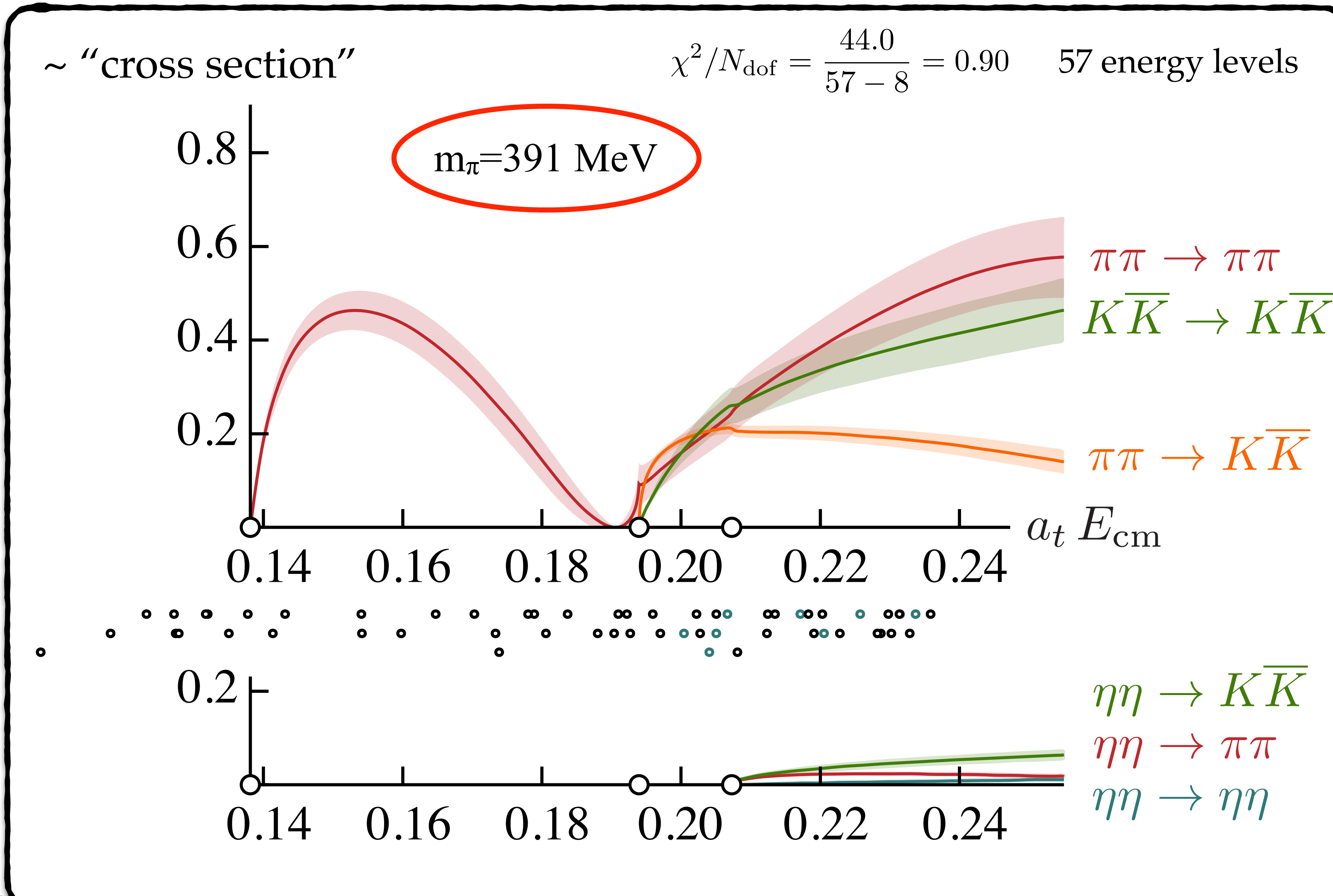


State-of-the-art: coupled channels

- S-wave above $2m_\pi$, $2m_K$, and $2m_\eta$

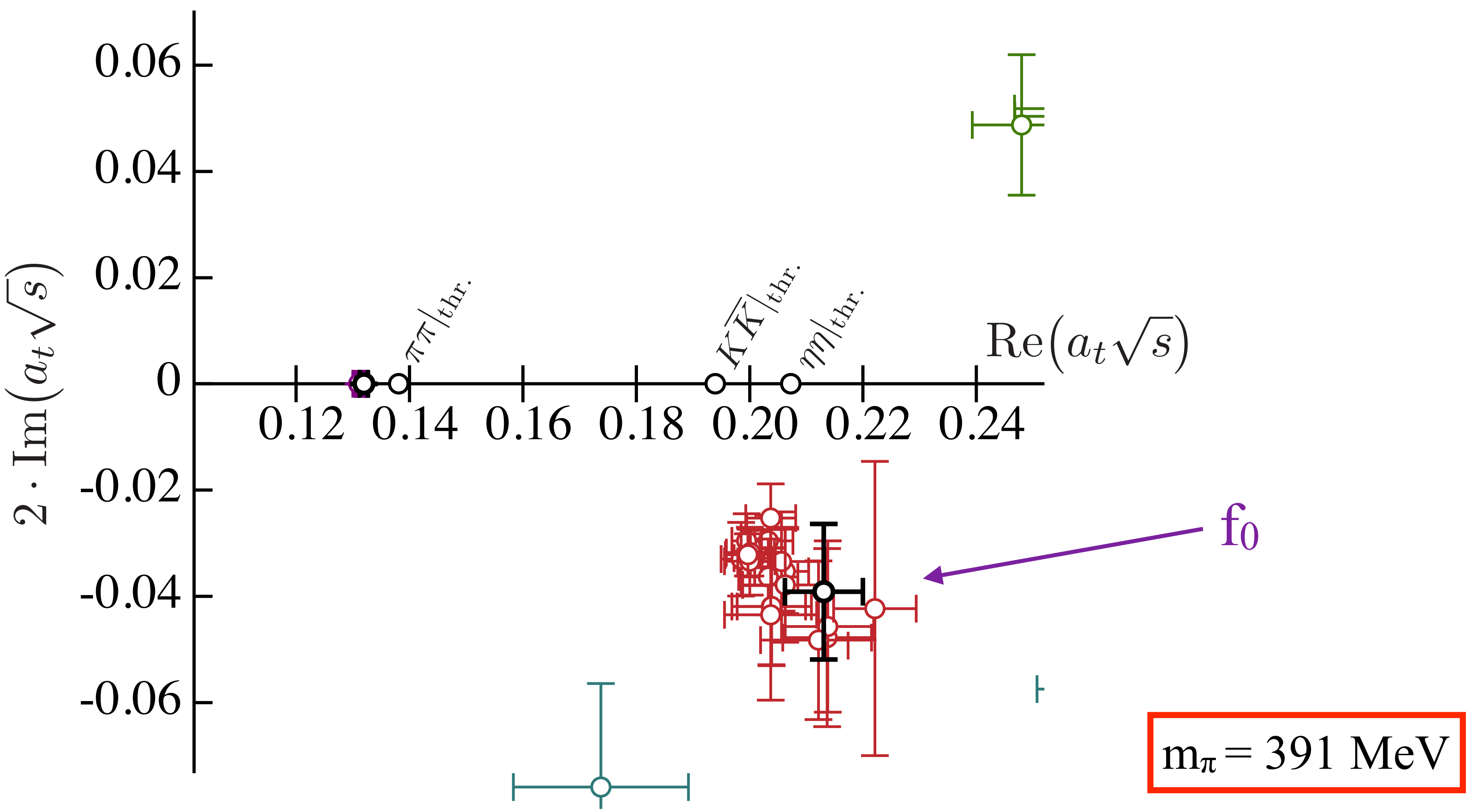
[Briceño, Dudek, Edwards, & Wilson arXiv:1708.06667]

- Ansatz $\mathbf{K}^{-1}(s) = \begin{pmatrix} a+bs & c+ds & e \\ c+ds & f & g \\ e & g & h \end{pmatrix}$



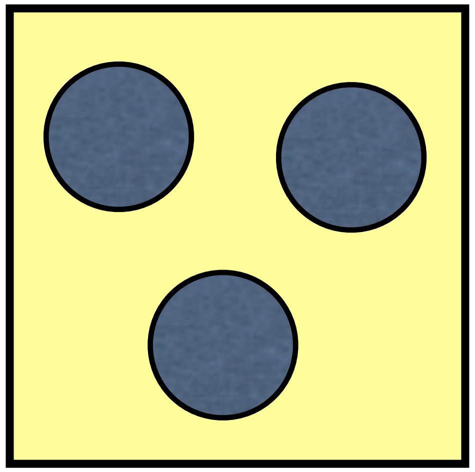
State-of-the-art: coupled channels

- Parametrization dependence of pole positions



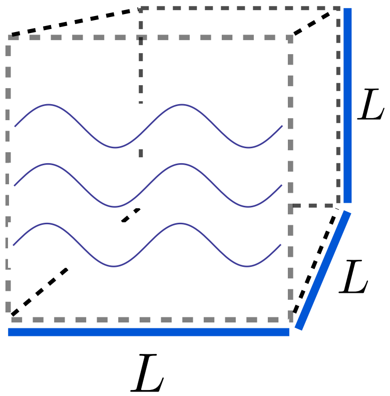
[Briceño, Dudek, Edwards & Wilson arXiv:1708.06667]

3-particle quantization condition



Two-step method

2 & 3 particle
spectrum from LQCD



Quantization conditions

$$\det\left[F_2^{-1} + \mathscr{K}_2\right] = 0$$

$$\det\left[F_3^{-1} + \mathscr{K}_{\mathrm{df},3}\right] = 0$$

ediate, unphysical
ering quantity

Integral equ
infinite vo

Need for two steps
common to all approaches
(though intermediate
quantities differ)

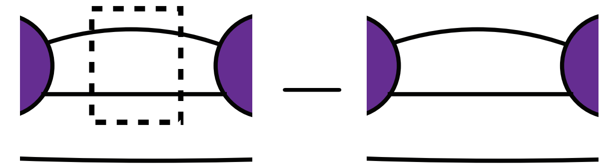
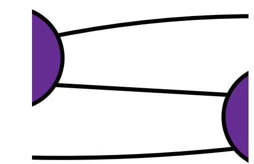
Scattering amplitudes

$$\mathscr{M}_{22}\,,\quad \mathscr{M}_{23}\,,\quad \mathscr{M}_{32}\,,\quad \mathscr{M}_{23}$$

Meaning of quantization condition

$$\det\left[F_3^{-1} + \mathscr{K}_{\mathrm{df},3}\right] = 0$$

$$F_3 \equiv \frac{F}{2\omega L^3}\left[-\frac{2}{3} + \frac{1}{1 + [1 + \mathcal{K}_2 G]^{-1}\mathcal{K}_2 F}\right]$$

$F =$  $G =$ 

- All quantities are infinite-dimensional matrices with indices describing 3 on-shell particles

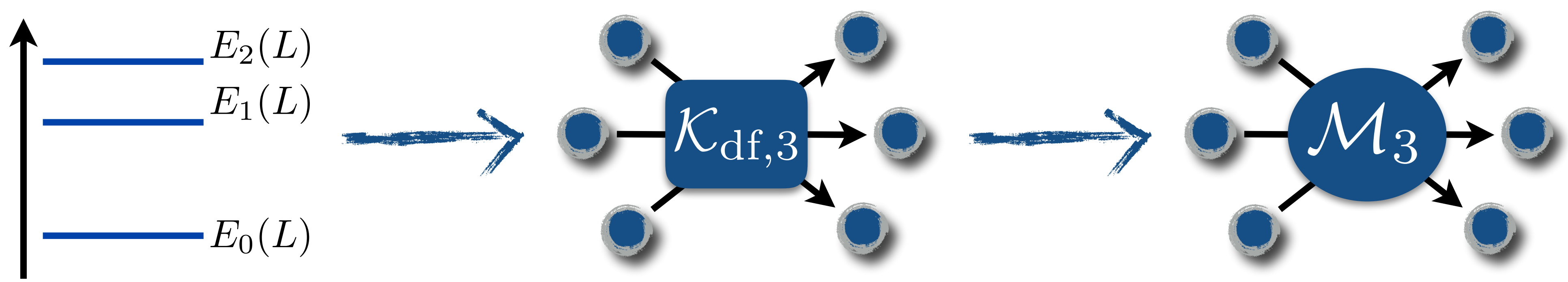
[finite volume "spectator" momentum: $\mathbf{k}=2\pi\mathbf{n}/L$] x [2-particle CM angular momentum: l,m]

- F (closely related to $F_{\rm PV}$) and G are known kinematic functions depending on L and E
- F_3 contains effects of two-particle scattering, entering through $\mathcal{K}_2$
- For large spectator-momentum **k**, the other two particles are below threshold; we must include such configurations by analytic continuation up to a cut-off at k~m

Status of relativistic approach

- Original work applied to scalars with G-parity & no subchannel resonances [Hansen & SRS, arXiv:1408.5933 & 1504.04248]

$$\det\left[F_3^{-1}+\mathcal{K}_{\mathrm{df},3}\right]=0$$



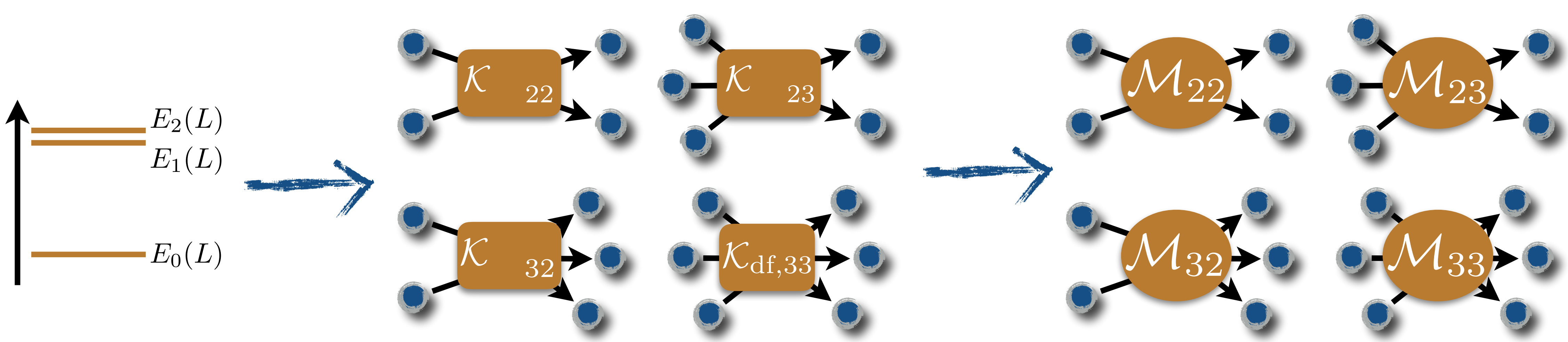
UPDATE: subchannel resonances may be allowed by adopting a variant of the PV pole-prescription

Status of relativistic approach

- Second major step: removing G-parity constraint, allowing 2↔3 processes [Briceño, Hansen & SRS, arXiv:1701.07465]

F_2 appears in 2-particle quantization condition

$$\det\left[\begin{pmatrix} F_2 & 0 \\ 0 & F_3 \end{pmatrix}^{-1} + \begin{pmatrix} \mathcal{K}_{22} & \mathcal{K}_{23} \\ \mathcal{K}_{32} & \mathcal{K}_{\mathrm{df},33} \end{pmatrix}\right] = 0$$



Status of relativistic approach

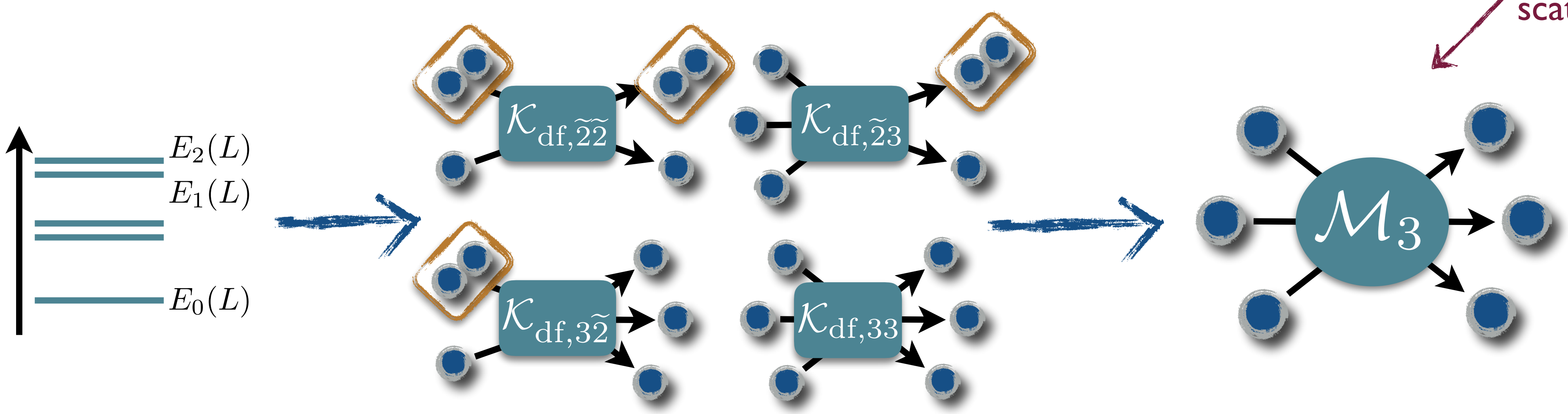
- Final major step: allowing subchannel resonance (i.e. pole in $\mathcal{K}_2$)
  [Briceño, Hansen & SRS, arXiv:1810.01429]

Determined by K_2 & Lüscher finite-volume zeta functions

$$\det\left[\begin{pmatrix} F_{\tilde{2}\tilde{2}} & F_{\tilde{2}3} \\ F_{3\tilde{2}} & F_{33} \end{pmatrix}^{-1} + \begin{pmatrix} \mathscr{K}_{\mathrm{df},\tilde{2}\tilde{2}} & \mathscr{K}_{\mathrm{df},\tilde{2}3} \\ \mathscr{K}_{\mathrm{df},3\tilde{2}} & \mathscr{K}_{\mathrm{df},33} \end{pmatrix}\right] = 0$$

resonance + particle channel (not physical, but forced on us by derivation)

