

# The physics of the QCD critical point

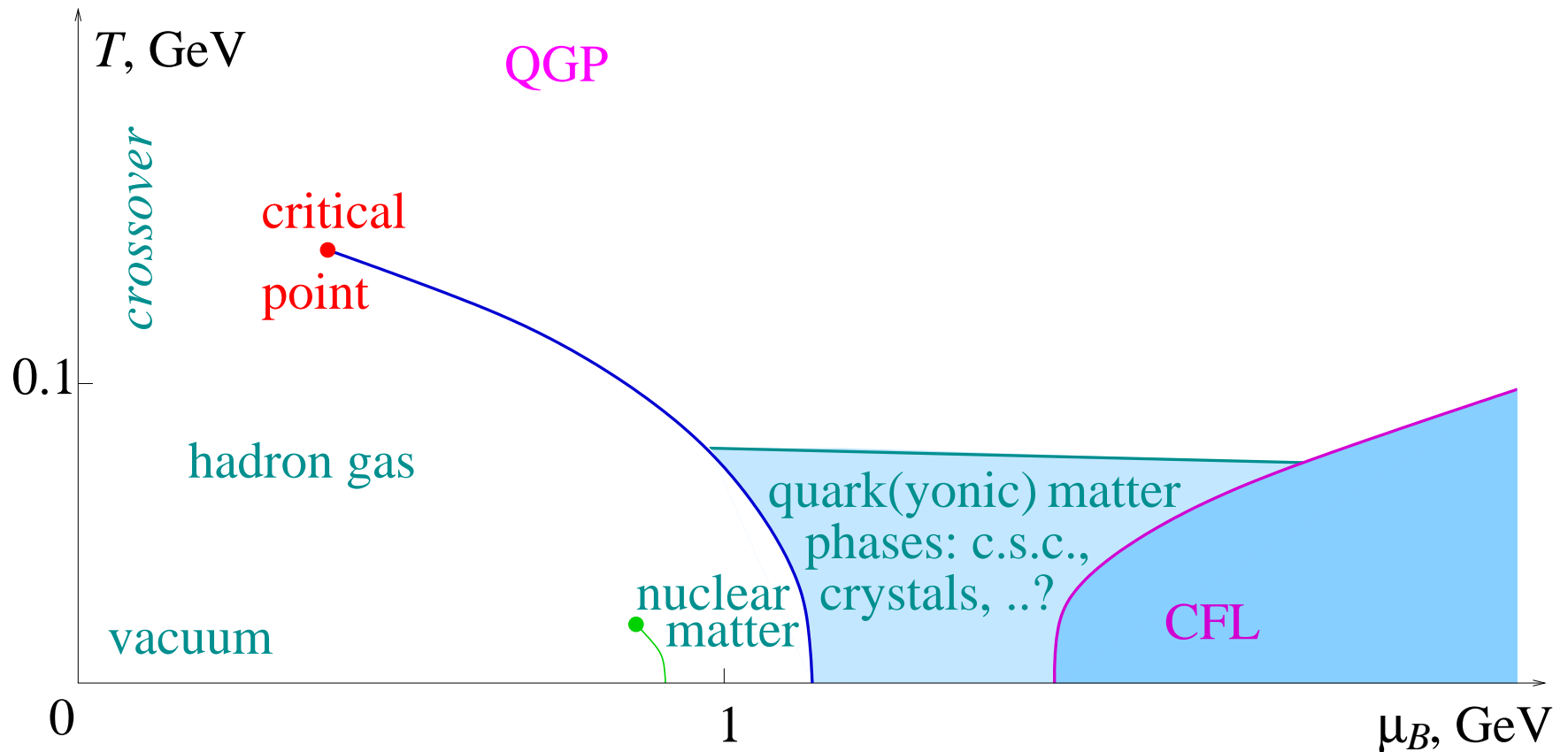
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# Outline

- Introduction: QCD phase diagram and heavy ion collisions.
- Fluctuation signatures of the critical point: non-Gaussian moments.
- ● Evolution of fluctuations: stochastic Boltzmann with soft mode  
– concept and application.
- Concluding remarks.

# QCD phase diagram (what we think we know)

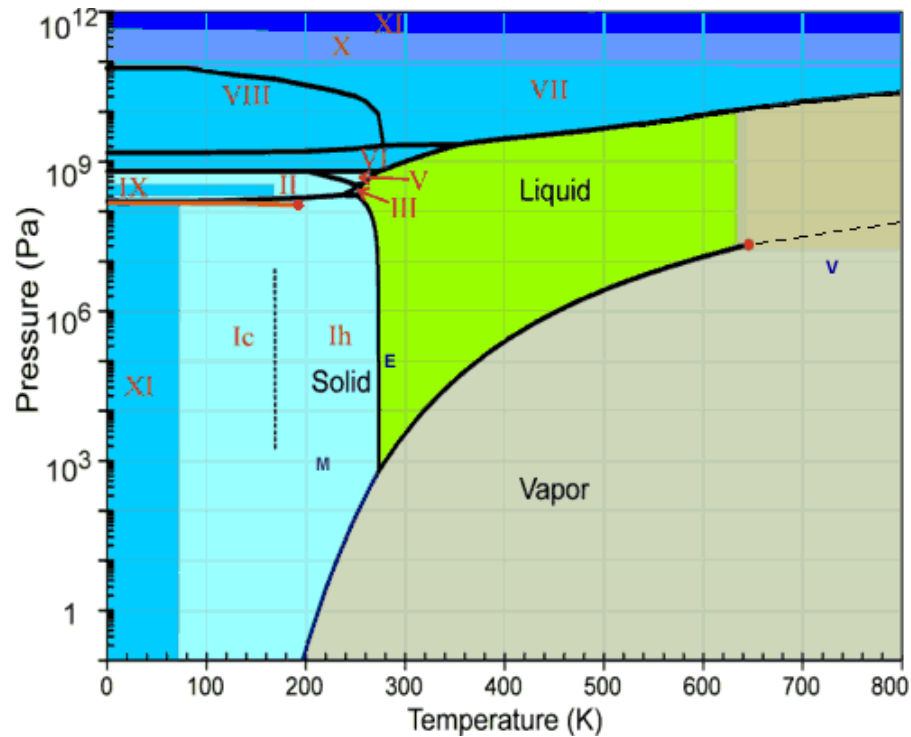


- Models (and lattice) suggest the transition becomes 1st order at some  $\mu_B$ .
- Can we observe the **critical point** in heavy ion collisions, and how?