No unphysical channel in final scattering amplitude

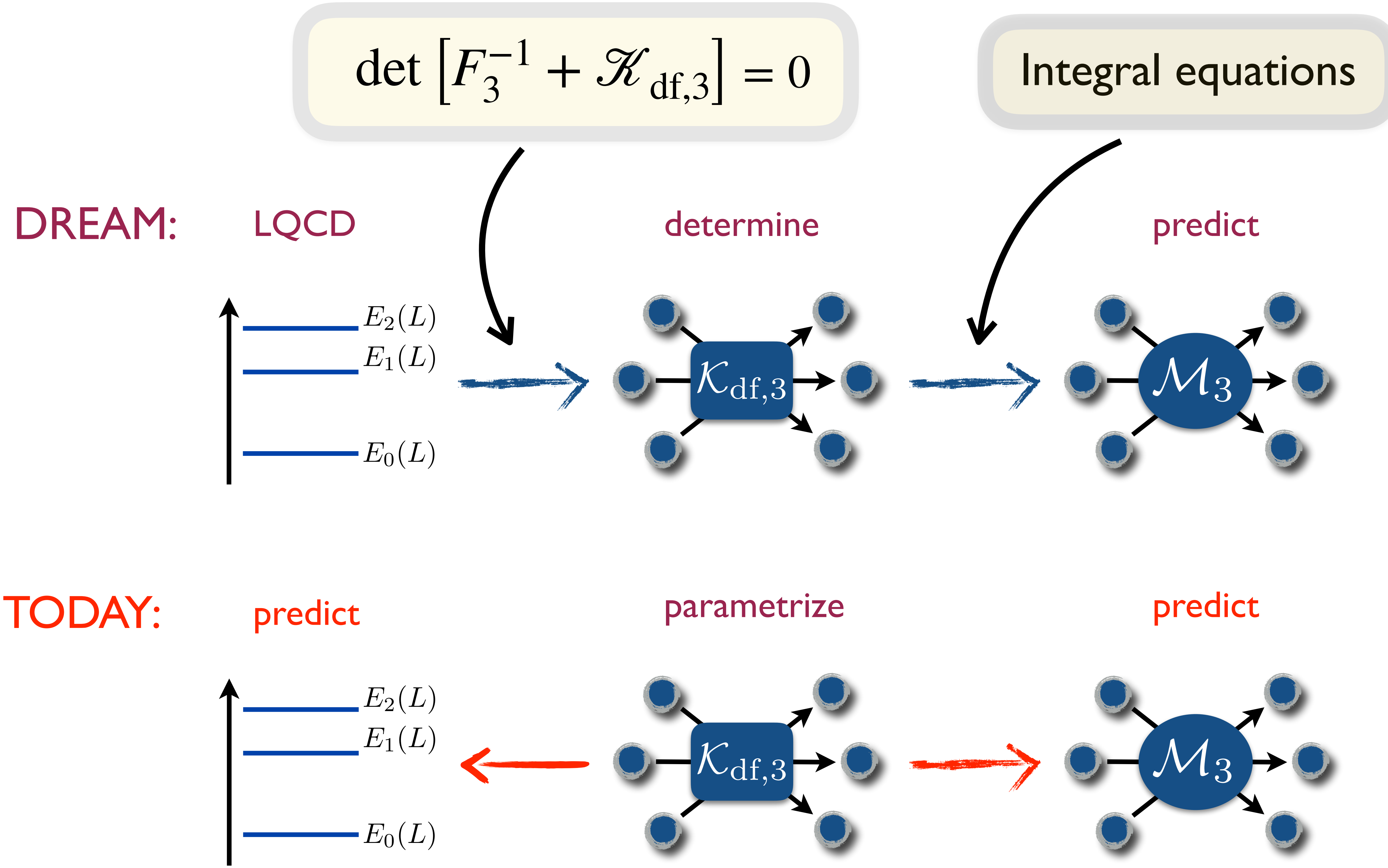


UPDATE: this elaboration may be avoidable

Numerical implementation: isotropic approximation

[Briceño, Hansen & SRS, arXiv:1803.04169]

Overview



$\det\left[F_3^{-1} + \mathcal{K}_{\text{df},3}\right] = 0$
Integral equations
DREAM:
LQCD
determine
predict
$E_2(L)$
$E_1(L)$
$E_0(L)$
$\mathcal{K}_{\text{df},3}$
$\mathcal{M}_3$
TODAY:
predict
parametrize
predict
$E_2(L)$
$E_1(L)$
$E_0(L)$
$\mathcal{K}_{\text{df},3}$
$\mathcal{M}_3$

Truncation

$$\det\left[F_3^{-1} + \mathcal{K}_{\text{df},3}\right]$$

matrices with indices:

[finite volume "spectator" momentum: $\mathbf{k}=2\pi\mathbf{n}/L$] x [2-particle CM angular momentum: l,m]

- To use quantization condition, one must truncate matrix space, as for the two-particle case
- Spectator-momentum space is truncated by cut-off function H(**k**)
- Need to truncate sums over l,m in $\mathcal{K}_2$ & $\mathcal{K}_{\text{df},3}$

Isotropic low-energy approximation

[Hansen & SRS]

- Scalar particles with G parity so no 2⟷3 transitions and no subchannel resonances (e.g. 3 π^+)
- 2-particle interactions are purely s-wave, and determined by the scattering length, a, alone
  - Avoiding poles in $\mathcal{K}_2$ restricts scattering length to $1 > a > -\infty$, implying no two-particle bound states
- Point-like three-particle interaction $\mathcal{K}_{df,3}$ independent of momenta, although can depend on $s=(E_{cm})^2$
- Consider only **P**=0 (though formalism applies for all **P**)
- Analog in our formalism of the approximations used in other approaches: [Hammer, Pang, Rusetsky, 1706.07700; Mai & Döring, 1709.08222; Döring et al., 1802.03362; Mai & Döring, 1807.04746]

Isotropic low-energy approximation

[Briceño, Hansen & SRS, 1803.04169]

- Reduces problem to 1-d quantization condition, with intermediate matrices involve finite-volume momenta up to cutoff |k|~m

$$\boxed{\det\left[F_3^{-1} + \mathscr{K}_{\mathrm{df},3}\right] = 0} \longrightarrow 1/\mathcal{K}^{\mathrm{iso}}_{\mathrm{df},3}(E^*) = -F_3^{\mathrm{iso}}[E, \vec{P}, L, \mathcal{M}_2^s]$$

$$F_3^{\mathrm{iso}}(E,L) = \langle \mathbf{1}|F_3^s|\mathbf{1}\rangle = \sum_{k,p}[F_3^s]_{kp} \qquad [F_3^s]_{kp} = \frac{1}{L^3}\left[\frac{\tilde{F}^s}{3} - \tilde{F}^s \frac{1}{1/(2\omega\mathcal{K}_2^s) + \tilde{F}^s + \tilde{G}^s}\tilde{F}^s\right]_{kp}$$

- Relation of $\mathcal{K}_{\mathrm{df},3}$ to $\mathcal{M}_3$ (matrix equation that becomes integral equation when L→∞)

$$\mathcal{M}_3 = \mathcal{S}\left[\mathcal{D} + \mathcal{L}\frac{1}{1/\mathcal{K}^{\mathrm{iso}}_{\mathrm{df},3} + F^{\mathrm{iso}}_{3,\infty}}\mathcal{R}\right]$$

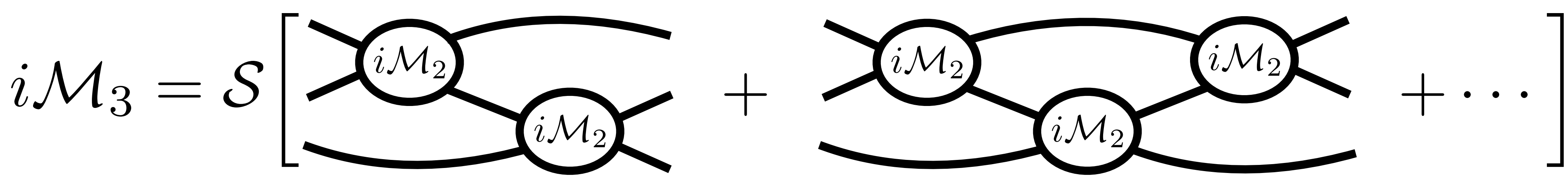
symmetrization

$\mathcal{D}$, $\mathcal{L}$ & $\mathcal{R}$ depend on $\mathcal{M}_2$ & kinematical factors

L→∞ limit of F_3^{iso} depends on $\mathcal{M}_2$ & kinematical factors

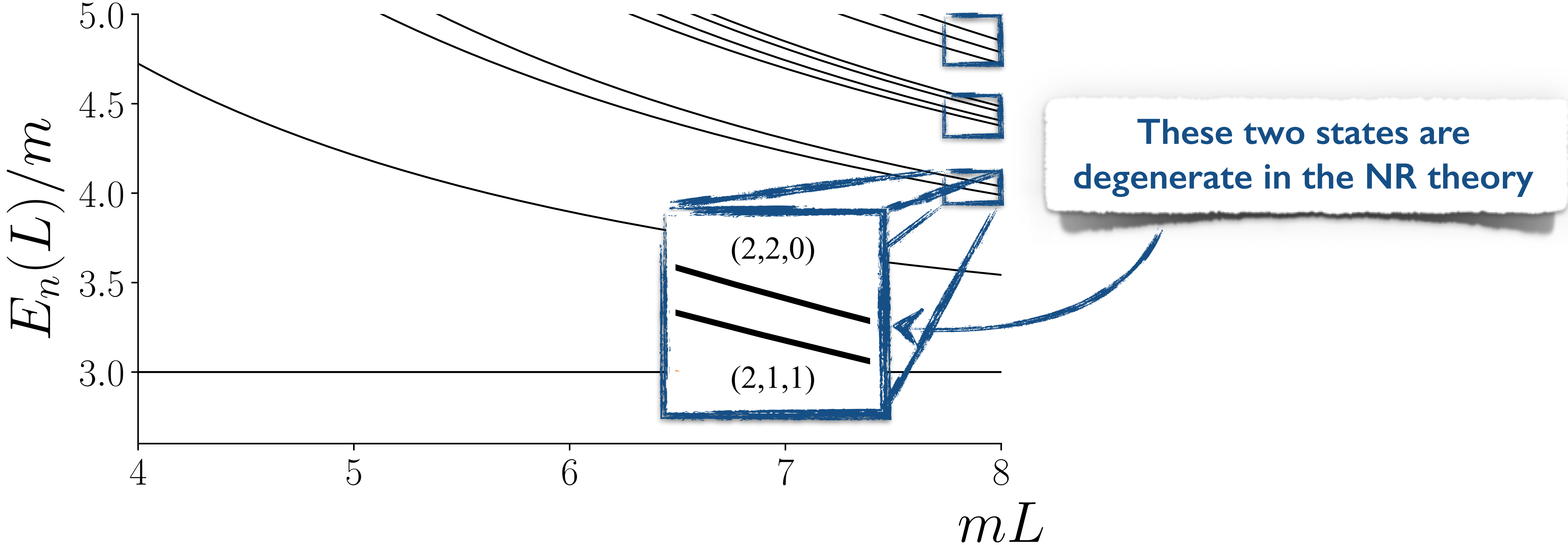
Solutions with $K_{df,3} = 0$

- Useful benchmark: deviations measure impact of 3-particle interaction
    - Caveat: scheme-dependent since $\mathcal{K}_{df,3}$ depends on cut-off function H
- Qualitative meaning of this limit for $\mathcal{M}_3$:



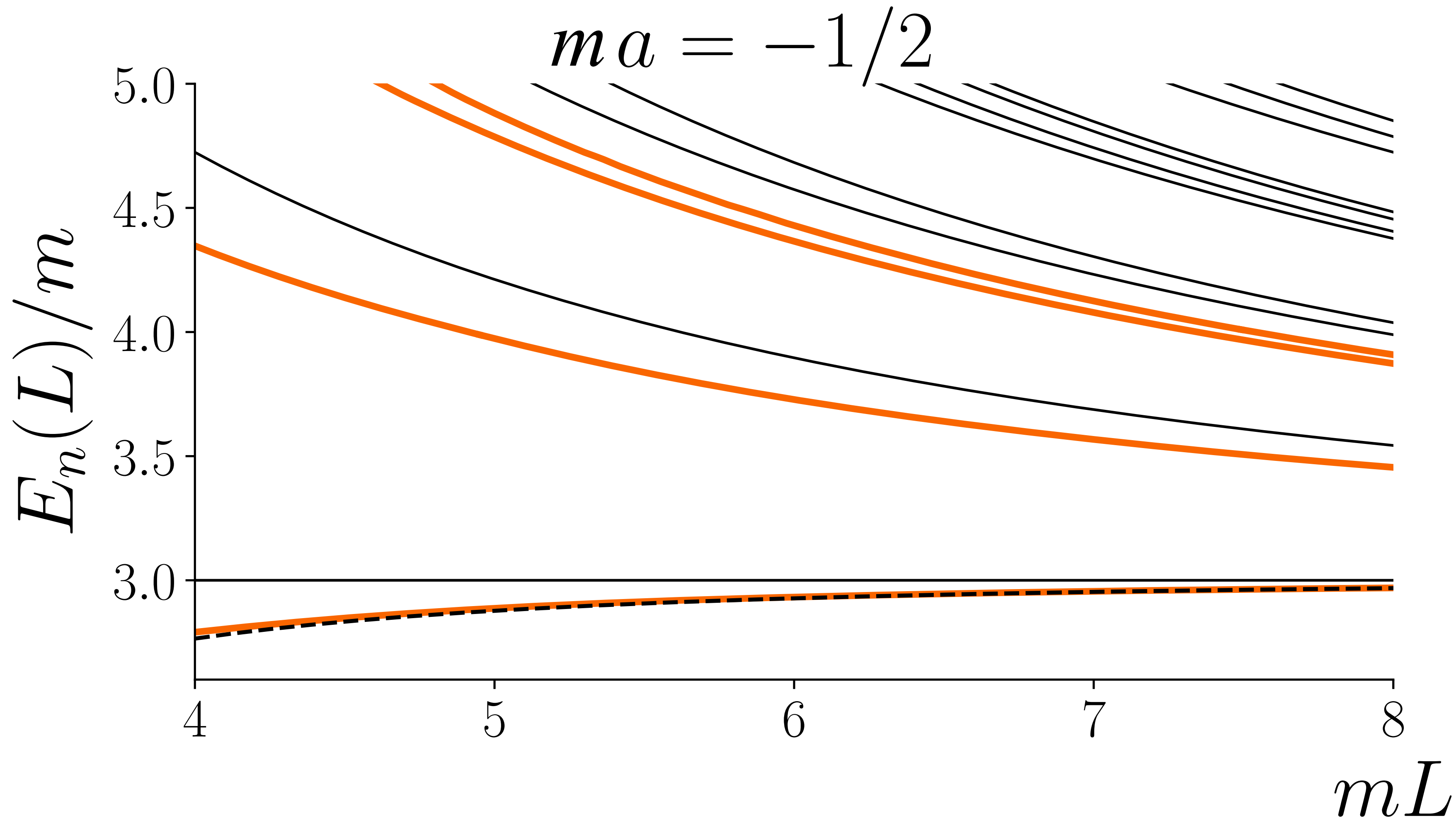
Solutions with $K_{df,3}=0$

- Non-interacting states



Solutions with $K_{df,3}=0$

- Weakly attractive two-particle interaction



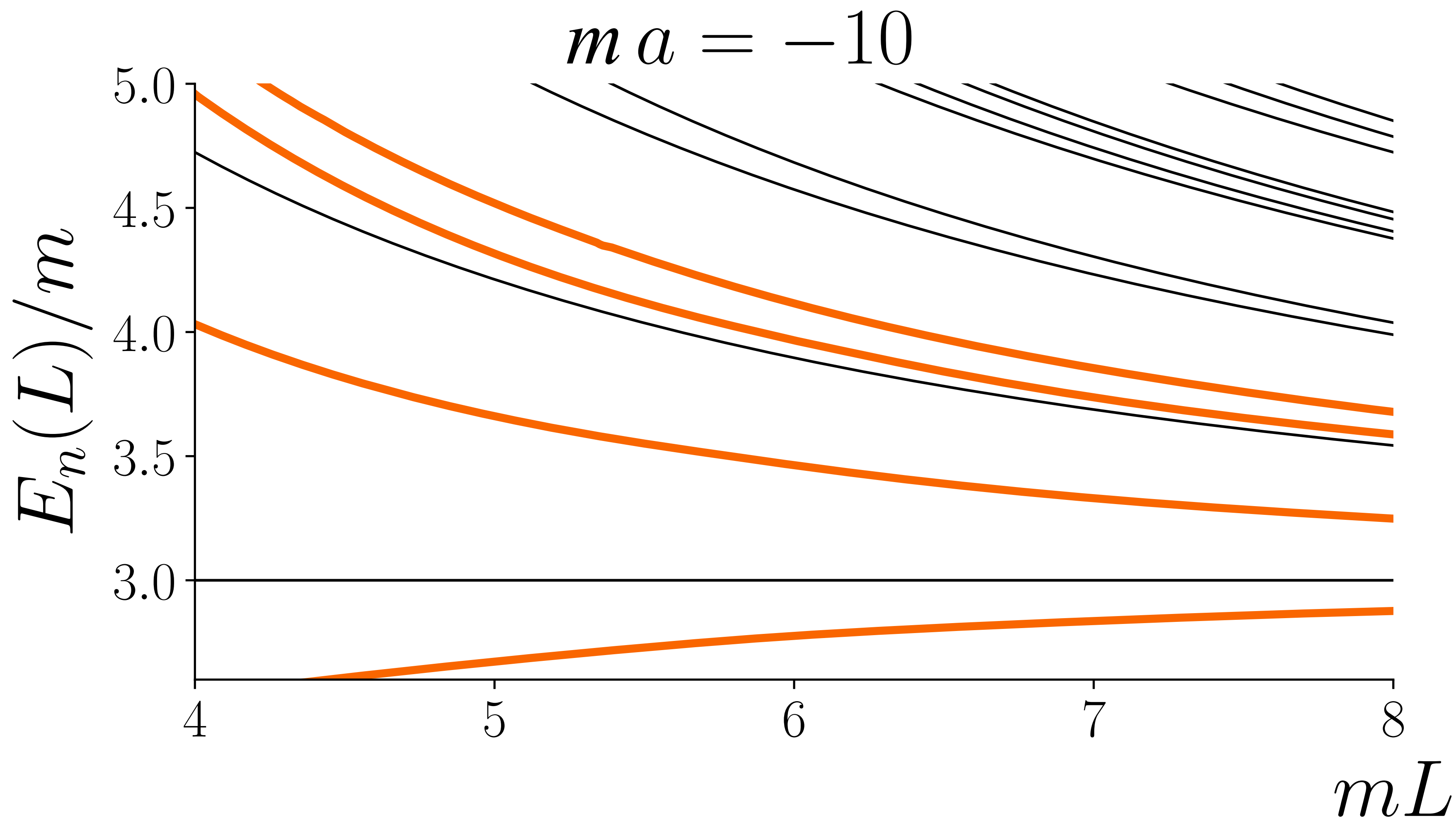
$1/L$ **expansion**

[Beane, Detmold, Savage; Tan; Hansen & SRS]

2-particle interaction enters at $1/L^3$, 3-particle interaction (and relativistic effects) enter at $1/L^6$

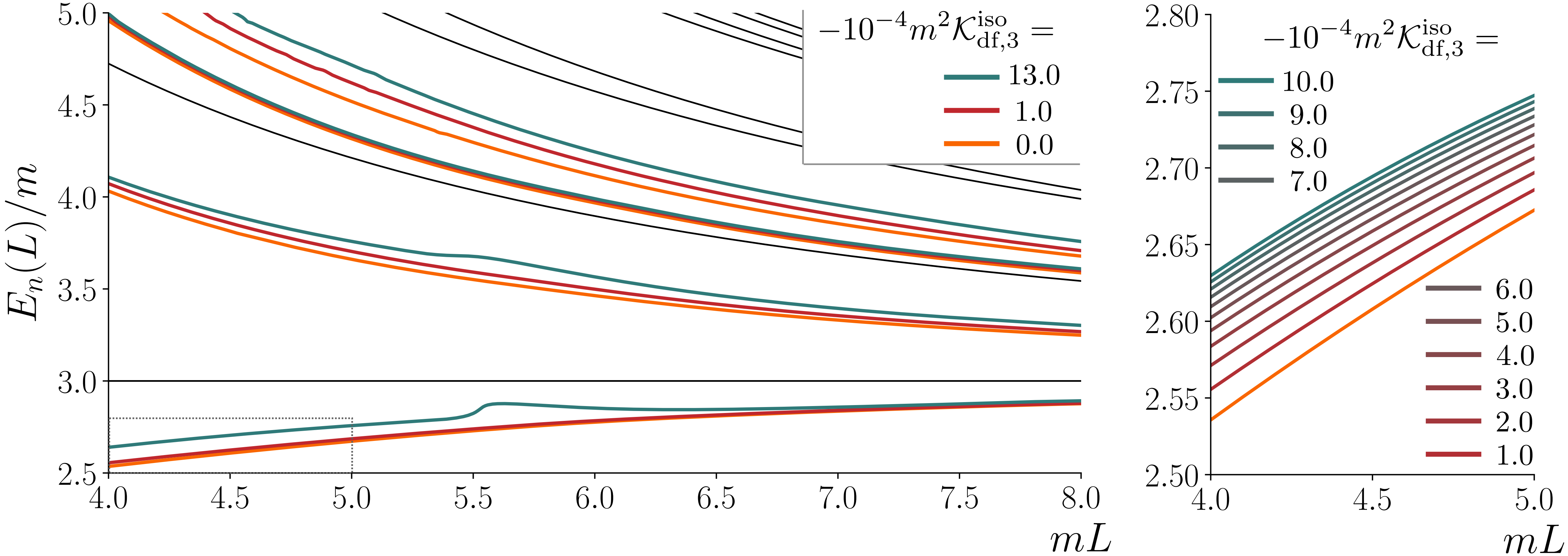
Solutions with $K_{df,3}=0$

- Strongly attractive two-particle interaction



Impact of $K_{df,3}$

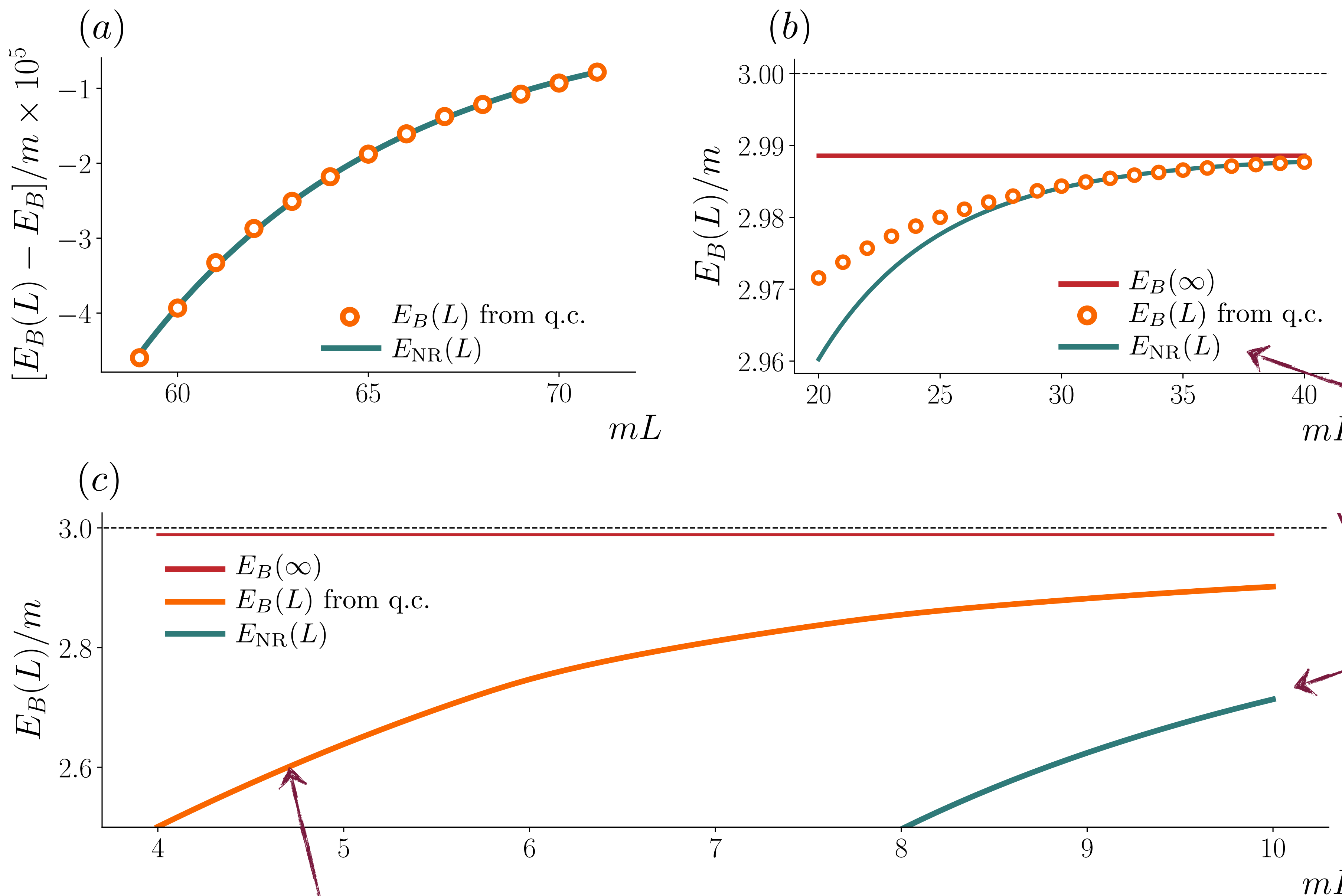
$ma = -10$ (strongly attractive interaction)



Local 3-particle interaction has significant effect on energies, especially in region of simulations (mL<5), and thus can be determined

Volume-dependence of 3-body bound state

$$am = -10^4 \ \& \ m^2\mathcal{K}_{\mathrm{df},3} = 2500 \quad \text{(unitary regime)}$$



Prediction of asymptotic volume-dependence from NRQM [Meißner, Rîos, Rusetsky]

Need quantization condition to determine finite-volume effects for realistic values of mL

Bound state wave-function

- Work in unitary regime (ma=−10⁴) and tune $\mathcal{K}_{\rm df,3}$ so 3-body bound state at E_B=2.98858 m

- Solve integral equations numerically to determine $\mathcal{M}_{\rm df,3}$ from $\mathcal{K}_{\rm df,3}$

- Determine wavefunction from residue at bound-state pole

$$\mathcal{M}_{\rm df,3}^{(u,u)}(k,p) \sim -\frac{\Gamma^{(u)}(k)\Gamma^{(u)}(p)^*}{E^2-E_B^2}$$

- Compare to analytic prediction from NRQM in unitary limit [Hansen & SRS, 1609.04317]

$$|\Gamma^{(u)}(k)_{\rm NR}|^2 = |c||A|^2 \frac{256\pi^{5/2}}{3^{1/4}} \frac{m^2\kappa^2}{k^2(\kappa^2+3k^2/4)} \frac{\sin^2\left(s_0 \sinh^{-1}\frac{\sqrt{3}k}{2\kappa}\right)}{\sinh^2\frac{\pi s_0}{2}}$$

Known constant

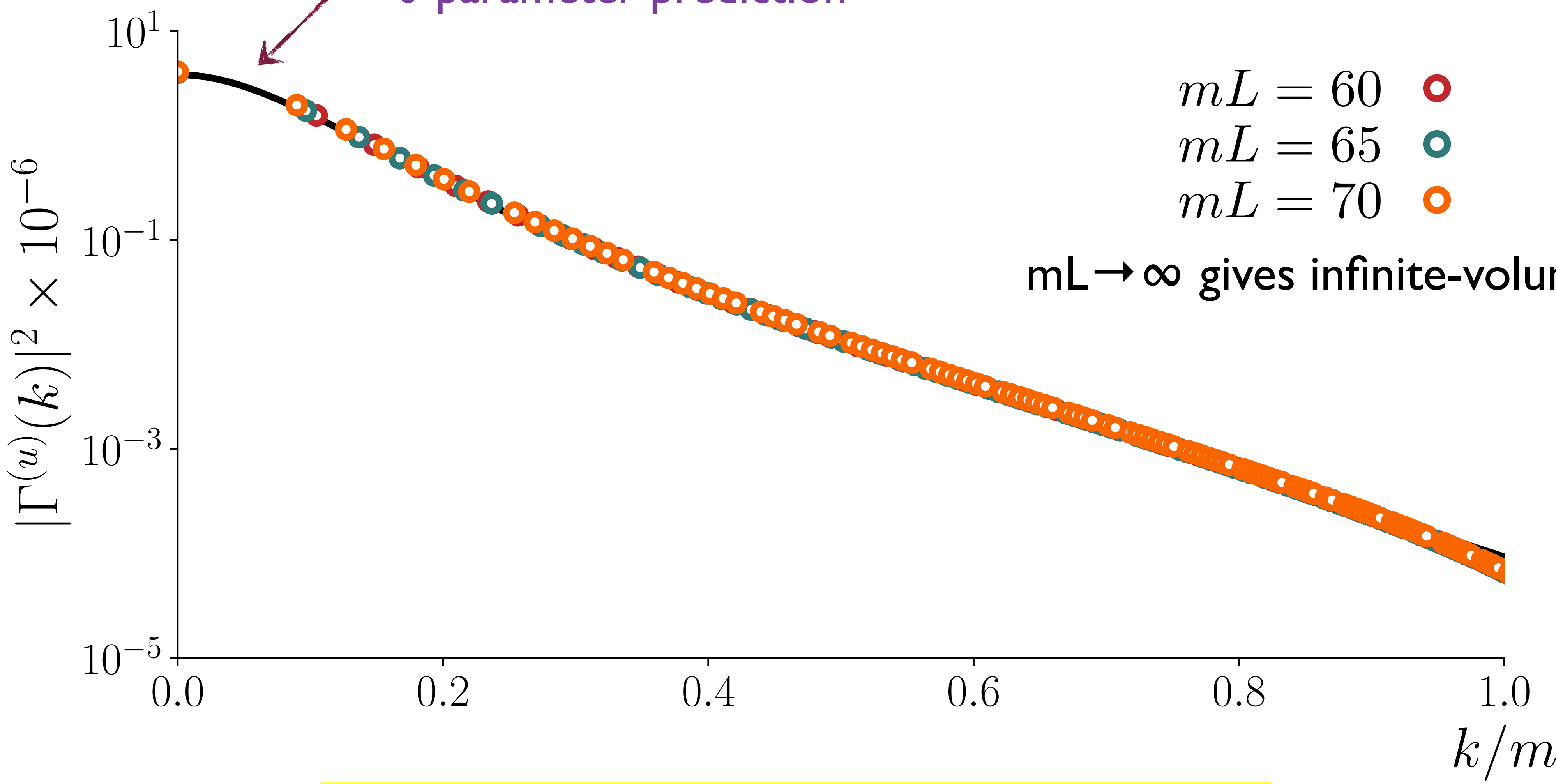
Determined by fit to volume-dependence of bound-state energy

Known constant

Bound state wave-function

$$|\Gamma^{(u)}(k)_{\rm NR}|^2 = |c||A|^2 \frac{256\pi^{5/2}}{3^{1/4}} \frac{m^2\kappa^2}{k^2(\kappa^2+3k^2/4)} \frac{\sin^2\left(s_0 \sinh^{-1}\frac{\sqrt{3}k}{2\kappa}\right)}{\sinh^2\frac{\pi s_0}{2}}$$

0-parameter prediction



mL→∞ gives infinite-volume result

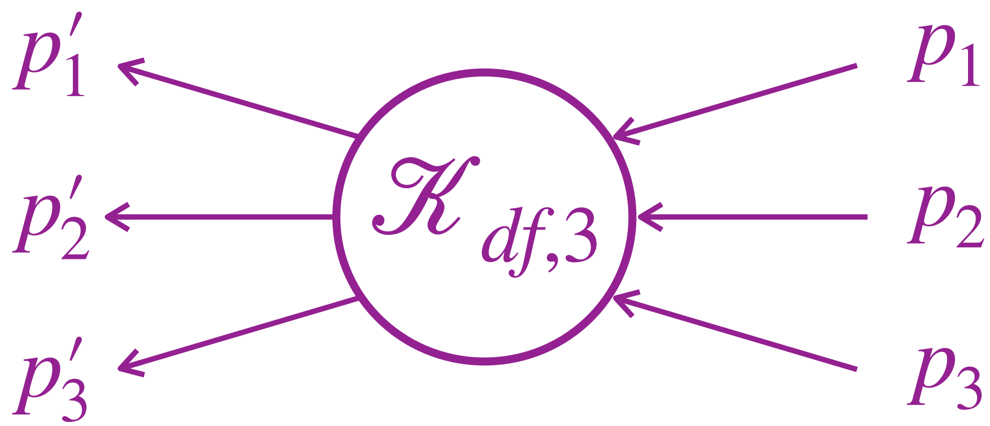
Works over many orders of magnitude
to expected accuracy

Beyond isotropic: including higher partial waves

[Blanton, Romero-López & SRS, 1901.07095]

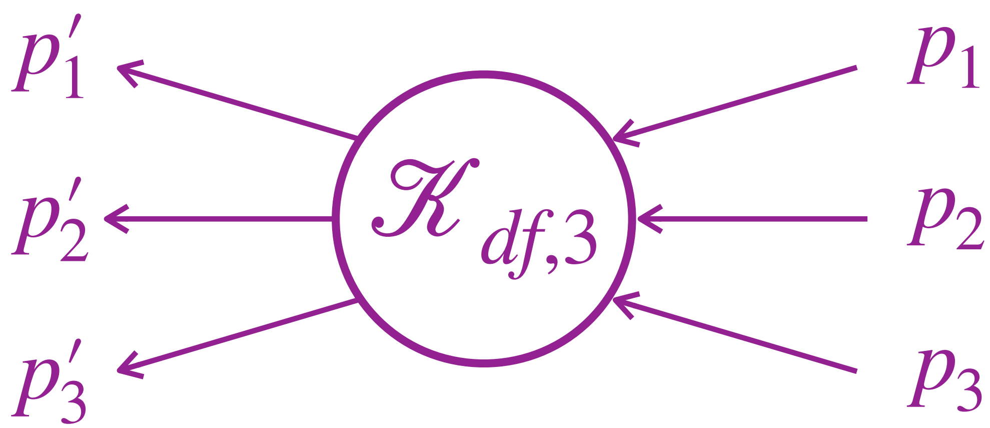
Beyond the isotropic approximation

- In 2-particle case, we know that s-wave scattering dominates at low energies; can then systematically add in higher waves (suppressed by q^{2l})
- We are implementing the same general approach for $\mathcal{K}_{\mathrm{df},3}$, making use of the facts that it is relativistically invariant and completely symmetric under initial- & final-state permutations, and expanding about threshold



- We work in the G-parity invariant theory with 3 identical scalars, so the first channel beyond s-wave has $l=2$ (d-wave)

Beyond the isotropic approximation



$$s_{ij} \equiv (p_i + p_j)^2$$

$$s'_{ij} \equiv (p'_i + p'_j)^2$$

$$t_{ij} \equiv (p_i - p'_j)^2$$

$$\Delta \equiv \frac{s - 9m^2}{9m^2}$$

$$\Delta_i \equiv \frac{s_{jk} - 4m^2}{9m^2}$$

$$\Delta'_i \equiv \frac{s'_{jk} - 4m^2}{9m^2}$$

$$\widetilde{t}_{ij} \equiv \frac{t_{ij}}{9m^2}$$

$$m^2 \mathcal{K}_{\text{df},3} = \mathcal{K}^{\text{iso}} + \mathcal{K}^{(2,A)}_{\text{df},3} \Delta^{(2)}_A + \mathcal{K}^{(2,B)}_{\text{df},3} \Delta^{(2)}_B + \mathcal{O}(\Delta^3)$$

$$\mathcal{K}^{\text{iso}} = \mathcal{K}^{\text{iso}}_{\text{df},3} + \mathcal{K}^{\text{iso},1}_{\text{df},3} \Delta + \mathcal{K}^{\text{iso},2}_{\text{df},3} \Delta^2$$