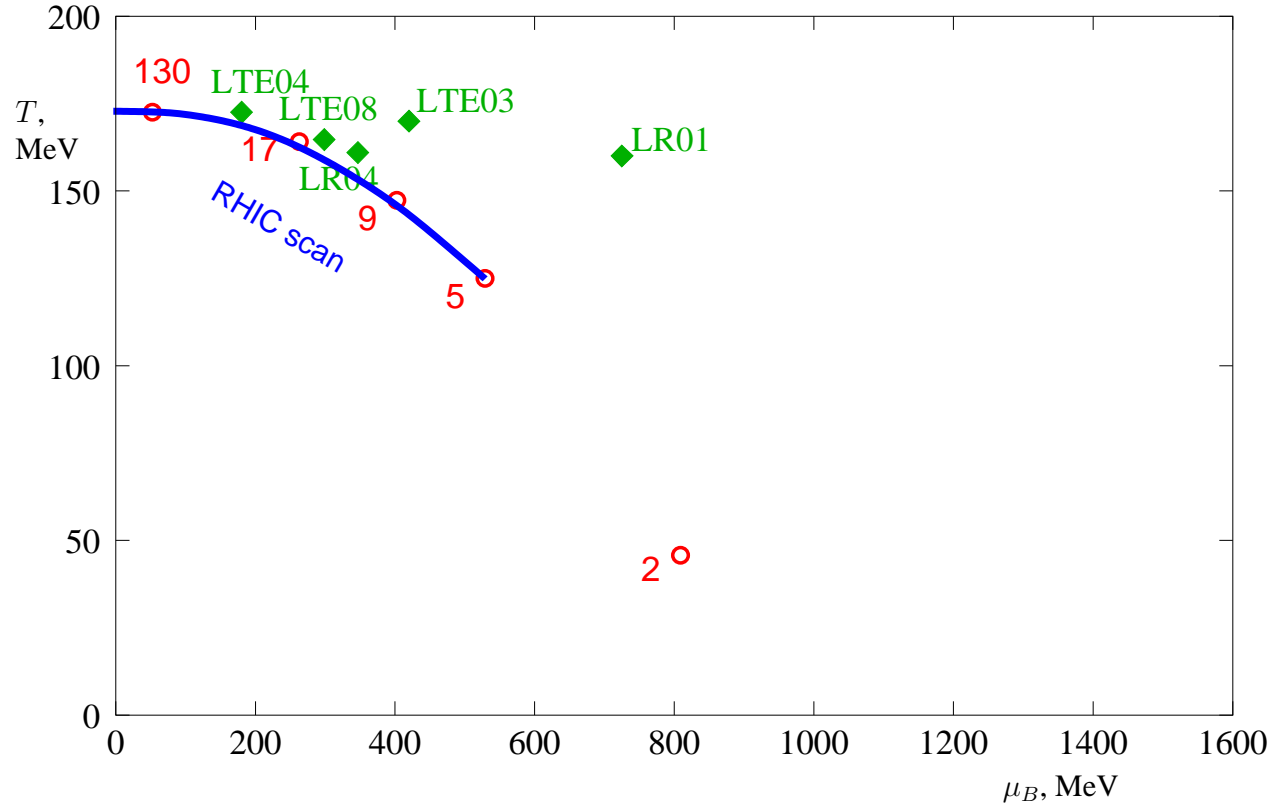
# Critical point(s) in known liquids

Critical point is a common phenomenon in many liquids (critical opalescence).

Water:



# What do we need to discover the QCD critical point?



- Experiments: RHIC, NA61(SHINE), FAIR/GSI, NICA
- Lattice: understand systematic errors.  
Predictions from different approaches must converge.
- Understand critical phenomena in the dynamical environment of a h.i.c.,  
develop optimal signatures.

# Phase transitions and fluctuations

- Thermodynamic quantities fluctuate, but  $\delta\varepsilon \sim \frac{1}{\sqrt{V}}$ . Central limit theorem.  
I.e., fluctuations are small when  $V$  is large.
- Large compared to what? More precisely: when  $V/\xi^3$  is large.
- When  $\xi$  increases, so do the fluctuations.
- At the critical point  $\xi \rightarrow \infty$  (more precisely  $\xi \rightarrow$  system size).  
Fluctuations get as large as they can.  
This is the signature of a critical point.

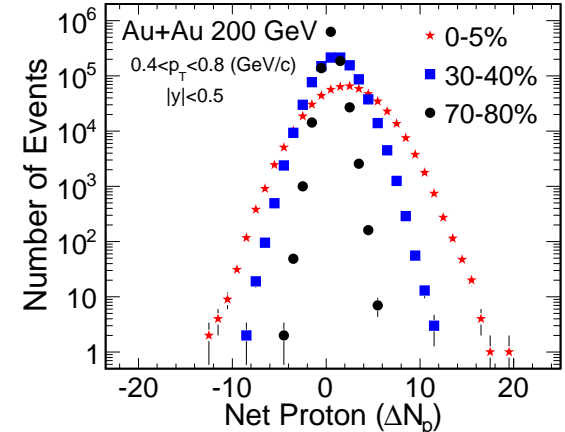
# Fluctuation signatures

- Experiments give for each event: multiplicities  $N_\pi$ ,  $N_p$ , ..., set of momenta  $p$ , etc.

These quantities fluctuate event-by-event.

- Measure – sq. var., e.g.,  $\langle(\delta N)^2\rangle, \langle(\delta p_T)^2\rangle$ .

- What is the magnitude of these fluctuations near the QCD C.P.? (Rajagopal-Shuryak-MS, 1998)



- Universality* tells us how it grows at the critical point:  $\langle(\delta N)^2\rangle \sim \xi^2$ .

Correlation length is a universal measure of the “distance” from the c.p.

It diverges as  $\xi \sim (\Delta\mu \text{ or } \Delta T)^{-2/5}$  as the c.p. is approached.

- Magnitude of  $\xi$  is limited  $< \mathcal{O}(2\text{--}3 \text{ fm})$  (Berdnikov-Rajagopal).

- “Shape” of the fluctuations can be measured: non-Gaussian moments.

As  $\xi \rightarrow \infty$  fluctuations become less Gaussian ( $\xi \rightarrow \infty$  vs  $N \rightarrow \infty$ ).