Only three coefficients needed at quadratic order

<u>only</u> term at linear order

leading order term: only term kept in isotropic approx

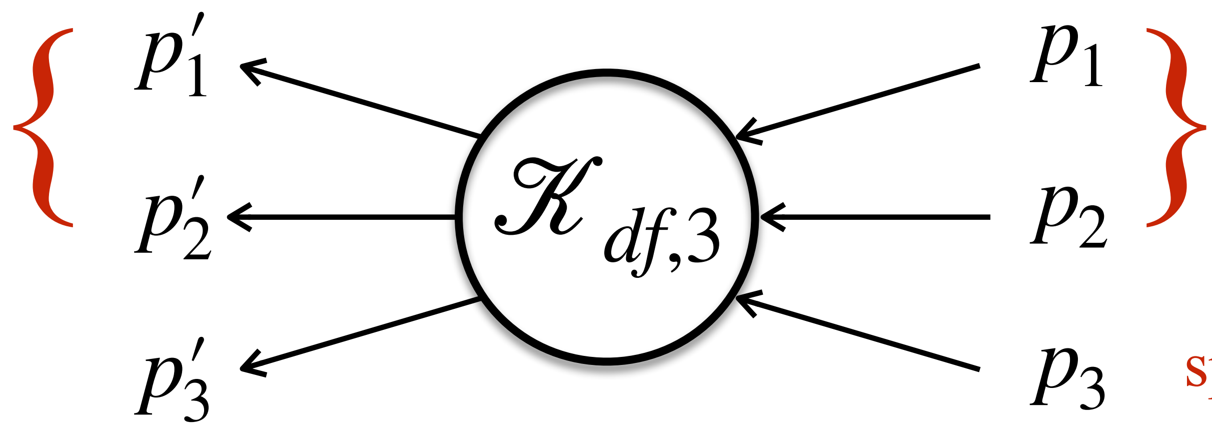
$$\Delta^{(2)}_A = \sum_{i=1}^{3} (\Delta_i^2 + \Delta_i'^2) - \Delta^2$$

$$\Delta^{(2)}_B = \sum_{i,j=1}^{3} \widetilde{t}_{ij}^{\,2} - \Delta^2$$

Decomposing into spectator/dimer basis

l',m'

spectator momentum



Decompose into harmonics in dimer CM frame: l,m

spectator momentum

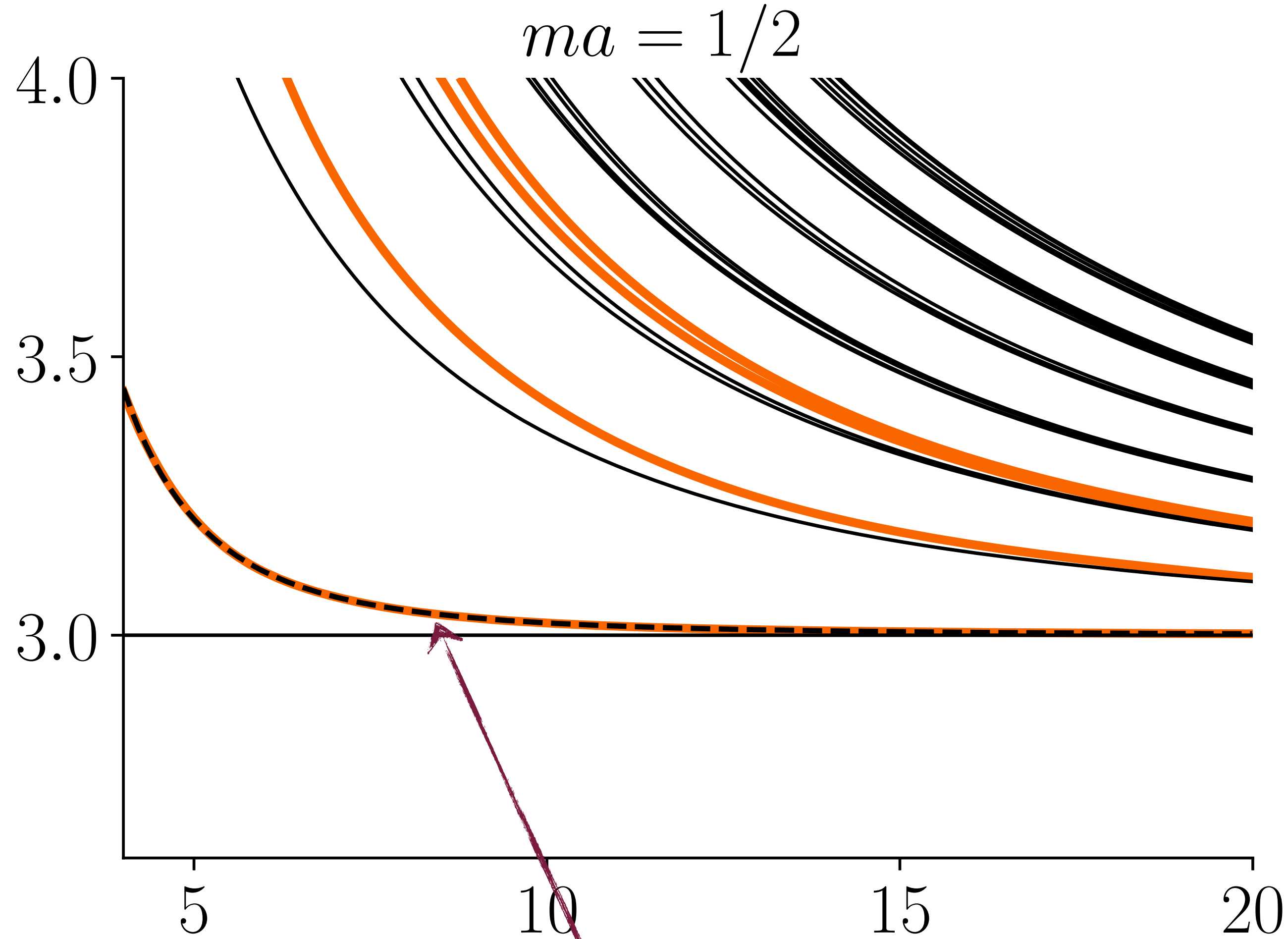
- Quadratic terms $\Delta_A^{(2)}$ & $\Delta_B^{(2)}$ $\Rightarrow$ l'=0,2 & l=0,2
- For consistency, need

$$\frac{1}{\mathcal{K}_2^{(0)}} = \frac{1}{16\pi E_2}\left[\frac{1}{a_0} + r_0\frac{q^2}{2} + P_0 r_0^3 q^4\right] \qquad \frac{1}{\mathcal{K}_2^{(2)}} = \frac{1}{16\pi E_2}\frac{1}{q^4}\frac{1}{a_2^5}$$

Implemented quantization condition through quadratic order, for P=0, including projection onto overall cubic group irreps

First results including l=2

Results from Isotropic approximation with $\mathscr{K}_{df,3} = 0$

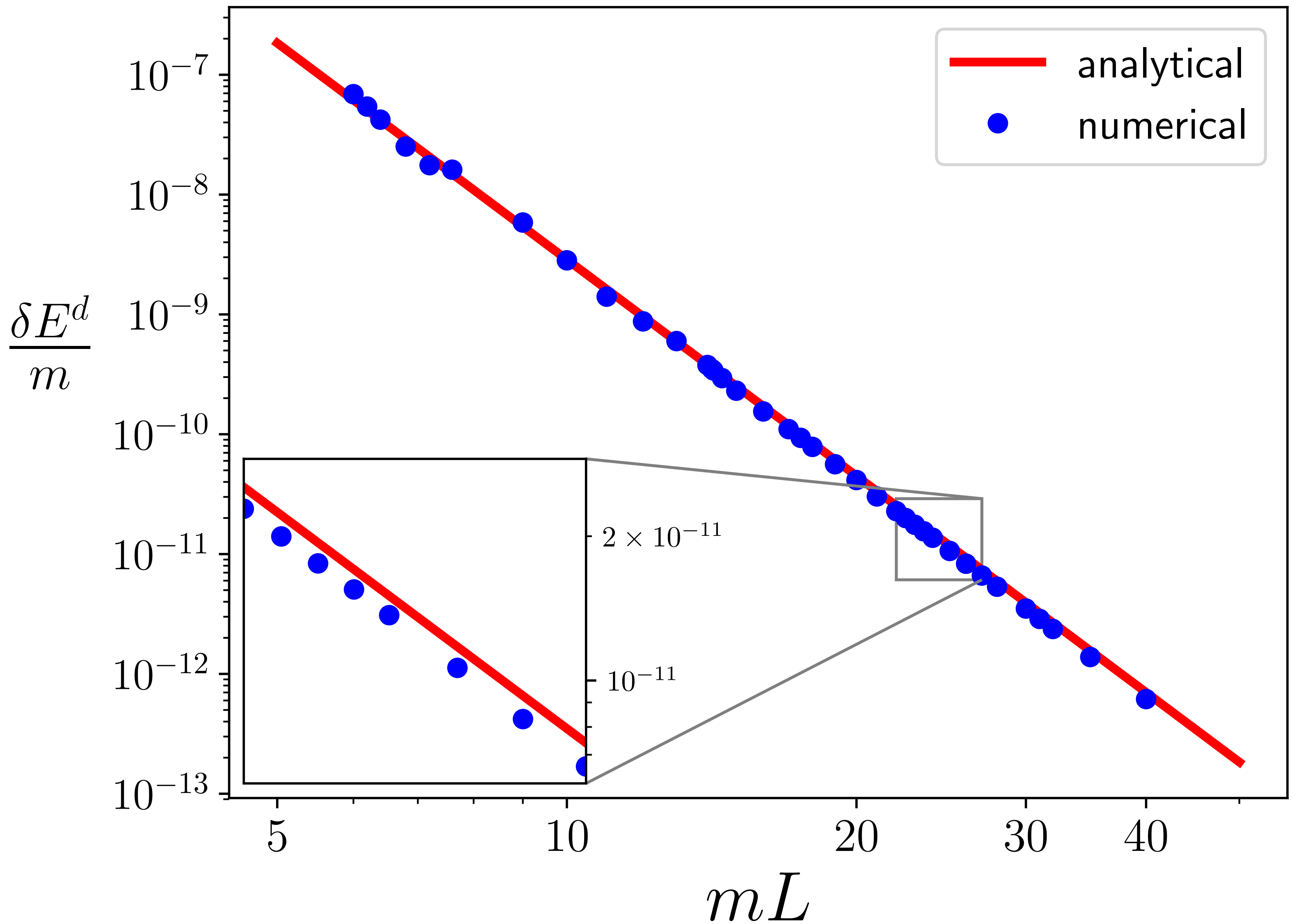


Threshold expansion works well.
What happens to this level as a_2 is turned on?

First results including l=2

Determine $\delta E^d = \left[E(a_2, L) - E(a_2 = 0, L)\right]$ using quantization condition

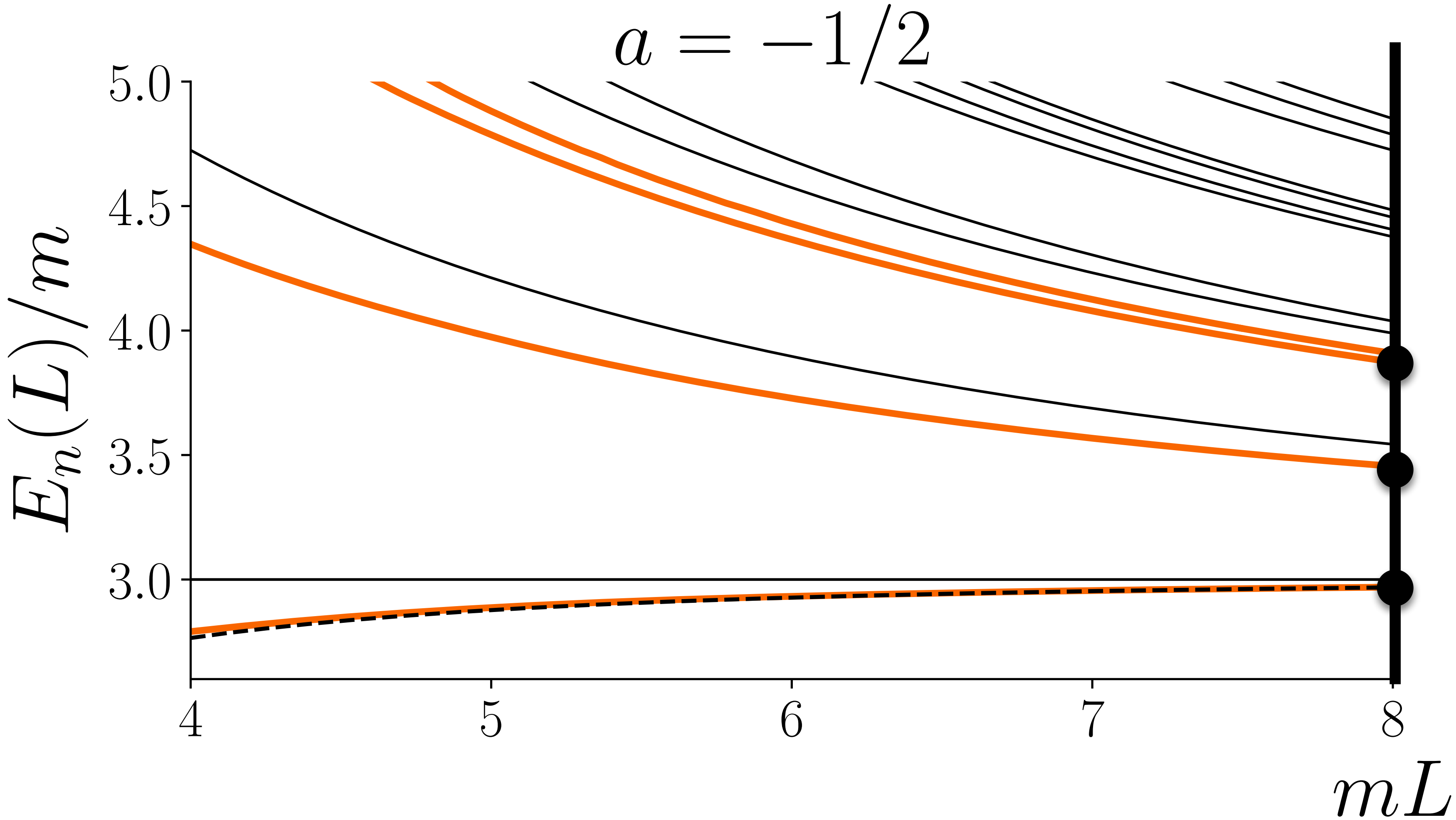
Compare to prediction: $\delta E^d = 294 \frac{(a_0 m)^2 (a_2 m)^5}{(mL)^6} + \mathcal{O}(a_0^3/L^6, 1/L^7)$



Works well (also for a_0 and a_2 dependence)
Tiny effect, but checks our numerical implementation

First results including *l*=2

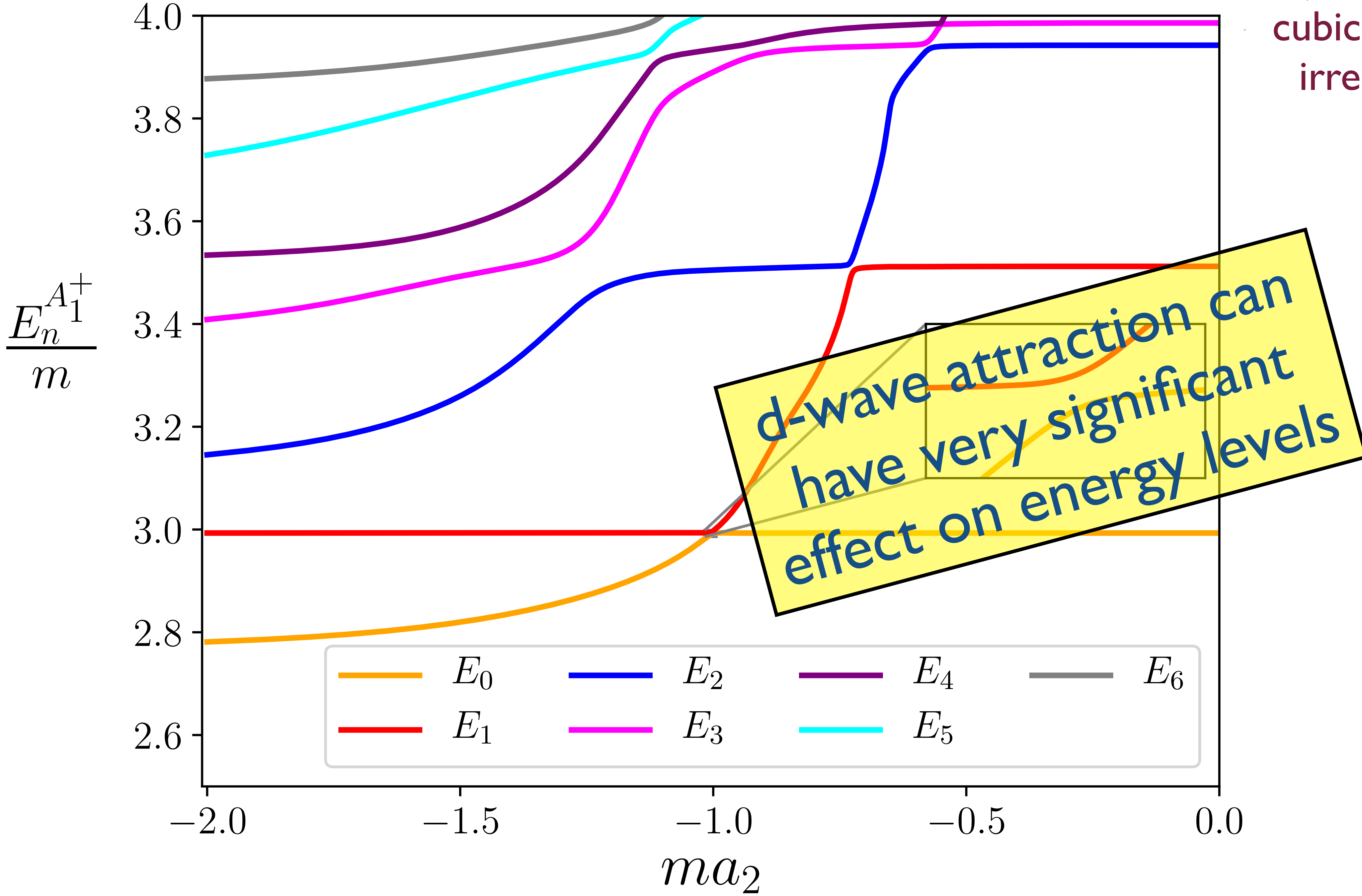
Results from Isotropic approximation with $\mathscr{K}_{df,3} = 0$



What happens to these levels as a_2 is turned on?