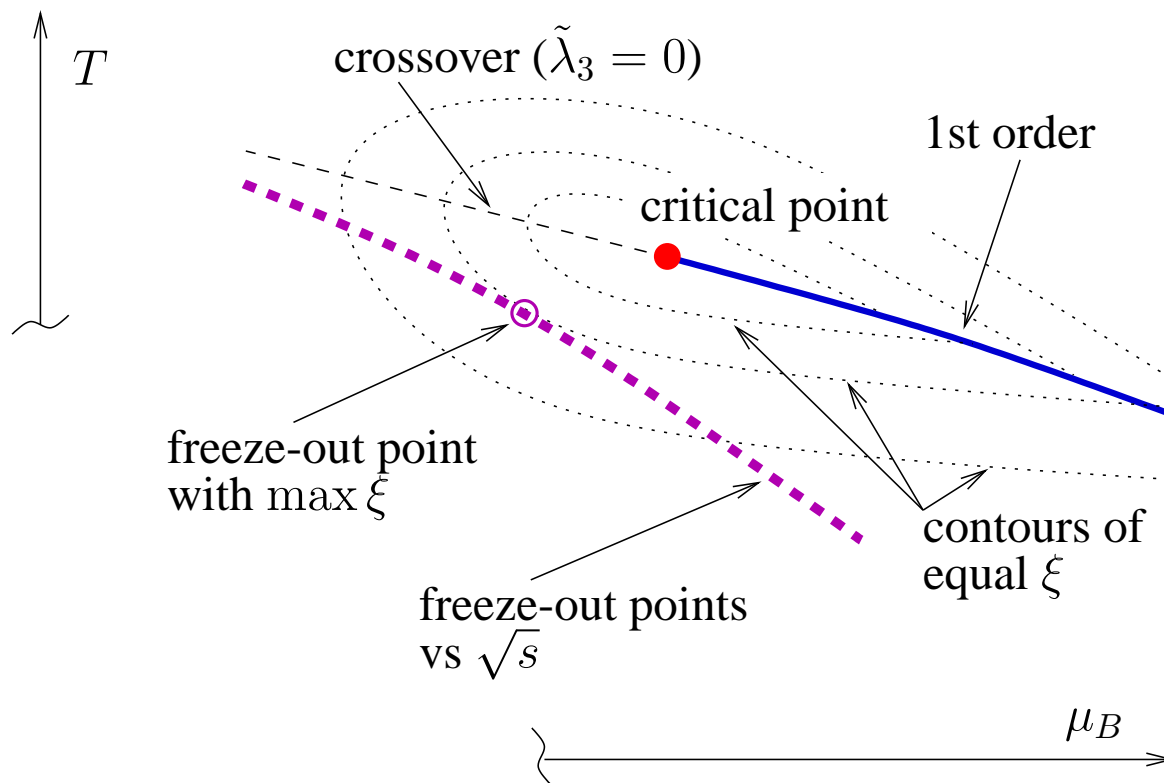
- Higher cumulants show even stronger dependence on  $\xi$

(MS, PRL 102:032301,2009):

$$\langle(\delta N)^3\rangle \sim \xi^{4.5}, \quad \langle(\delta N)^4\rangle - 3\langle(\delta N)^2\rangle^2 \sim \xi^7$$

which makes them more sensitive signatures of the critical point.

# Energy scan and fluctuation signatures



- The 3rd cumulant ( $\sim$  skewness) receives contribution from the critical point.
- Although “only”  $\xi^{4.5}$  (vs  $\xi^7$  for the 4th cumulant) it might give better signal/noise.  $\langle(\delta N)^4\rangle_c = \langle(\delta N)^4\rangle - 3\langle(\delta N)^2\rangle^2 = \mathcal{O}(N^2) - \mathcal{O}(N^2)$ .

# Critical fluctuations away from equilibrium

# Evolution of fluctuations

- In equilibrium, fluctuations are distributed as  $e^S$  (Einstein, 1910).
- In a heavy ion collision the “equilibrium” state evolves with time.

How do the fluctuations evolve?

- Typically, fluctuations adjust to the new equilibrium on the scale of  $\tau_{\text{thermalization}}$ .

But some may be much slower: hydrodynamic “slow” modes.

Hydrodynamics can describe those fluctuations and their slow evolution.  
What more do we need?

- Most observables are *not* conserved quantities. E.g., mean  $p_T$ . Can we describe their evolution?
  - Relevant long-standing problem: how quickly are the *critical*  $p_T$  fluctuations “washed out” after chemical freezeout? How much remains until kinetic freezeout?
  - Contrast with particle number (conserved after chem. freezeout), which can only be changed by (much slower) diffusion.



# The approach: Boltzmann++

- Primary degree of freedom in such a description should be particle distribution in phase space. Governed by Boltzmann equation.
- Fluctuations are encoded in correlators of the fluctuating phase space distribution. Can be described by stochastic Boltzmann eqn.: Fox and Uhlenbeck, 1970.
- To describe special effects of the critical point we need to couple the particles to the “soft” mode. The soft mode will also become stochastic – Langevin.
- We are led to consider a system of Vlasov-like equations *with noise*. What is the dynamics of fluctuations in such a system?  
(MS, PRD81:054012,2010)

# Model: particles in a scalar field, or variable mass.

- E.O.M. follow from action: field + particle of variable mass. E.g.,  $M = g\sigma$ .

$$\mathcal{S} = \int d^3\mathbf{x} \frac{1}{2} (\partial_\mu \sigma \partial^\mu \sigma - U(\sigma)) - \int ds M(\sigma),$$

Particle e.o.m.:

$$dp^\mu / d\tau = \partial^\mu M(\sigma), \quad \text{with} \quad p^\mu = M dx^\mu / d\tau,$$

Field e.o.m.:

$$\partial^2 \sigma + dU/d\sigma + \int ds (dM/d\sigma) = 0;$$

- $\Rightarrow$  Boltzmann equation, with effect of collisions added:

$$\frac{p^\mu}{M} \frac{\partial f}{\partial x^\mu} + \partial^\mu M \frac{\partial f}{\partial p^\mu} + \mathcal{C}[f] = 0,$$

+ field equation:

$$\partial^2 \sigma + dU/d\sigma + (dM/d\sigma) \int_{\mathbf{p}} f/\gamma = 0.$$

# Linearized equations and noise

- Linearize around equilibrium  $f_\sigma(\mathbf{p}) = e^{\mu/T} e^{-\gamma(\mathbf{p})M/T}$  ;  
define  $h(x; p)$  as  $f = f_\sigma(1 + h)$ , and  $g \equiv dM/d\sigma$ :

$$\dot{h} - \dot{\sigma} g/(\gamma T) + \mathbf{v} \cdot \nabla h + \mathcal{I}[h] = 0, \quad (\text{note: } \nabla_\sigma \text{ cancels})$$

$$\ddot{\sigma} - \nabla^2 \sigma + m^2 \sigma + g \int_{\mathbf{p}} f_0 h/\gamma = 0. \quad (\text{note: } m^2 = U''(0) + (g \int_{\mathbf{p}} f_\sigma/\gamma)').$$

- Introduce noise: (note:  $\sigma$  is stochastic due to mixing, even if  $\eta = 0$ )

$$\dot{h} - \pi g/(\gamma T) + \mathbf{v} \cdot \nabla h + \mathcal{I}[h] = \xi; \quad (\text{Fox-Uhlenbeck, 1970})$$

$$\dot{\pi} - \nabla^2 \sigma + m^2 \sigma + g \int_{\mathbf{p}} f_0 h/\gamma + \Gamma_0 \pi = \eta; \quad (\text{"soft" mode eqn.})$$

$$\dot{\sigma} - \pi = 0.$$

- Write the system of eqs. in matrix form

$$\dot{\mathbf{h}} + \mathbb{G}\mathbf{h} = \boldsymbol{\xi}, \quad \mathbf{h} = \begin{pmatrix} h \\ \pi \\ \sigma \end{pmatrix}, \quad \boldsymbol{\xi} = \begin{pmatrix} \xi \\ \eta \\ 0 \end{pmatrix}.$$

Langevin-like matrix eqn.  $\mathbb{G}\mathbf{h}$  – generalized “drift”.

# Noise and fluctuation-dissipation

- Expand the entropy function to  $\mathcal{O}(h^2)$ :

$$S[f, \sigma] = \int d^3x \left[ - \int_p f (\log f - 1) - \frac{1}{T} \left( \frac{\pi^2}{2} + \frac{(\nabla \sigma)^2}{2} + U(\sigma) + \int_p (M(\sigma)\gamma - \mu)f \right) \right] = -\frac{1}{2} \mathbf{h} \cdot \mathbb{E} \mathbf{h} + \mathcal{O}(h^3)$$

Then find the matrix of noise correlations

$$\langle \xi(t_1) \otimes \xi(t_2) \rangle = 2\mathbb{Q} \delta(t_1 - t_2),$$

such that, in equilibrium,  
 $\langle \mathbf{h} \otimes \mathbf{h} \rangle = \mathbb{E}^{-1}$ .

$\mathbb{Q}$  is given by

$$2\mathbb{Q} = \mathbb{G} \mathbb{E}^{-1} + \mathbb{E}^{-1} \mathbb{G}^\dagger.$$

Thus

$$\begin{aligned} \langle \xi(x_1, p_1) \xi(x_2, p_2) \rangle \\ = (\mathcal{K} + \mathcal{K}^\dagger) \delta_{p_1, p_2}^3 \delta^4(x_1 - x_2); \end{aligned}$$

$$\langle \eta(x_1) \eta(x_2) \rangle = 2\Gamma_0 T \delta^4(x_1 - x_2);$$

$$\langle \xi(x_1, p_1) \eta(x_2) \rangle = 0.$$

where  $\mathcal{K}$  is linearized  $\mathcal{C}$ :  $\mathcal{K}[h] = \mathcal{I}[h/f_0]$

- $\langle \xi \xi \rangle$  same in Fox-Uhlenbeck.
- Noise is uncorrelated, local, etc.

# Evolution of correlator

The correlator of fluctuating d.o.f.  $\mathbf{h} \equiv (h, \pi, \sigma)$ , defined as

$$\langle \mathbf{h} \otimes \mathbf{h} \rangle \equiv \Sigma.$$

obeys equation

$$\dot{\Sigma} = -\mathbb{G}\Sigma - \Sigma\mathbb{G}^\dagger + 2\mathbb{Q}.$$

which is easily integrated using  $\mathbb{V}(t, t') = \mathcal{T} \exp \left( - \int_{t'}^t dt'' \mathbb{G}(t'') \right)$  :

$$\Sigma(t) = \mathbb{V}(t, 0)\Sigma(0)\mathbb{V}^\dagger(t, 0) + 2 \int_0^t dt' \mathbb{V}(t, t')\mathbb{Q}(t')\mathbb{V}^\dagger(t, t'),$$

- $\Sigma \rightarrow \mathbb{E}^{-1}$  in equilibrium;
- The rate and the slowest mode of this equilibration is determined by the lowest e.v. of the “drift” matrix  $\mathbb{G}$ .

# Zero (slow) mode

- If particle number is conserved by  $\mathcal{C}$ , i.e.,  $\mathcal{I}[\text{const}] = 0$ , at  $q = 0$  (homogeneous fluctuations) “drift” matrix  $\mathbb{G}$  has a *zero mode*:

$$\mathbf{h}_{\lambda_0} \sim \begin{pmatrix} -m^2 \\ 0 \\ gn_0 \mathcal{M}[1/\gamma] \end{pmatrix}, \quad \text{where } n_0 \equiv \int_{\mathbf{p}} f_0 \quad \text{and } \mathcal{M}[h] \equiv \frac{\int_{\mathbf{p}} f_0 h}{n_0}.$$

- Corresponds to particle number conservation, but is a mixture of  $h$  and  $\sigma$ . (Recall  $f = f_\sigma(1 + h)$ .)

When  $g \rightarrow 0$ :  $\mathbf{h}_{\lambda_0} \sim (1, 0, 0)$ , i.e., pure  $h$ ,

but when  $m \rightarrow 0$ :  $\mathbf{h}_{\lambda_0} \sim (0, 0, 1)$ , i.e., pure  $\sigma$ .

- Consequences of the mixing: at  $q \neq 0$

$$\lambda_0 = Dq^2 + \mathcal{O}(q^4), \quad \text{where } D = \frac{m^2}{\Delta^2 + m^2} D_0.$$

$D_0$  - diffusion coeff. at  $g = 0$  and  $\Delta^2 \equiv \frac{g^2 n_0}{T} \mathcal{M}[1/\gamma]^2$ .

- This agrees with hydro result:  $D = \bar{\sigma}/\chi$  (and  $\chi \sim 1/m^2$ ).

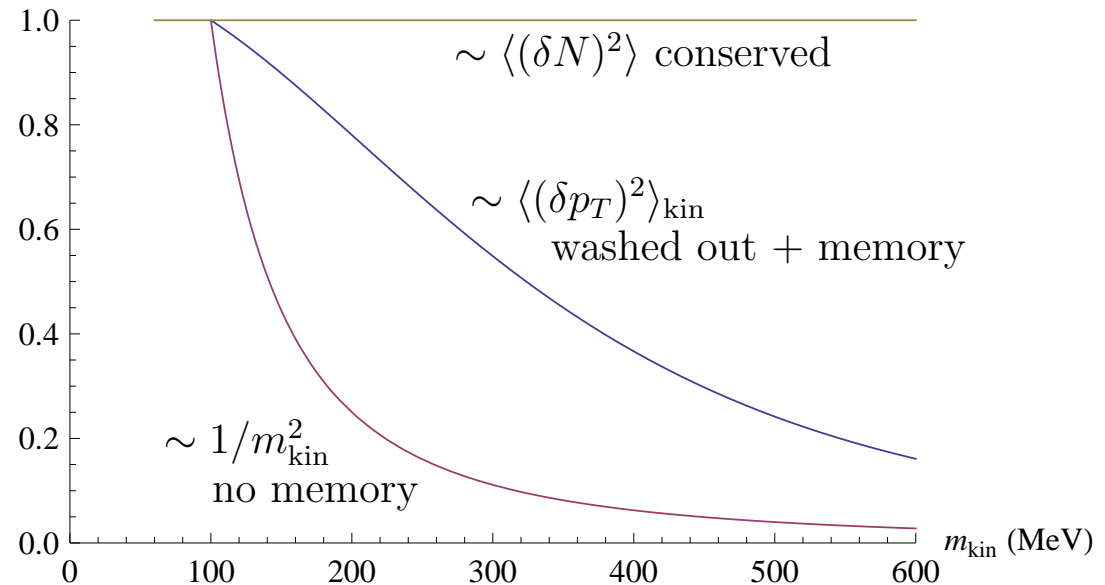
# Mixing with slow mode and “memory” effect