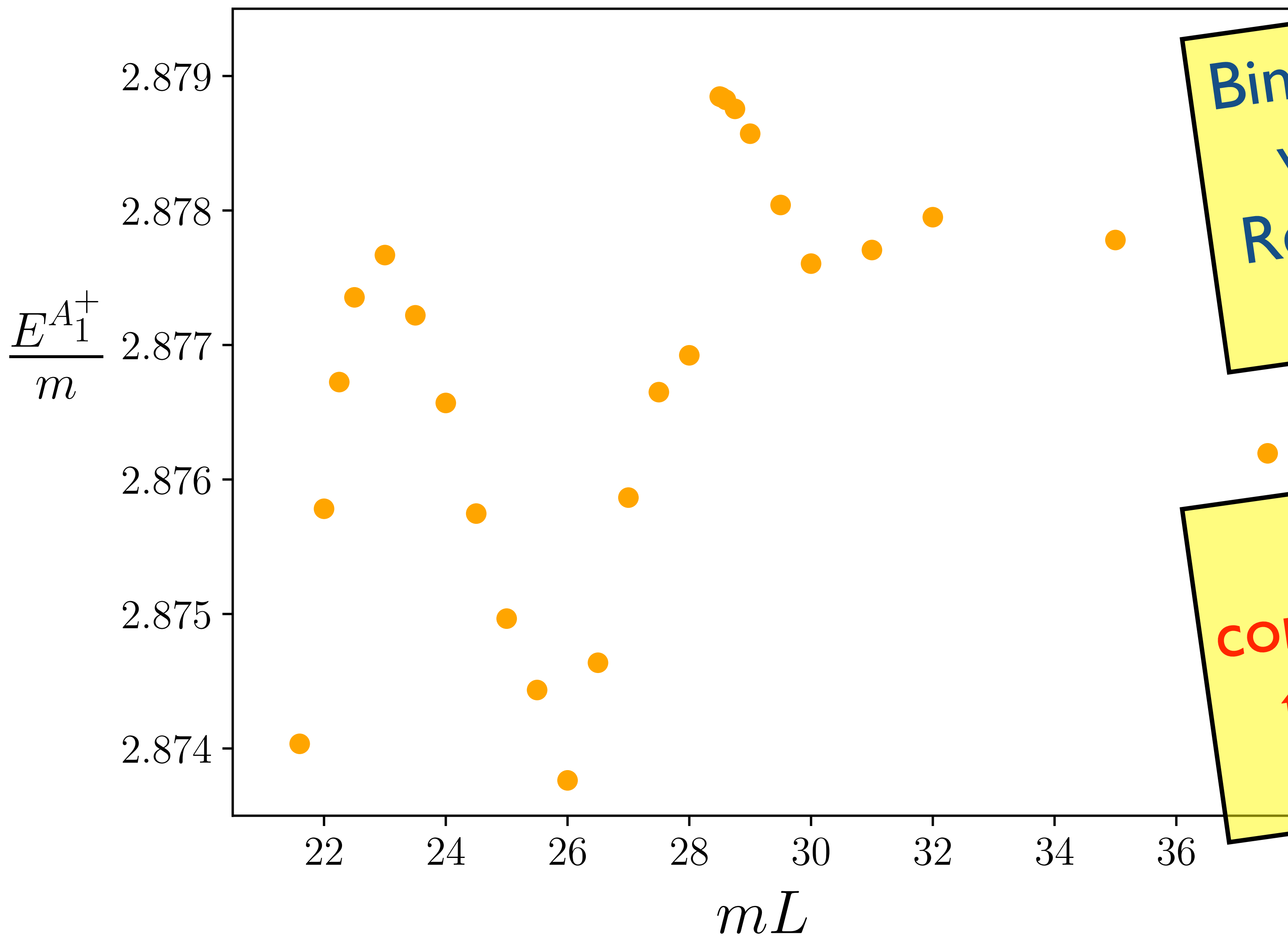
First results including l=2

Projected onto cubic-group irrep A_1^+



$$mL = 8.1,\ ma_0 = -0.1,\ r_0 = P_0 = \mathcal{K}_{df,3} = 0$$

Evidence for 3-particle bound state



Binding caused by d-wave attraction! Relevant for atomic physics?

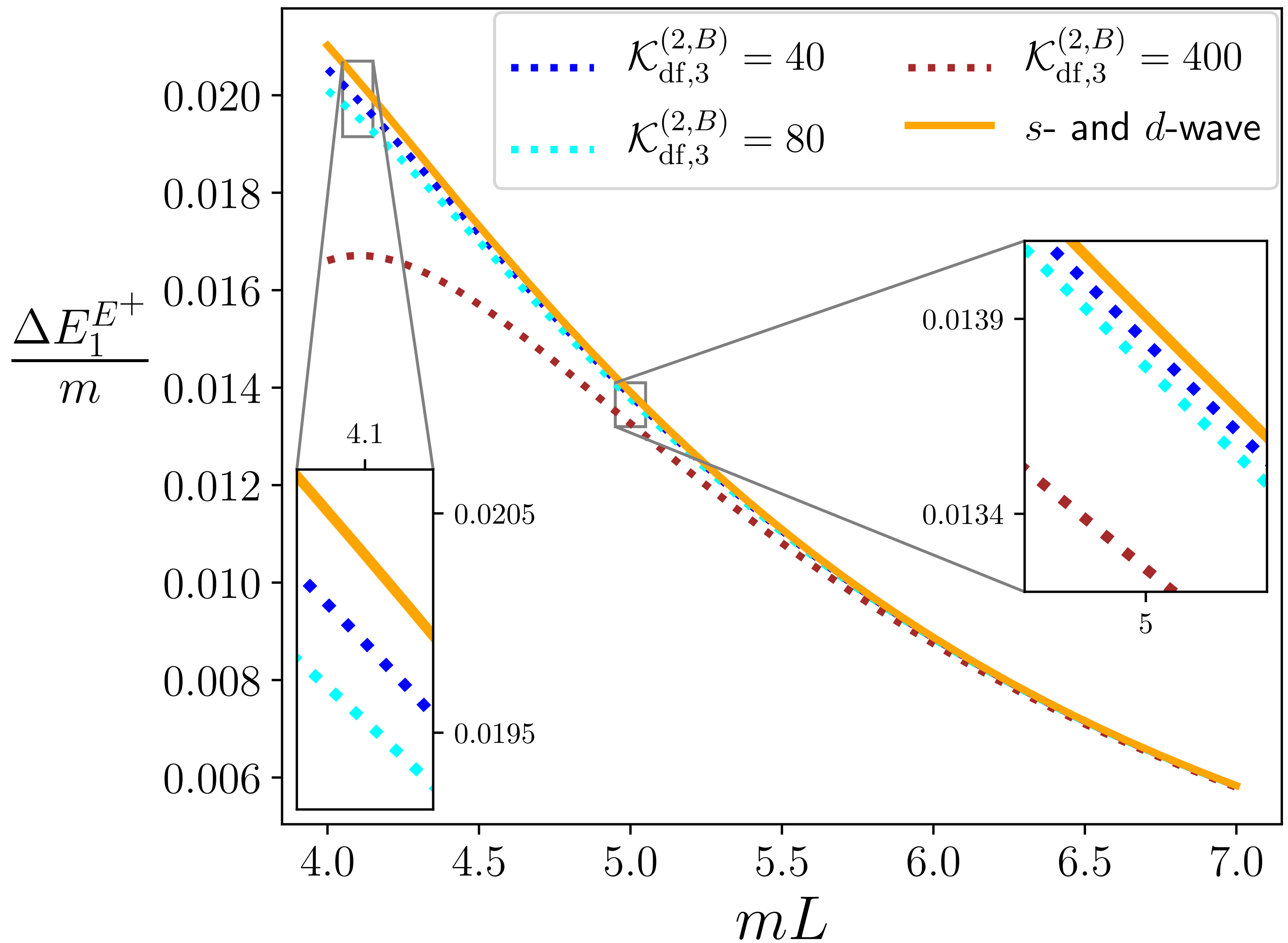
Quantization condition is useful as tool for studying infinite-volume!

$$ma_0 = -0.1,\ ma_2 = -1.3,\ r_0 = P_0 = \mathscr{K}_{df,3} = 0$$

Impact of quadratic terms in $\mathcal{K}_{df,3}$

a_0, r_0, P_0, & a_2 set to physical values for $3\pi^+$

Energy shift relative to noninteracting energy for first excited state. Projected into E^+ irrep.



Energies of $3\pi^+$ states need to be determined very accurately to be sensitive to $\mathcal{K}_{df,3}^{(2,B)}$, but this is achievable in ongoing simulations

Numerical implementation: isotropic approximation including two-particle bound states

[Blanton, Briceño, Hansen, Romero-López & SRS, in progress]

Isotropic approximation: v2

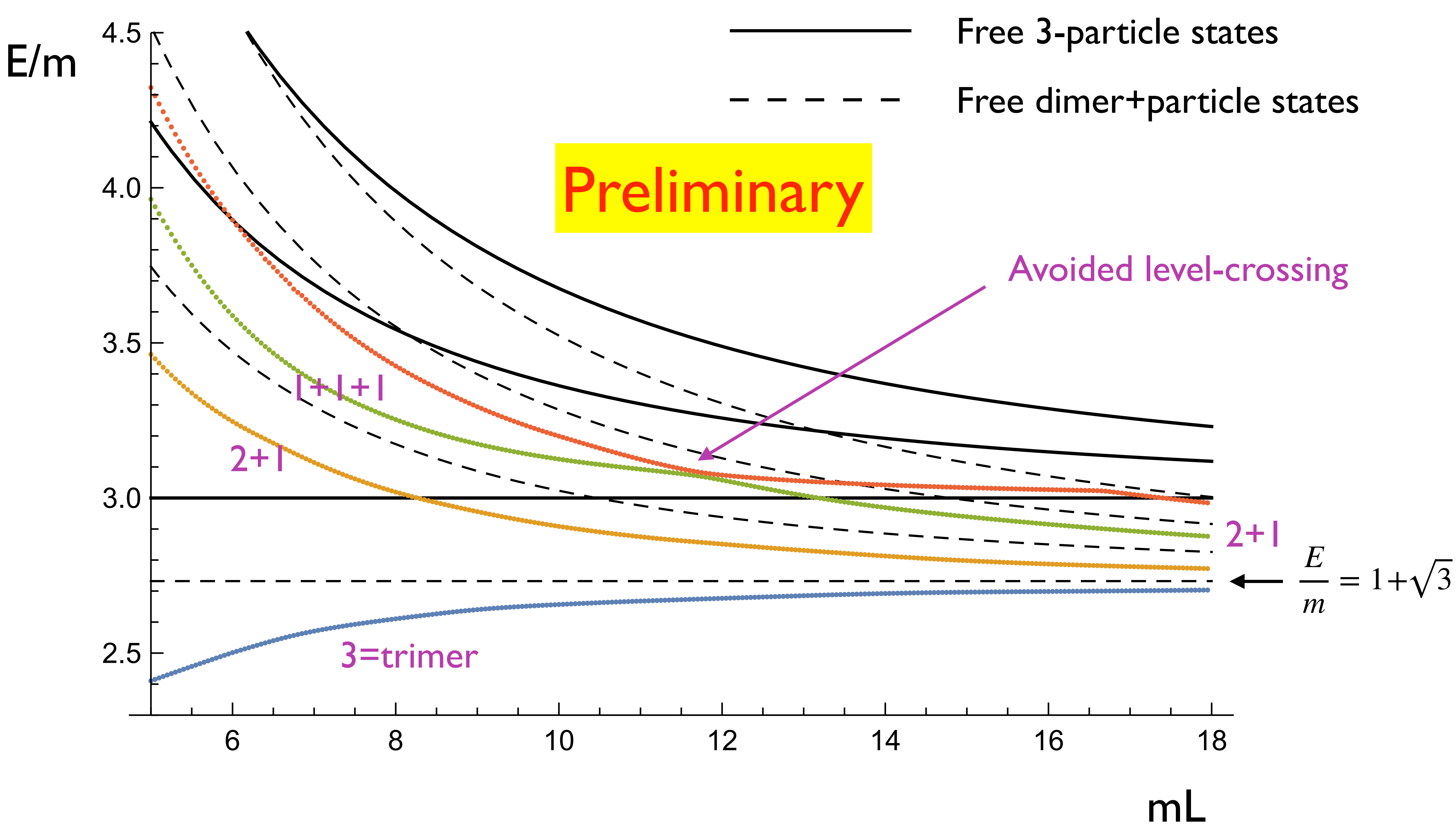
Preliminary

- Same set-up as in [Briceño, Hansen & SRS, 1803.04169], except that by tweaking the PV pole-prescription, the formalism works for a > 1
  - Allows us to study cases where, in infinite-volume, there is a two-particle bound state ("dimer")

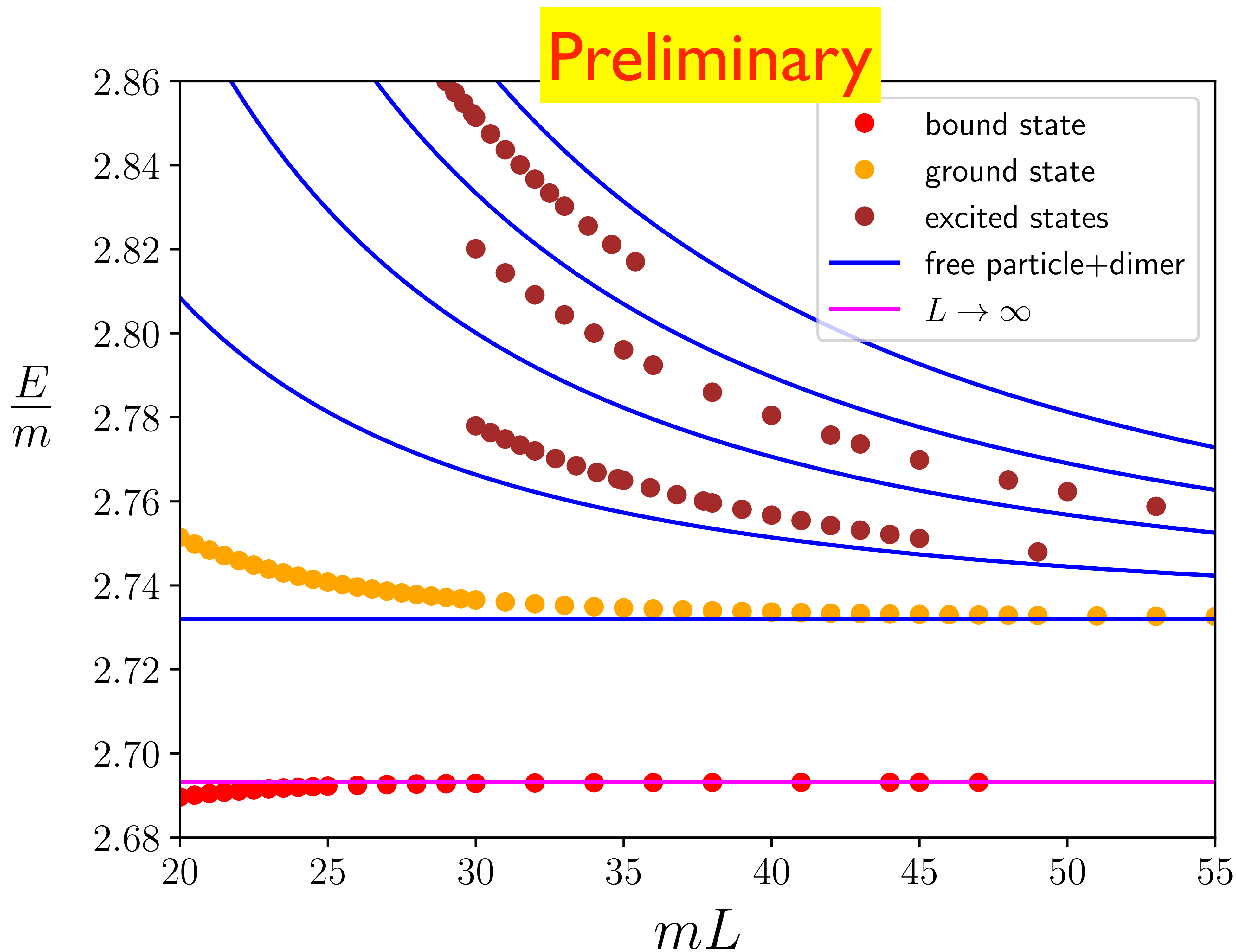
$$E_B/m = 2\sqrt{1 - 1/a^2} \xrightarrow{a=2} \sqrt{3}$$

- Interesting case: choose parameters so that there is both a dimer and a trimer
  - This is the analog (without spin) of studying the n+n+p system in which there are neutron + deuteron and tritium bound states
  - The finite-volume states will have components of all three types

Isotropic approximation: a=2, $\mathcal{K}_{df,3}=0$



Isotropic approximation: a=2, $\mathcal{K}_{\mathrm{df},3}$=0

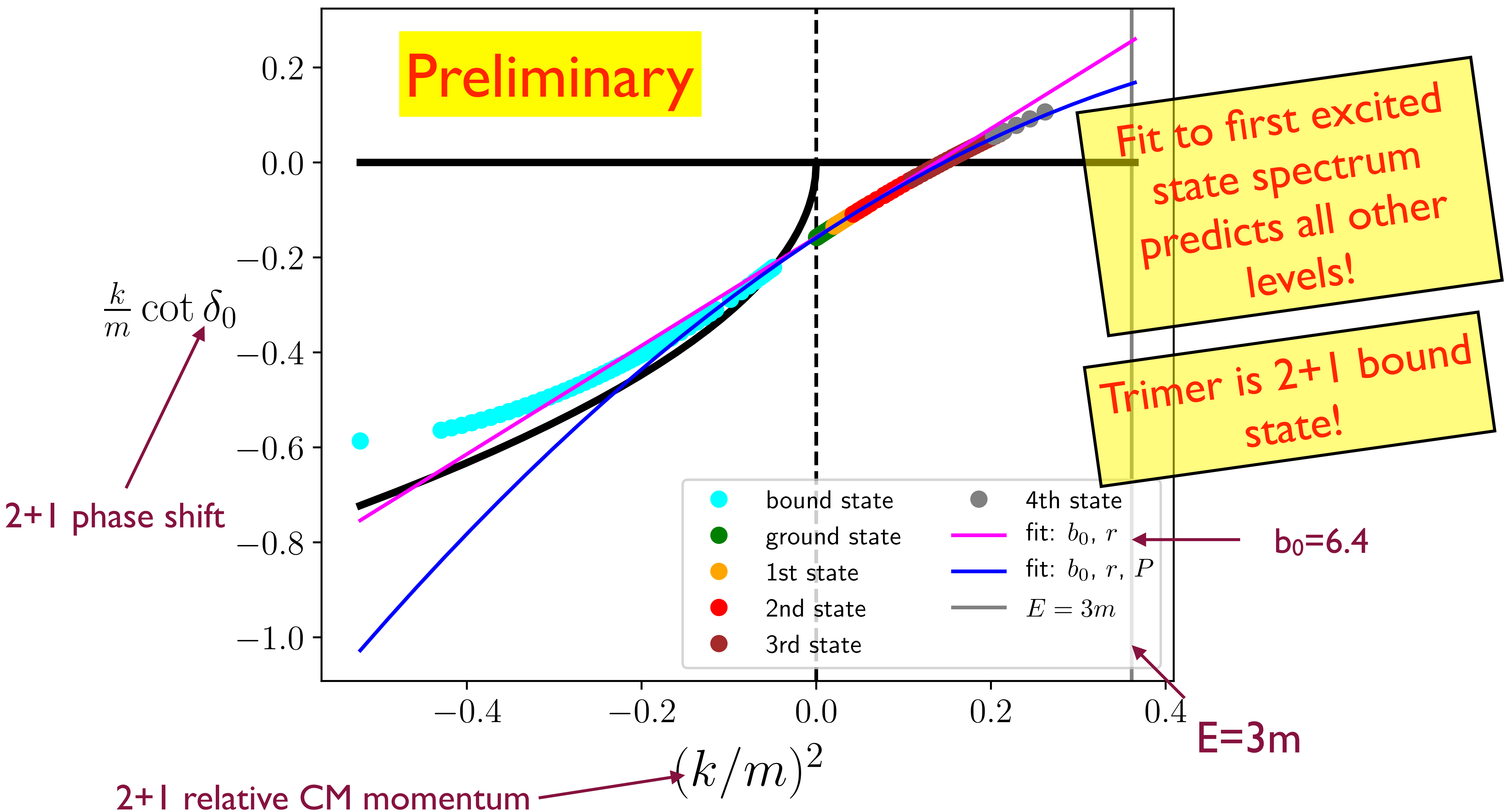


Dominantly 2+1 states

trimer!