- Idealize:  $\tau_{\text{coll}} \ll \tau_{\text{evol}} \ll \tau_D$ . As a function of  $m_{\text{kin}}$ , the fluctuation at kin. freezeout is the following fraction of that at chem. freezeout ( $m_{\text{kin}} > m_{\text{chem}}$ )

$$\frac{\langle (\delta p_T)^2 \rangle_{\text{kin}}}{\langle (\delta p_T)^2 \rangle_{\text{chem}}} = \frac{m_{\text{chem}}^2}{m_{\text{kin}}^2} + \underbrace{\left(1 - \frac{m_{\text{chem}}^2}{m_{\text{kin}}^2}\right) \times \left(\frac{\Delta^2}{\Delta^2 + m_{\text{kin}}^2}\right)^2}_{\text{off equilibrium contribution of slow mode}} \times e^{-\tau/\tau_D}$$

For example, let

- $m_{\text{chem}} = 100 \text{ MeV}$  ( $\xi = 2 \text{ fm}$ )
- neglecting diffusion, i.e.,  $\tau_{\text{coll}} \ll \tau_{\text{evol}} \ll \tau_D$
- and using  $\Delta \approx 460 \text{ MeV}$  for SPS top energy  $T$  and  $\mu$



- This *analytical* solution requires simplifying assumptions. More realistic treatment could be achieved numerically.

# Concluding remarks

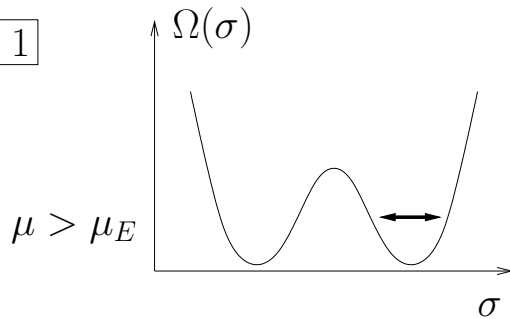
- Phase diagram of QCD is a fascinating challenge for both theory and experiment.
- On the theory side:
  - The sign problem is a big hurdle, but progress is being made towards locating the QCD critical point.
  - Higher non-gaussian moments offer many new signatures, more sensitive to the critical point.  
(See also 1006.4636[hep-ph] by Athansiou, Rajagopal, MS.)
  - The dynamic evolution of fluctuations in the vicinity of the critical point can be modelled.
- On the experimental side – heavy ion collisions are still to leave a mark on the QCD phase diagram. The beam energy scan at RHIC has the potential to discover the QCD critical point. The reach will be extended by FAIR/GSI towards higher densities.

# Higher moments of fluctuations

# Critical mode and equilibrium fluctuations



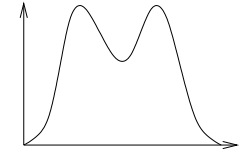
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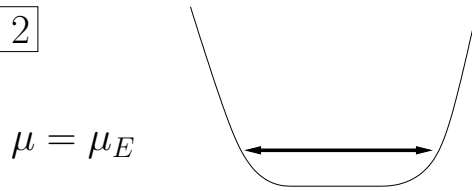
Consider "soft mode"  $\sigma$   
e.g.,  $\sigma \sim \bar{\psi}\psi - \langle \bar{\psi}\psi \rangle$

$$\langle \sigma^2 \rangle \sim (\Omega'')^{-1}$$

Einstein, 1910:  
probability  $\sim$  number  
of states with that  $\sigma$   
i.e.,  $e^S$ , or  $e^{-\Omega/T}$

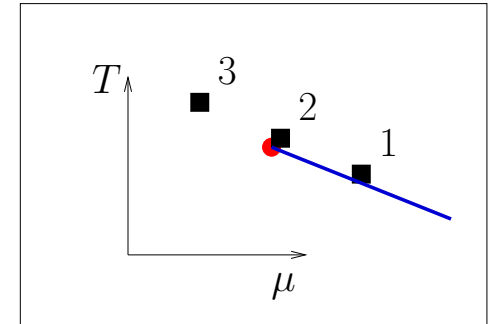


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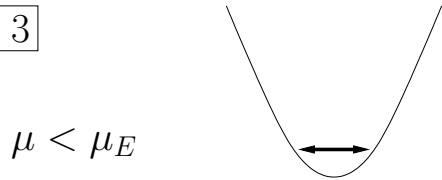


$$(\Omega'')^{-1} \rightarrow \infty$$

large **equilibrium** fluctuations



3



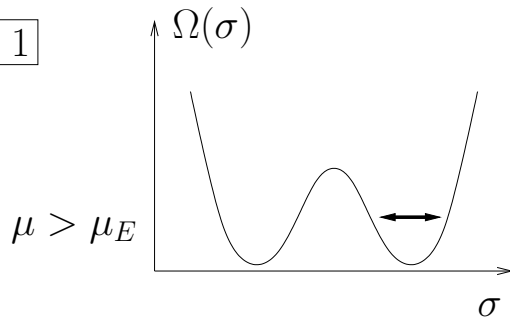
Magnitude of fluctuation and correlation length:

$$\langle \sigma(\mathbf{x})\sigma(\mathbf{0}) \rangle \sim \begin{cases} e^{-|\mathbf{x}|/\xi} & \text{for } |\mathbf{x}| \gg \xi \\ 1/|\mathbf{x}| & \end{cases}$$

# Critical mode and equilibrium fluctuations



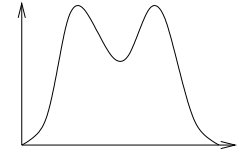
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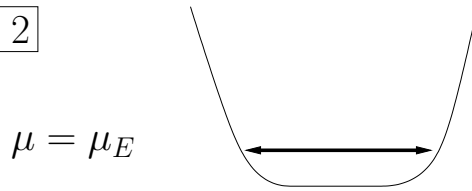
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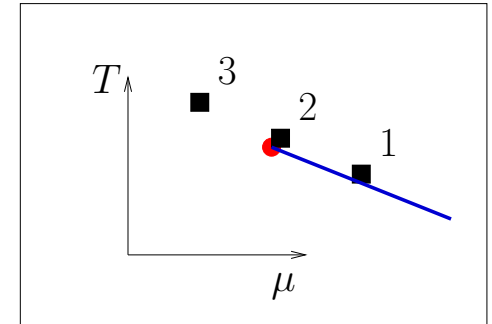


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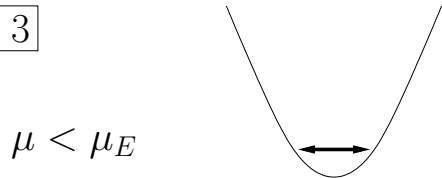


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Magnitude of fluctuation and correlation length:

$$\langle \sigma(\mathbf{x})\sigma(\mathbf{0}) \rangle \sim \begin{cases} e^{-|\mathbf{x}|/\xi} & \text{for } |\mathbf{x}| \gg \xi \\ 1/|\mathbf{x}|^{1+\eta} & \text{for } |\mathbf{x}| \ll \xi \end{cases}$$

$$\langle \sigma_0^2 \rangle = \int d^3x \langle \sigma(\mathbf{x})\sigma(\mathbf{0}) \rangle \sim \xi^{2-\eta}$$

critical singularity is a *collective* phenomenon



$\sigma$  or  $n_B$  or  $T^{00}$ ? Because they mix, only *one* linear combination is critical.