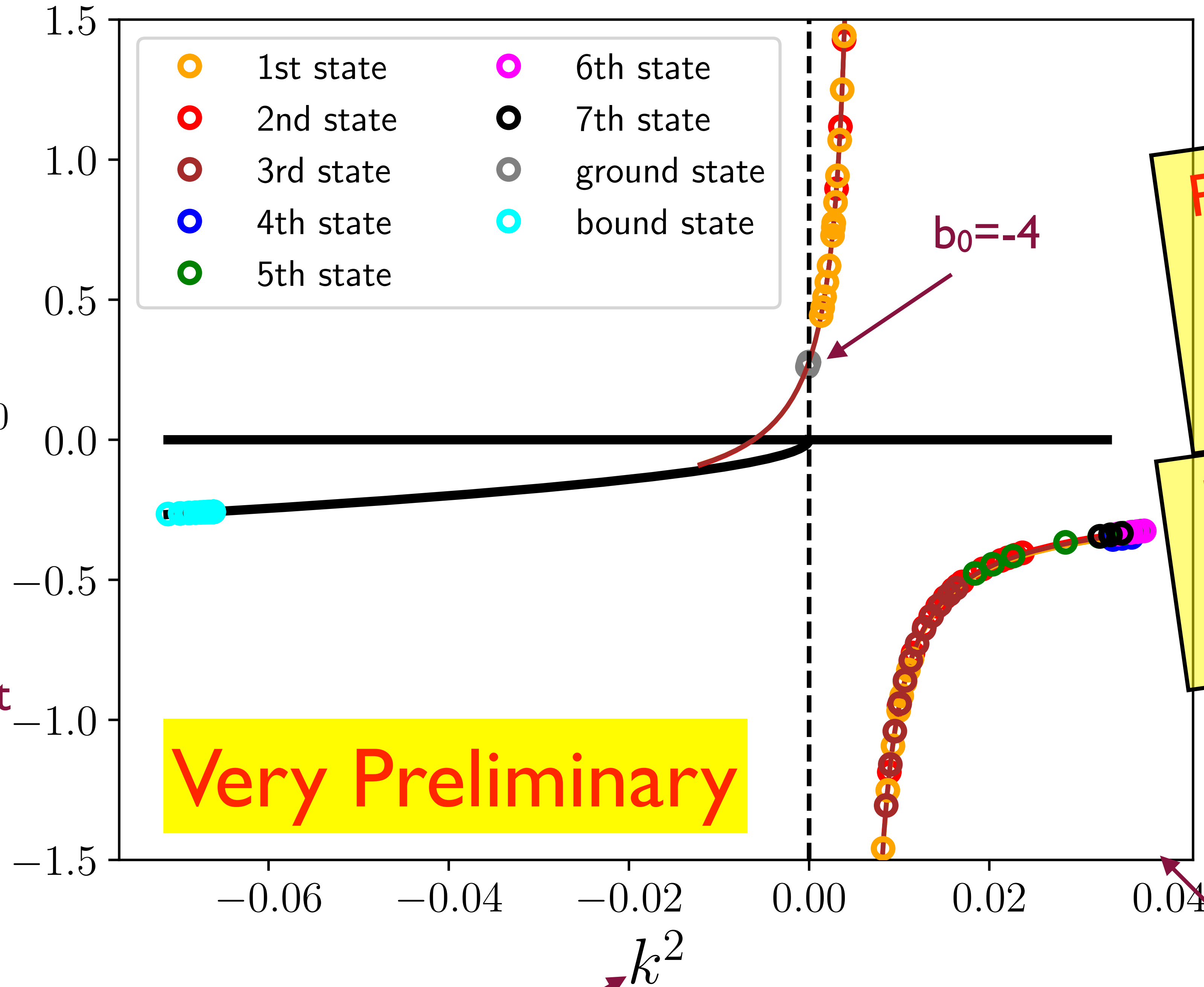
Isotropic approximation: a=2, $\mathcal{K}_{\mathrm{df},3}=0$

2+1 EFT: solve 2-particle quant. cond. for nondegenerate particles



Isotropic approximation: a=6, $\mathcal{K}_{\mathrm{df},3}=0$

2+1 EFT: solve 2-particle quant. cond. for nondegenerate particles



2+1 phase shift

2+1 relative CM momentum

E=3m

Pole reminiscent of that found in n+d and p+d spin doublet scattering

Trimer is probably not a 2+1 bound state!

Outlook

Outlook

- Substantial progress on three-particle formalism
  - Relationship to the other methods (NREFT & FV Unitarity) now well understood [Hansen & SRS (review)]
  - Freedom in PV prescription extends range of original formalism; allows study of cases with two particle bound states and resonances [Blanton, Briceño, Hansen, Romero-López, SRS]
  - Similar freedom allows the use of a higher cutoff, which can be used to investigate unphysical solutions [Blanton, Briceño, Hansen, Romero-López, SRS]
  - Relation of M_3 to $K_{df,3}$ provides an alternative infinite-volume description of M_3 that is unitary—may be useful in data analysis [Briceño, Hansen, SRS & Szczepaniak]
- Extensions to higher spins, nonidentical particles, multiple K-matrix poles, and Lellouch-Lüscher factors are needed, but will likely be straightforward

Outlook

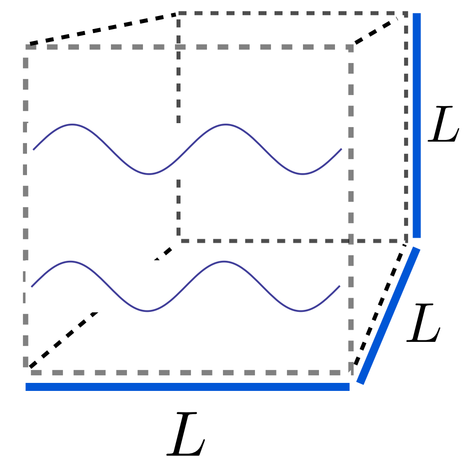
- The major issue is how to make the formalism practical
  - Need physics-based parametrizations of $\mathcal{K}_{df,3}$
  - Need to implement relation between $\mathcal{K}_{df,3}$ and $\mathcal{M}_3$ above threshold
  - Successful extraction of 3-body amplitude from simulations of φ^4 theory [Roméro-Lopez et al.]; application to QCD simulations is underway [HADSPEC collab.]
- Moving to 4+ particles in this fashion looks challenging but does not obviously introduce new theoretical issues

Backup slides

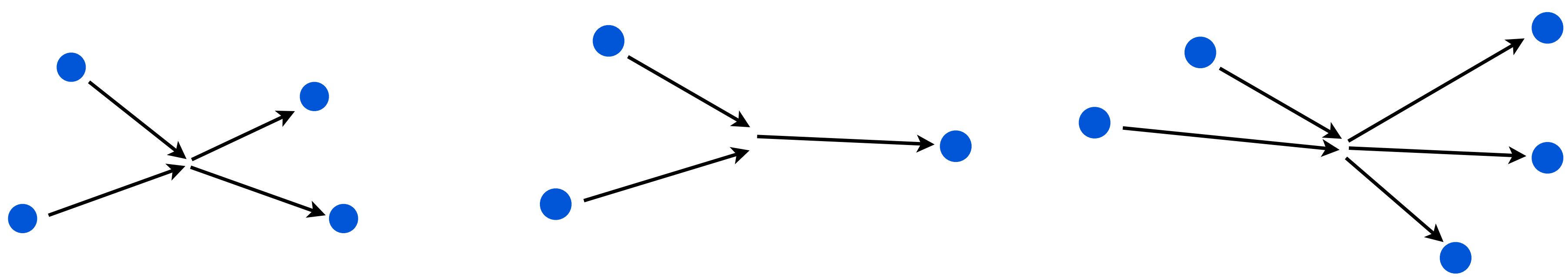
Sketch of derivation of 2-particle quantization condition

[Kim, Sachrajda & SRS 05]

Setup



- Work in continuum (assume that LQCD can control discretization errors)
- Cubic box of size L with periodic BC, and infinite (Minkowski) time
  - Spatial loops are sums: $\frac{1}{L^3}\sum_{\vec{k}}$ $\vec{k} = \frac{2\pi}{L}\vec{n}$
- Consider identical scalar particles with physical mass m, interacting ***arbitrarily*** in a general relativistic effective field theory



Methodology

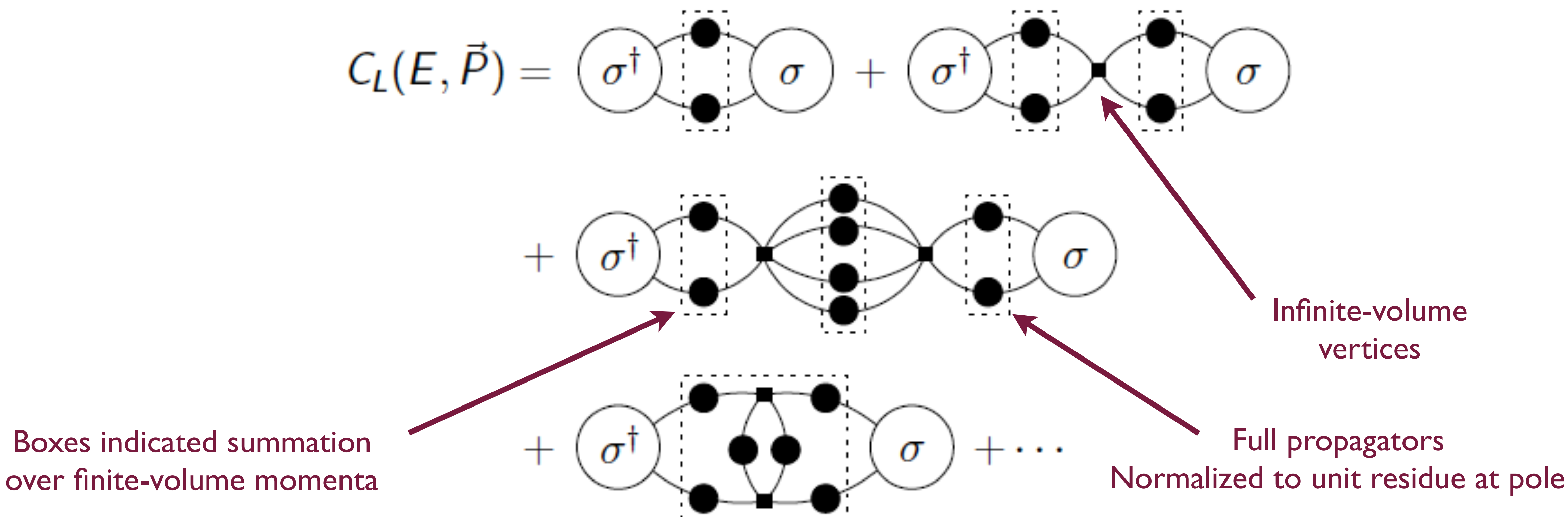
- Calculate (for some **P**=2π**n**$_{\mathbf{P}}$/L)

$$C_L(E,\vec{P}) \equiv \int_L d^4x\, e^{-i\vec{P}\cdot\vec{x}+iEt}\langle\Omega|T\sigma(x)\sigma^\dagger(0)|\Omega\rangle_L$$

CM energy is $E^*=\sqrt{(E^2-P^2)}$

- Poles in C_L occur at energies of finite-volume spectrum: consider m < E* < 3m

- E.g. for 2 particles, σ ~ π^2 :



Step 1

- Replace loop sums with integrals where possible
  - Drop exponentially suppressed terms (~e^{-ML}, $e^{-(ML)^2}$, etc.) while keeping power-law dependence

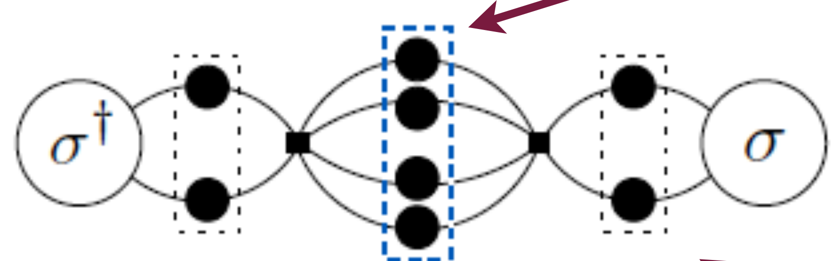
$$\frac{1}{L^3}\sum_{\vec{k}} g(\vec{k}) = \int \frac{d^3k}{(2\pi)^3} g(\vec{k}) + \sum_{\vec{l}\neq\vec{0}} \cancel{\int \frac{d^3k}{(2\pi)^3} e^{iL\vec{l}\cdot\vec{k}} g(\vec{k})}$$

Exp. suppressed if g(k) is smooth
and scale of derivatives of g is ~1/M

- Possible whenever no physical, on-shell cut through loop
  - Can show using time-ordered PT

can replace sum with integral here

$P = (E, \vec{P}) \longrightarrow$



but not here!

Step 2

- Use "sum=integral + [sum-integral]" if integrand has pole, using

$$\left(\int \frac{dk_0}{2\pi}\frac{1}{L^3}\sum_{\vec{k}} - \widetilde{PV}\!\!\int \frac{d^4k}{(2\pi)^4}\right) f(k)\frac{1}{k^2-m^2+\cancel{i\epsilon}}\frac{1}{(P-k)^2-m^2+\cancel{i\epsilon}}g(k)$$

$$= \int d\Omega_{q^*} d\Omega_{q^{*\prime}}\, f^*(\hat{q}^*)\mathcal{F}_{\widetilde{PV}}(q^*, q^{*\prime})g^*(\hat{q}^{*\prime}) \quad + \text{exp. suppressed}$$

q^* is relative momentum of pair on left in CM

Kinematic function

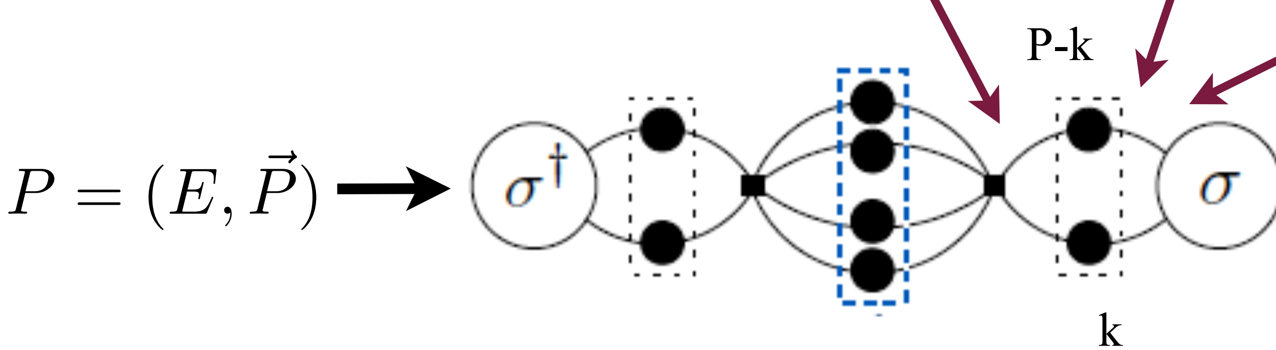
f & g evaluated for ON-SHELL momenta
Depend only on direction in CM

- Example

Focus on this loop

f is left-hand part of integrand

g is right-hand part of integrand

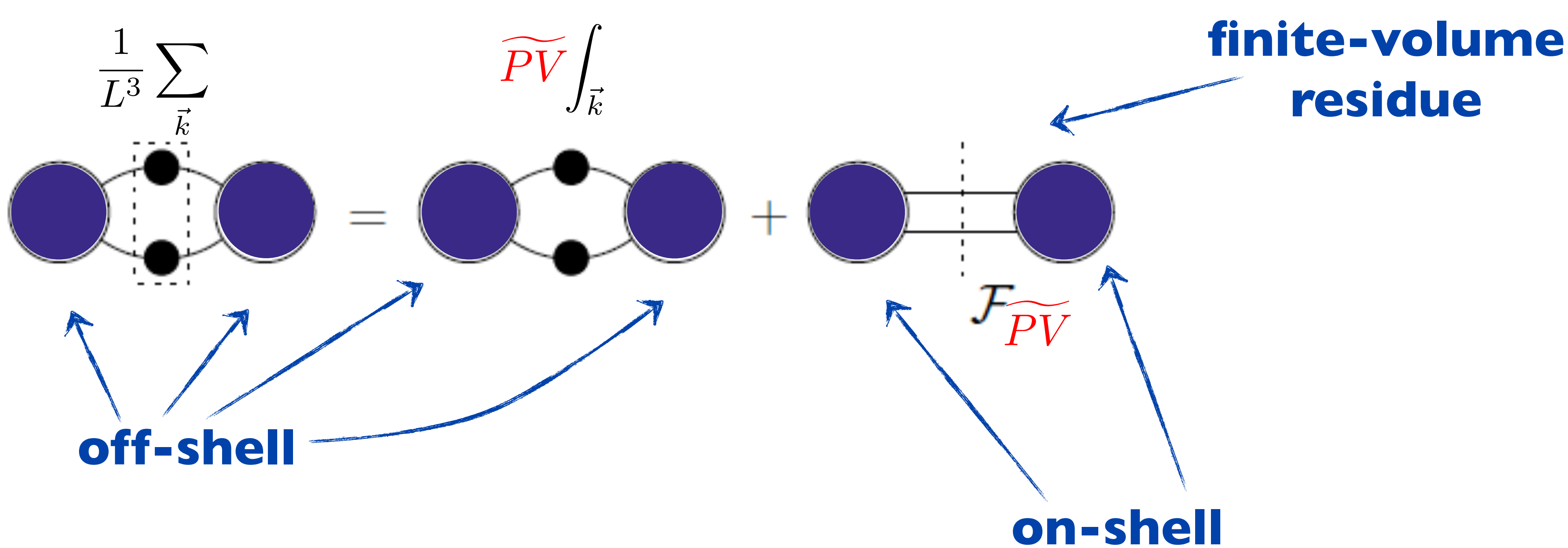


Step 2

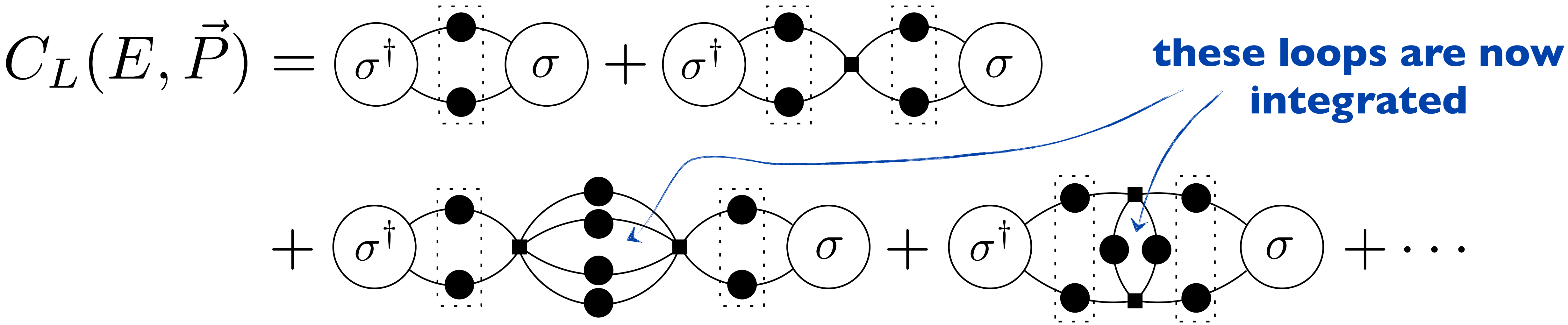
- Use "sum=integral + [sum-integral]" where integrand has pole, with [KSS]

$$\left(\int \frac{dk_0}{2\pi} \frac{1}{L^3} \sum_{\vec{k}} - \widetilde{PV}\!\!\int \frac{d^4k}{(2\pi)^4} \right) f(k) \frac{1}{k^2 - m^2 + \cancel{i\epsilon}} \frac{1}{(P-k)^2 - m^2 + \cancel{i\epsilon}} g(k)$$
$$= \int d\Omega_{q^*} d\Omega_{q^{*\prime}} f^*(\hat{q}^*) \mathcal{F}_{\widetilde{PV}}(q^*, q^{*\prime}) g^*(\hat{q}^{*\prime})$$