# Relation between $\sigma$ fluctuations and observables

Consider example: fluctuations of multiplicity of pions (or protons).

- Free gas:  $n_p^0$  – fluctuating occupation number of momentum mode  $p$ .  
Ensemble (event) average  $\langle n_p^0 \rangle = f_p$  and

$$n_p^0 = f_p + \delta n_p^0; \quad \langle \delta n_p^0 \delta n_k^0 \rangle = f'_p \delta_{pk}; \quad f_p = (e^{\omega_p/T} \mp 1)^{-1}; \quad f'_p \equiv f_p(1 \pm f_p).$$

- Couple these particles to  $\sigma$  field:  $G\sigma\pi\pi$  (or  $g\sigma\bar{N}N$ ).  
Think of  $m^2 \equiv m_0^2 + 2G\sigma$  as “fluctuating mass”. Then

$$\delta n_p = \delta n_p^0 + \frac{\partial f_p}{\partial m^2} 2G\sigma = \delta n_p^0 + \frac{f'_p}{\omega_p} \frac{G}{T} \sigma$$

- Using  $\langle \delta n_p^0 \sigma \rangle = 0$  and  $\langle \sigma^2 \rangle = (T/V)\xi^2$ .

$$\langle \delta n_p \delta n_k \rangle = f'_p \delta_{pk} + \frac{1}{VT} \frac{f'_p}{\omega_p} \frac{f'_k}{\omega_k} G^2 \xi^2.$$

More formal derivation: PRD65:096008,2002

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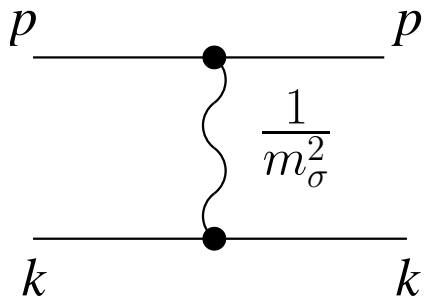
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# 4-point function

- The 2-particle correlator measures 4-point function at  $q = 0$  (for  $p \neq k$ ). Singularity appears at  $q = 0$  due to vanishing  $\sigma$  screening mass  $m_\sigma \rightarrow 0$ . (i.e.,  $\xi = 1/m_\sigma \rightarrow \infty$ ).



$$\langle \delta n_p \delta n_k \rangle_\sigma = \frac{1}{T} \frac{f_p(1+f_p)}{\omega_p} \frac{f_k(1+f_k)}{\omega_k} \frac{G^2}{m_\sigma^2}.$$

Check:  $\langle \delta n_p \delta n_k \rangle = \langle n_p n_k \rangle - \langle n_p \rangle \langle n_k \rangle > 0$  — as in attraction. Attraction lowers the energy of a pair (making it more likely) by  $\langle H_{\text{interaction}} \rangle \sim$  forward scattering amplitude.

- Consider baryon number susceptibility, which should diverge:  $\chi_B \sim \xi^{2-\eta}$

$$\chi_B \sim \langle \delta B \delta B \rangle_\sigma = \langle (\delta N_p - \delta N_{\bar{p}} + \delta N_n - \delta N_{\bar{n}})^2 \rangle_\sigma = \langle \delta N_p \delta N_p \rangle_\sigma + \dots$$

Each term on r.h.s. is  $\sim \frac{1}{m_\sigma^2}$ ,  $\Rightarrow \langle \delta B \delta B \rangle \sim 1/m_\sigma^2 = \xi^2$ .

- ● It is enough to measure protons  $\langle \delta N_p \delta N_p \rangle$  (Hatta, MS, PRL91:102003,2003)

# Limitations on $\xi$ in heavy-ion collisions

How big can  $\xi$  grow?

Limited by:

- Proximity of the critical point
- Finite size of the system  $\xi < 6$  fm.
- Finite *time*:  $\tau \sim 10$  fm.

Critical slowing down:  $\tau_{\text{equilibration}} \sim \xi^z$ .  
 $z > 1$  – dynamical critical exponent.

$$\xi_{\text{max}} \sim \tau^{1/z} \sim (2 - 3)\text{fm}$$

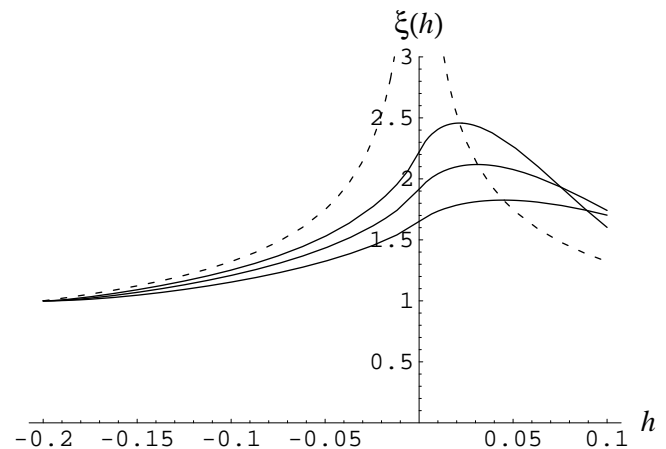
Dynamic universality class of liquid-gas phase transition, i.e.,  $z \approx 3$ :

— Critical mode – diffusive:

$$\omega = -iDq^2,$$

—  $D = \frac{\lambda_B}{\chi_B} \rightarrow 0$  at c.p.  $2 + 1 = 3$ .

(Son, MS, PRD70:056001,2004)



(Berdnikov, Rajagopal,  
PRD61:105017,2000)

# Higher moments (cumulants) of fluctuations

- Consider probability distribution for the order-parameter field:

$$P[\sigma] \sim \exp \{ -\Omega[\sigma]/T \},$$

$\Omega$  – effective potential:

$$\Omega = \int d^3x \left[ \frac{1}{2} (\nabla \sigma)^2 + \frac{m_\sigma^2}{2} \sigma^2 + \frac{\lambda_3}{3} \sigma^3 + \frac{\lambda_4}{4} \sigma^4 + \dots \right]. \quad \Rightarrow \quad \xi = m_\sigma^{-1}$$

- Moments of zero-momentum mode  $\sigma_0 \equiv \int d^3x \sigma(x)/V$ .

$$\kappa_2 = \langle \sigma_0^2 \rangle = \frac{T}{V} \xi^2; \quad \kappa_3 = \langle \sigma_0^3 \rangle = \frac{2\lambda_3 T^2}{V^2} \xi^6;$$

$$\kappa_4 = \langle \sigma_0^4 \rangle_c \equiv \langle \sigma_0^4 \rangle - \langle \sigma_0^2 \rangle^2 = \frac{6T^3}{V^3} [2(\lambda_3 \xi)^2 - \lambda_4] \xi^8.$$

- Tree graphs. Each zero-momentum propagator gives  $m_\sigma^{-2}$ , i.e.,  $\xi^2$ .



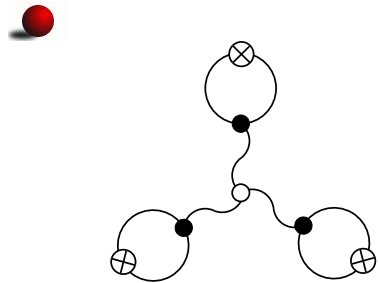
# Moments of *observables*

- Use multiplicity for an example. Since multiplicity is just the sum of all occupation numbers, and thus

$$\delta N = \sum_{\mathbf{p}} \delta n_{\mathbf{p}},$$

the cubic moment (skewness) of the pion multiplicity distribution is given by

$$\langle (\delta N)^3 \rangle = \sum_{\mathbf{p}_1} \sum_{\mathbf{p}_2} \sum_{\mathbf{p}_3} \langle \delta n_{\mathbf{p}_1} \delta n_{\mathbf{p}_2} \delta n_{\mathbf{p}_3} \rangle, \quad \text{where } \sum_{\mathbf{p}} = V \int d^3 \mathbf{p} / (2\pi)^3.$$



$$\langle \delta n_{\mathbf{p}_1} \delta n_{\mathbf{p}_2} \delta n_{\mathbf{p}_3} \rangle_{\sigma} = \frac{2\lambda_3}{V^2 T} \left( \frac{G}{m_{\sigma}^2} \right)^3 \frac{v_{\mathbf{p}_1}^2}{\omega_{\mathbf{p}_1}} \frac{v_{\mathbf{p}_2}^2}{\omega_{\mathbf{p}_2}} \frac{v_{\mathbf{p}_3}^2}{\omega_{\mathbf{p}_3}}$$

$$v_{\mathbf{p}}^2 = \bar{n}_{\mathbf{p}} (1 \pm \bar{n}_{\mathbf{p}})$$

Similarly for  $\langle (\delta N)^4 \rangle_c$ .

- Since  $\langle (\delta N)^3 \rangle$  scales as  $V^1$  it is convenient to normalize it by the mean total multiplicity  $\bar{N}$  which scales similarly. Thus we define

$$\omega_3(N) \equiv \frac{\langle (\delta N)^3 \rangle}{\bar{N}}$$

# Moments of observables contd.

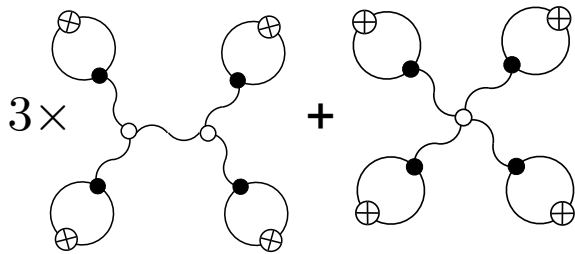
... and find

$$\omega_3(N)_\sigma = \frac{2\lambda_3}{T} \frac{G^3}{m_\sigma^6} \left( \int_{\mathbf{p}} \frac{v_{\mathbf{p}}^2}{\omega_{\mathbf{p}}} \right)^3 \left( \int_{\mathbf{p}} \bar{n}_{\mathbf{p}} \right)^{-1}.$$

● Similarly, for

$$\omega_4(N) \equiv \frac{\langle (\delta N)^4 \rangle_c}{\bar{N}}$$

from



we find

$$\omega_4(N)_\sigma = \frac{6}{T} \left[ 2 \frac{\lambda_3^2}{m_\sigma^2} - \lambda_4 \right] \frac{G^4}{m_\sigma^8} \left( \int_{\mathbf{p}} \frac{v_{\mathbf{p}}^2}{\omega_{\mathbf{p}}} \right)^4 \left( \int_{\mathbf{p}} \bar{n}_{\mathbf{p}} \right)^{-1}.$$

# Moments of observables contd.

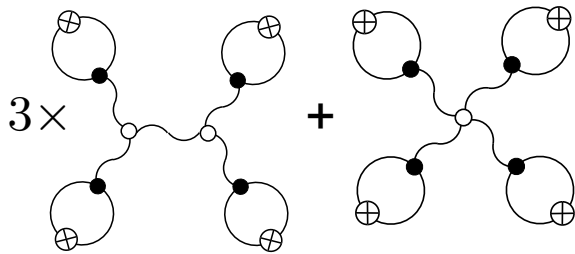
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# Scaling, $\lambda_n$

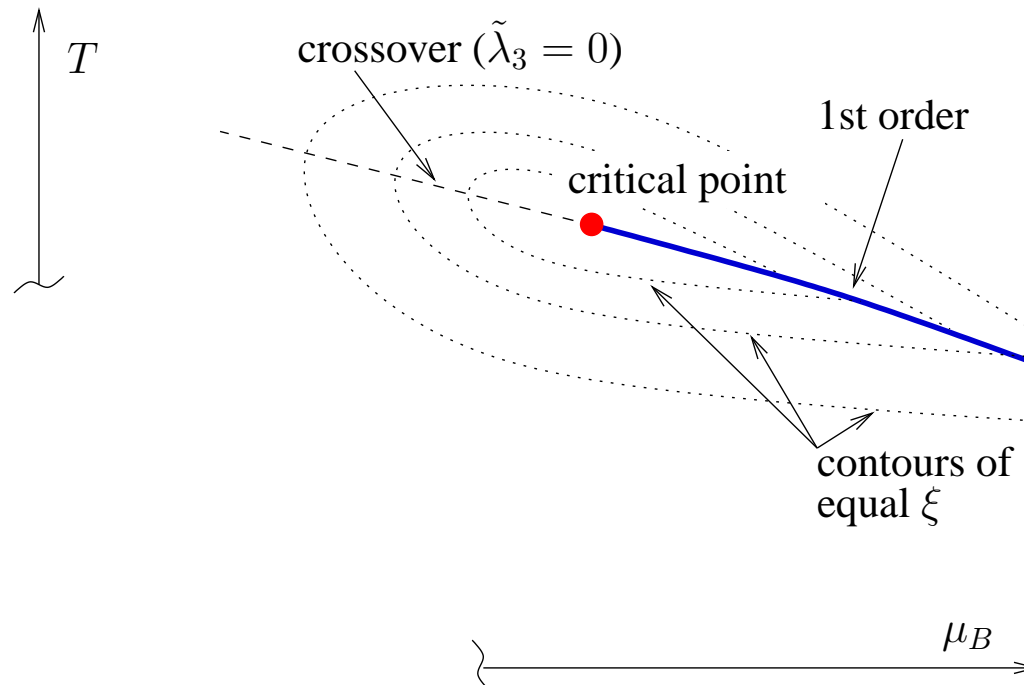
- Scaling requires that both  $\lambda_3$  and  $\lambda_4$  vanish with a power of  $\xi$  given by:

$$\lambda_3 = \tilde{\lambda}_3 T \cdot (T\xi)^{-3/2}, \quad \text{and} \quad \lambda_4 = \tilde{\lambda}_4 \cdot (T\xi)^{-1}, \quad (\eta \ll 1)$$