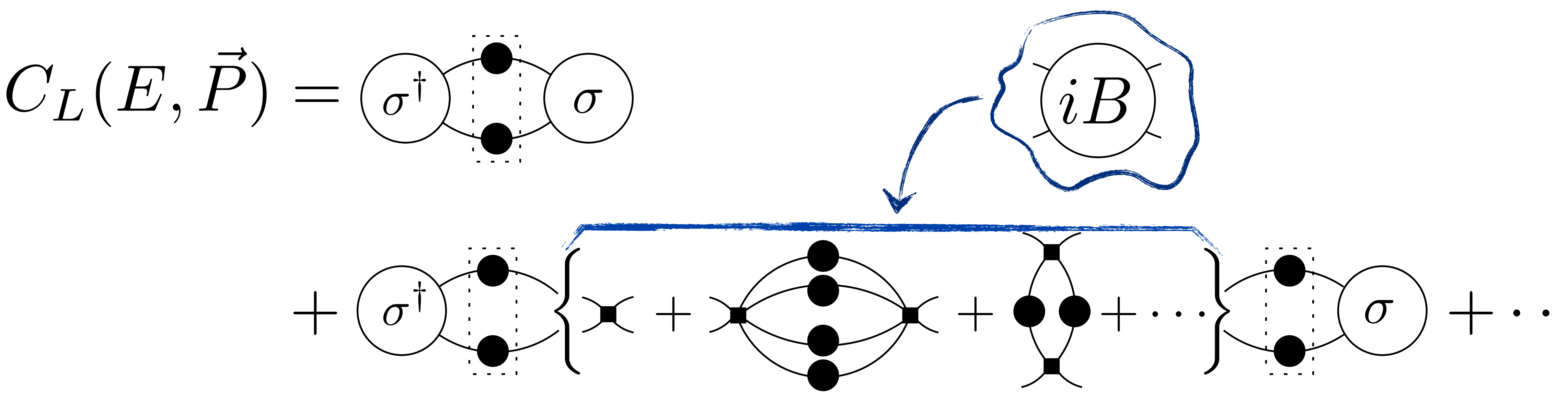
- Diagrammatically



- Apply previous analysis to 2-particle correlator (m < E* < 3m)

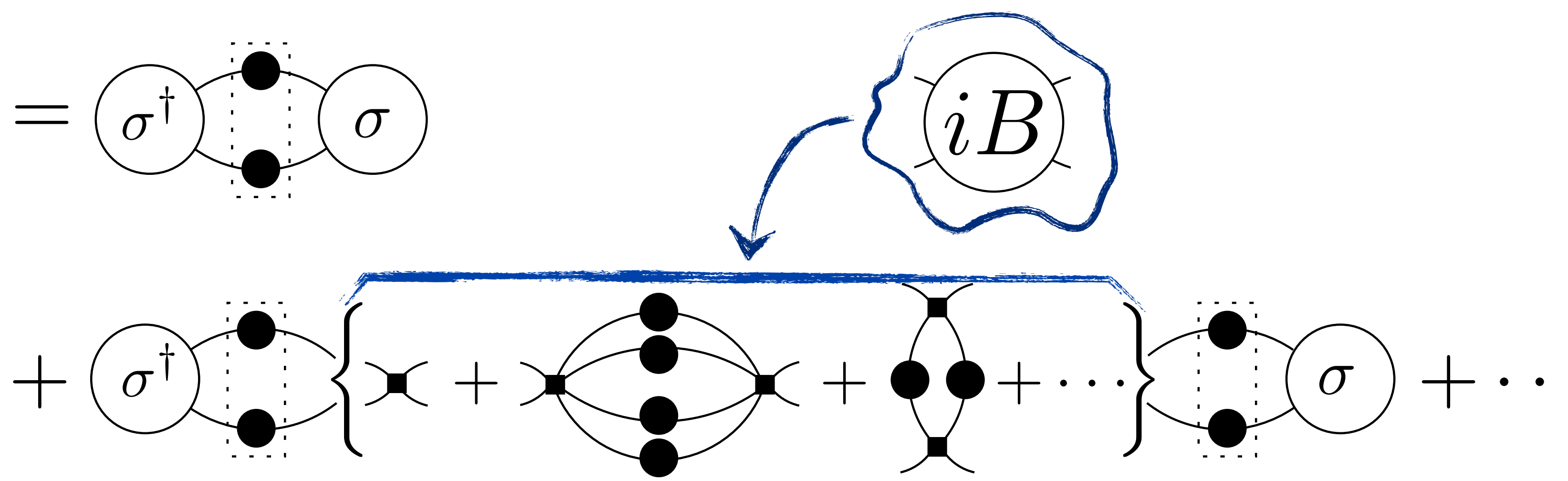


- Collect terms into infinite-volume Bethe-Salpeter kernels



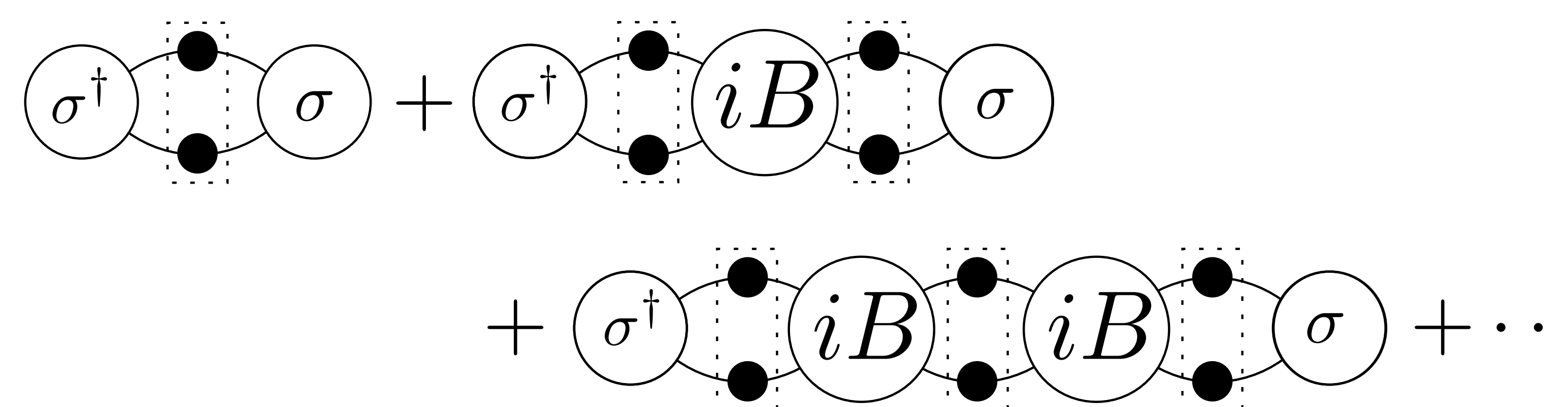
- Apply previous analysis to 2-particle correlator
- Collect terms into infinite-volume Bethe-Salpeter kernels

$C_L(E, \vec{P}) =$



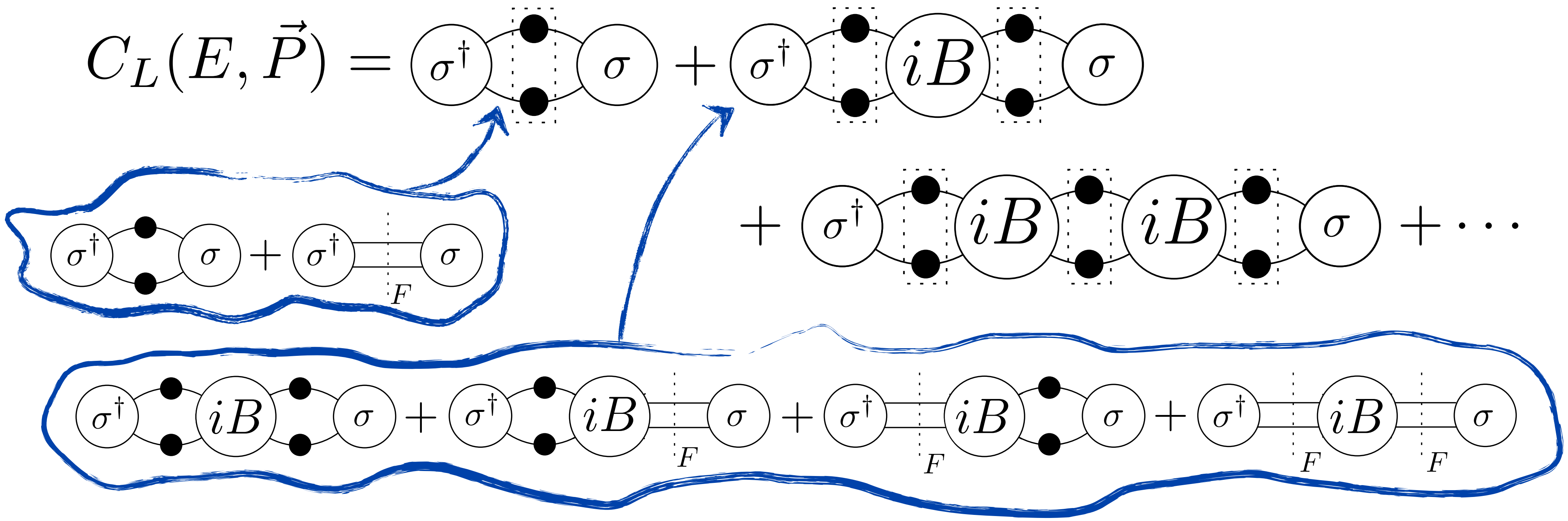
- Leading to

$C_L(E, \vec{P}) =$



- Next use sum identity

$$C_L(E,\vec{P}) =$$

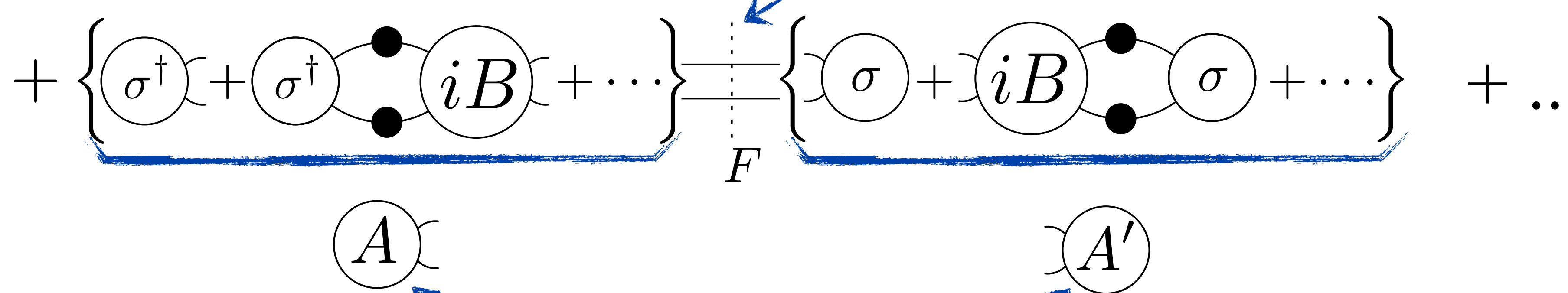


- And regroup according to number of "F cuts"

$$C_L(E,\vec{P}) = C_\infty(E,\vec{P})$$

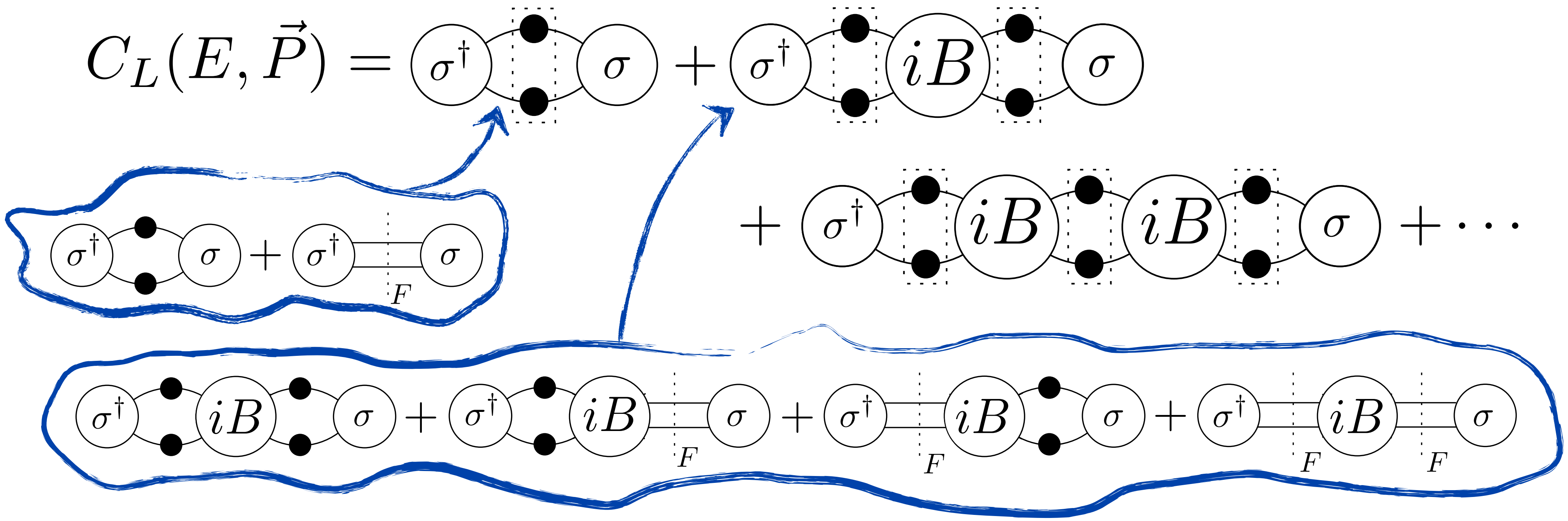
zero F cuts

one F cut

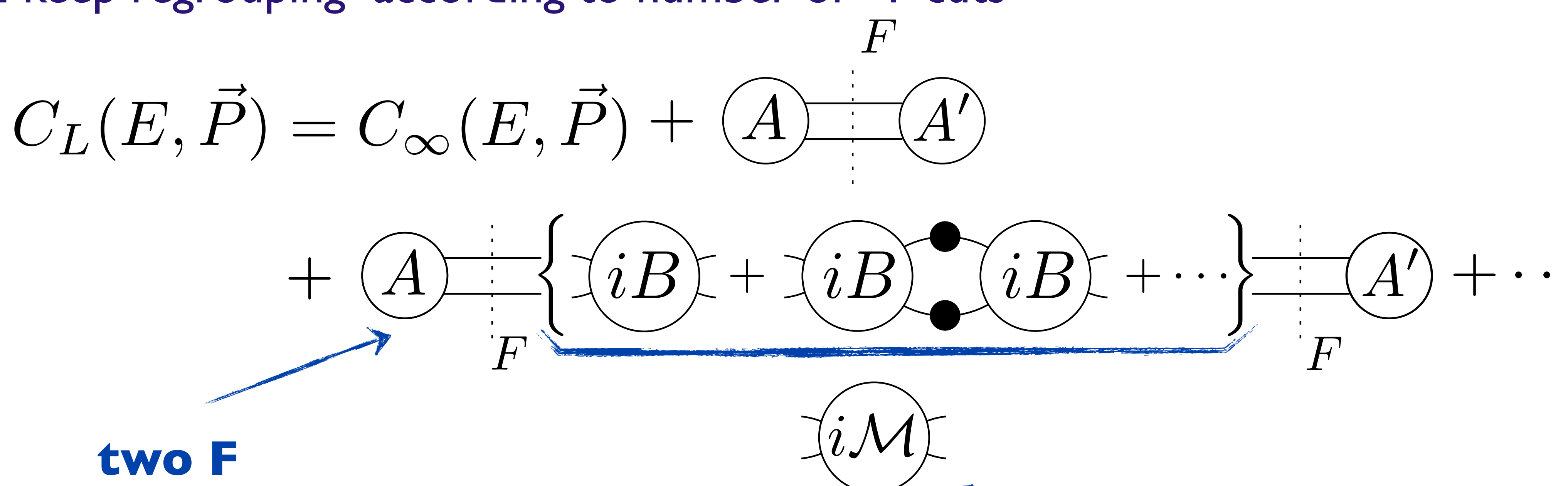


matrix elements:

- Next use sum identity



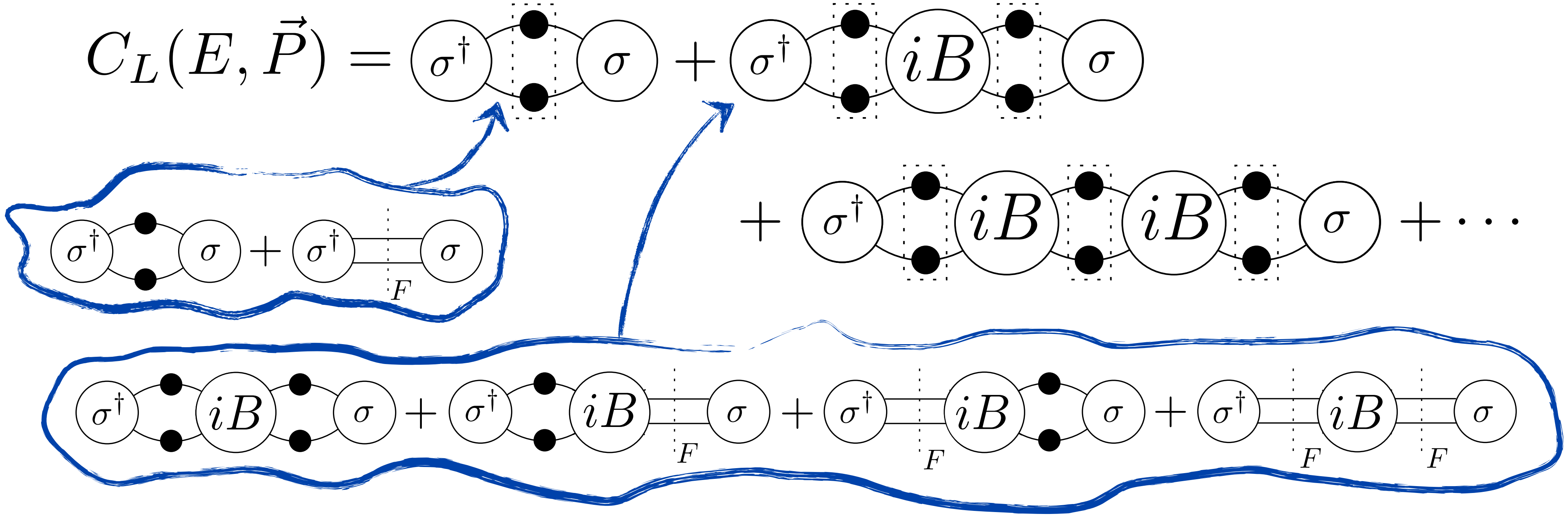
- And keep regrouping according to number of "F cuts"



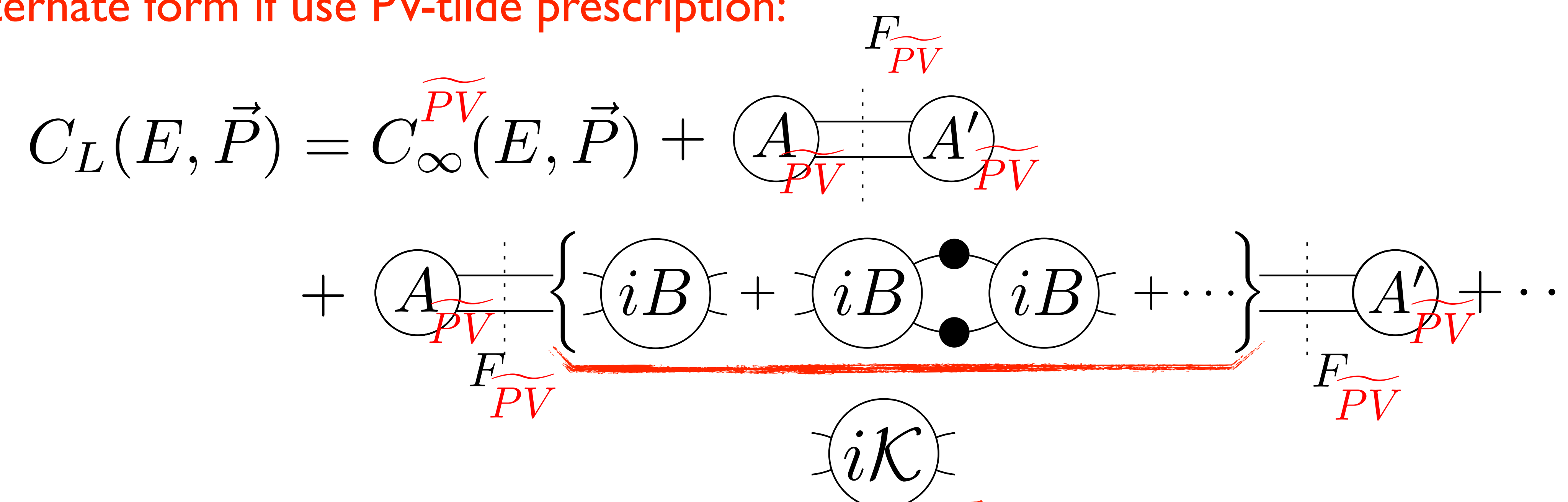
two F cuts

the infinite-volume, on-shell 2→2 scattering amplitude

- Next use sum identity



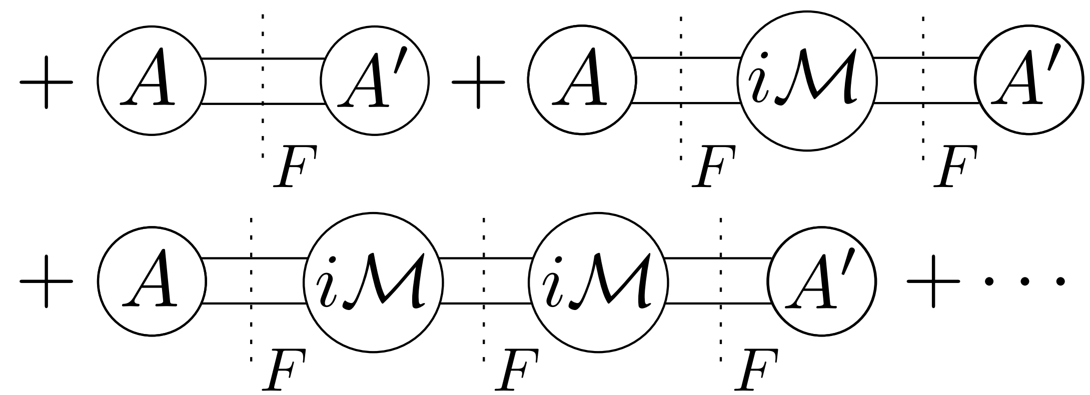
- Alternate form if use PV-tilde prescription:



the infinite-volume, on-shell 2→2 K-matrix

- Final result:

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P})$$

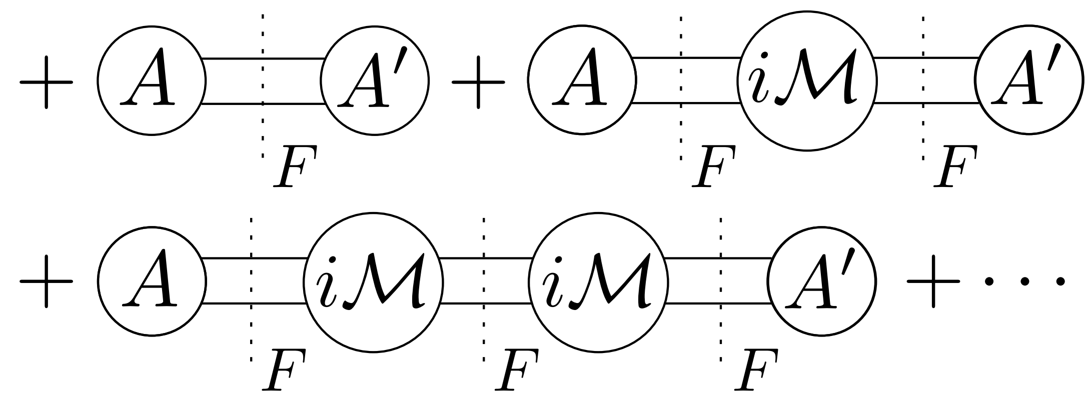


- $$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + \sum_{n=0}^{\infty} A' iF [i\mathcal{M}_{2\to 2} iF]^n A$$

- Correlator is expressed in terms of infinite-volume, physical quantities and kinematic functions encoding the finite-volume effects

- Final result:

$$C_L(E,\vec{P}) = C_\infty(E,\vec{P})$$



- $$C_L(E,\vec{P}) = C_\infty(E,\vec{P}) + \sum_{n=0}^{\infty} A' iF [i\mathcal{M}_{2\to2} iF]^n A$$

- $$C_L(E,\vec{P}) = C_\infty(E,\vec{P}) + A' iF \frac{1}{1 - i\mathcal{M}_{2\to2} iF} A$$

no poles, only cuts

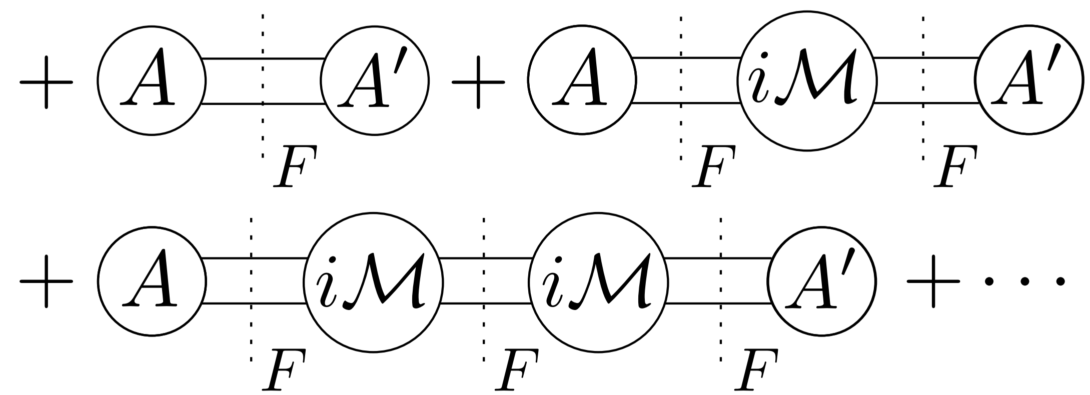
no poles, only cuts

matrices in l,m space

- $C_L(E,\vec{P})$ diverges whenever $iF \dfrac{1}{1 - i\mathcal{M}_{2\to2} iF}$ diverges

- Final result:

$$C_L(E,\vec P) = C_\infty(E,\vec P)$$



- $$C_L(E,\vec P) = C_\infty(E,\vec P) + \sum_{n=0}^{\infty} A' iF[i\mathcal{M}_{2\to2} iF]^n A$$

- $$C_L(E,\vec P) = C_\infty(E,\vec P) + A' iF \frac{1}{1 - i\mathcal{M}_{2\to2} iF} A$$

no poles, only cuts

no poles, only cuts

matrices in l,m space

$\Rightarrow$ $$\Delta_{L,\vec P}(E) = \det[(iF)^{-1} - i\mathcal{M}_{2\to2}] = 0$$

$\Rightarrow$ $$\Delta_{L,\vec P}(E) = \det[(F_{\widetilde{PV}})^{-1} + \mathcal{K}_2] = 0$$

Alternative form

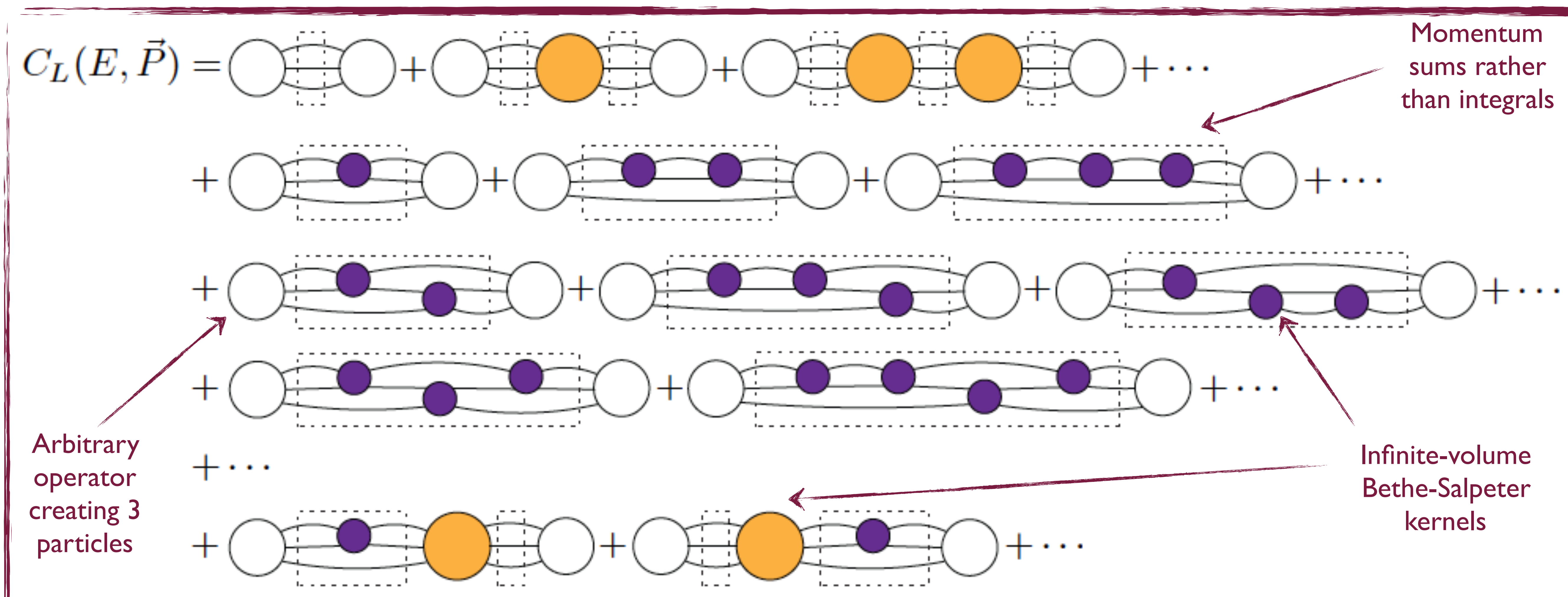
Sketch of derivation of 3-particle quantization condition

[Hansen & SRS, arXiv:1408.5933 & 1504.04248]

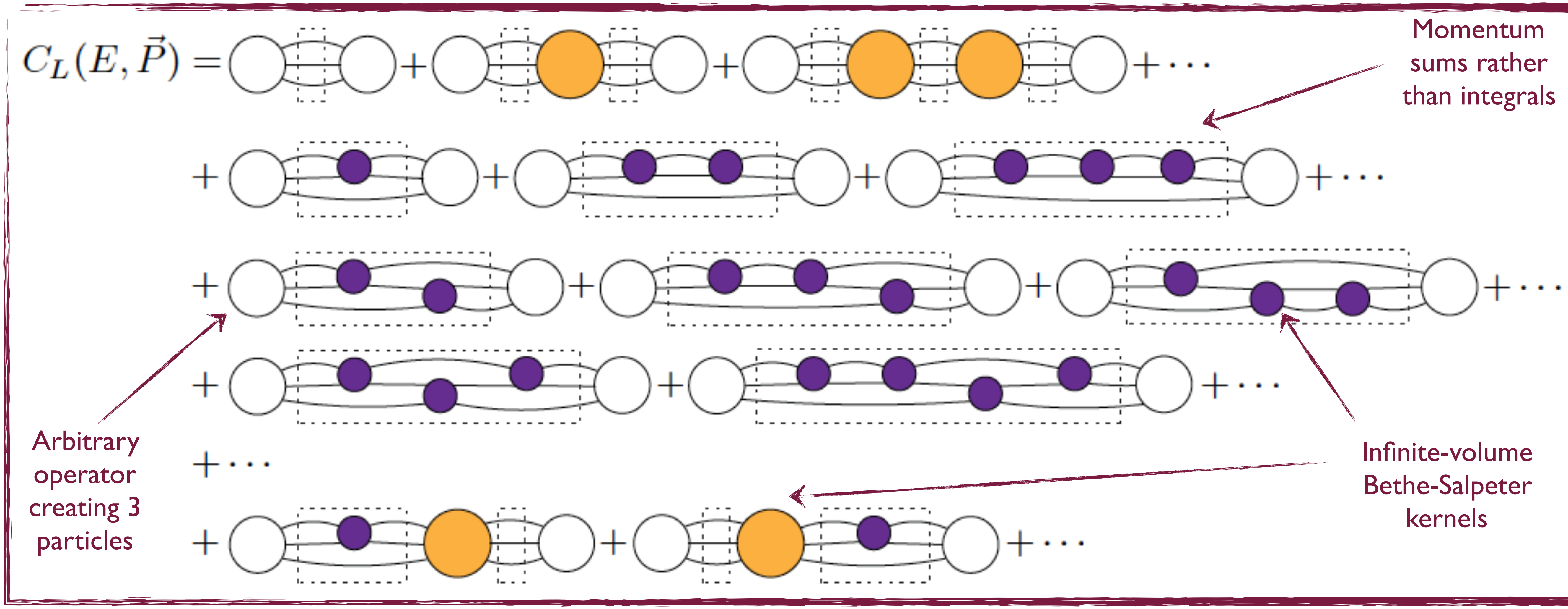
Derivation

(1)

- Generic relativistic EFT, working to all orders
  - Do not need a power-counting scheme
  - To simplify analysis: impose a global Z_2 symmetry (G parity) & consider identical scalars
- Obtain spectrum from poles in finite-volume correlator
  - Consider $E_{CM} < 5m$ so on-shell states involve only 3 particles



Derivation



- Replace sums with integrals plus sum-integral differences to extent possible
  - If summand has pole or cusp then difference ~$1/L^n$ and must keep (Lüscher zeta function)
  - If summand is smooth then difference ~ exp(-mL) and drop
- Avoid cusps by using PV prescription—leads to generalized 3-particle K matrix
- Subtract above-threshold divergences of 3-particle K matrix—leads to $\mathcal{K}_{df,3}$

(2)

Derivation

(3)

- Reorganize, resum, … to separate infinite-volume on-shell relativistically-invariant non-singular scattering quantities ($\mathcal{K}_2$, $\mathcal{K}_{\mathrm{df},3}$) from known finite-volume functions (F [Lüscher zeta function] & G ["switch function"])

$$\Rightarrow \boxed{\det\left[F_3^{-1} + \mathcal{K}_{\mathrm{df},3}\right] = 0}$$

Derivation

(4)

- Relate $\mathcal{K}_{df,3}$ to $\mathcal{M}_3$ by taking infinite-volume limit of finite-volume scattering amplitude
  - Leads to infinite-volume integral equations involving $\mathcal{M}_2$ & cut-off function H
  - Can formally invert equations to show that $\mathcal{K}_{df,3}$ (while unphysical) is relativistically invariant and has same properties under discrete symmetries (P,T) as $\mathcal{M}_3$

Involve only $\mathcal{M}_2$ and G
so "known"

$$i\mathcal{M}_{L,3\to3} = i\mathcal{D}_L + \mathcal{S}\left[\mathcal{L}_L \; i\mathcal{K}_{\mathrm{df},3\to3} \frac{1}{1 - iF_3 \; i\mathcal{K}_{\mathrm{df},3\to3}} \mathcal{R}_L\right]$$

$$i\mathcal{M}_{3\to3} = \lim_{L\to\infty}\Big|_{i\epsilon} \; i\mathcal{M}_{L,3\to3}$$

Sums over k go over
to integrals with iε pole prescription