(because  $[(\nabla\sigma)^2] = 3 \Rightarrow [\sigma] = 1/2$  and  $\Rightarrow [\lambda_n] = 3 - n/2$ )

Dimensionless couplings  $\tilde{\lambda}_3$  and  $\tilde{\lambda}_4$  are universal, and for the Ising universality class they have been measured on the lattice.

- $\lambda_3$  is nonzero:



# Scaling, $\lambda_n$

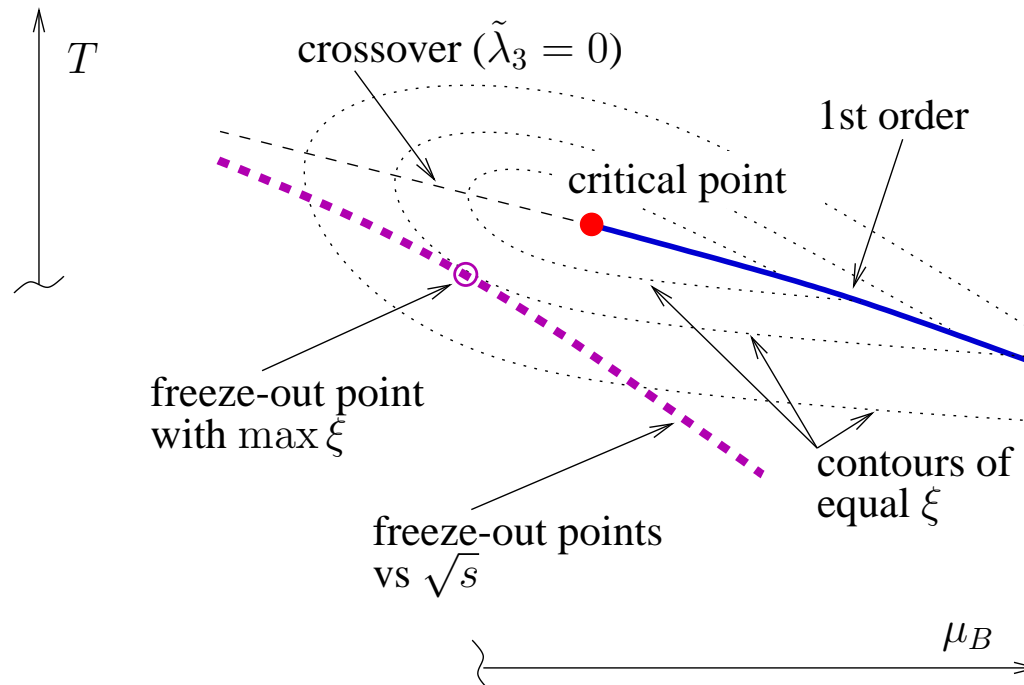
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Dimensionless couplings  $\tilde{\lambda}_3$  and  $\tilde{\lambda}_4$  are universal, and for the Ising universality class they have been measured on the lattice.

- $\lambda_3$  is nonzero:



# Estimates

Pions (top SPS):

$$\omega_3(N_\pi)_\sigma \equiv \frac{\langle(\delta N_\pi)^3\rangle}{\bar{N}_\pi} \approx 1. \left(\frac{\tilde{\lambda}_3}{4.}\right) \left(\frac{G}{300 \text{ MeV}}\right)^3 \left(\frac{\xi}{3 \text{ fm}}\right)^{9/2}$$

$$\omega_4(N_\pi)_\sigma \equiv \frac{\langle(\delta N_\pi)^4\rangle_c}{\bar{N}_\pi} \approx 12. \left(\frac{2\tilde{\lambda}_3^2 - \tilde{\lambda}_4}{50.}\right) \left(\frac{G}{300 \text{ MeV}}\right)^4 \left(\frac{\xi}{3 \text{ fm}}\right)^7$$

Protons (top SPS):

$$\omega_3(N_p)_\sigma \equiv \frac{\langle(\delta N_p)^3\rangle}{\bar{N}_p} \approx 3. \left(\frac{\tilde{\lambda}_3}{4.}\right) \left(\frac{g}{10.}\right)^3 \left(\frac{\xi}{1 \text{ fm}}\right)^{9/2}$$

$$\omega_4(N_p)_\sigma \equiv \frac{\langle(\delta N_p)^4\rangle_c}{\bar{N}_p} \approx 23. \left(\frac{2\tilde{\lambda}_3^2 - \tilde{\lambda}_4}{50.}\right) \left(\frac{g}{10.}\right)^4 \left(\frac{\xi}{1 \text{ fm}}\right)^7$$

Notes:

- Strong dependence on  $\xi$ , compared to  $\omega_2 \sim \xi^2$ .
- Significant uncertainty due to  $G, g$ .
- Crosscheck: same exponents as baryon number cumulants from scaling/universality:

$$\langle(\delta N_B)^k\rangle_c = VT^{k-1} \frac{\partial^k P(T, \mu_B)}{\partial \mu_B^k} \sim \xi^{k(5-\eta)/2-3}. \quad (\eta \ll 1)$$

# Concluding remarks I

- Sign of  $\omega_3$ ? *Positive* for  $N_\pi$  and  $N_p$ .

Crude argument:

(a)  $N_\pi$  and  $N_p$  are proxies for  $s$  and  $n_B$ , and

(b) e.g.,  $\langle (\delta S)^3 \rangle = T^2 \frac{d^2 S}{dT^2} > 0$  below C.P. because  $\frac{dS}{dT}$  peaks (Asakawa *et al*).

- Trivial background estimate:  $\omega_3(N)_{\text{BE}} = \overline{(1 + n_p)(1 + 2n_p)}$ .

This is about 1.3 ( $\approx 1 + 3\overline{n_p}$ ) for pions at  $T = 120$  MeV

$$\omega_4(N)_{\text{BE/FD}} = \overline{(1 \pm n_p)(1 \pm 6n_p(1 \pm n_p))}$$

- Note:  $\omega_k(N_{\pi+} + N_{\pi-})_{\text{BE}} = \omega_k(N_{\pi+})_{\text{BE}}$  – i.e., no cross-correlation.

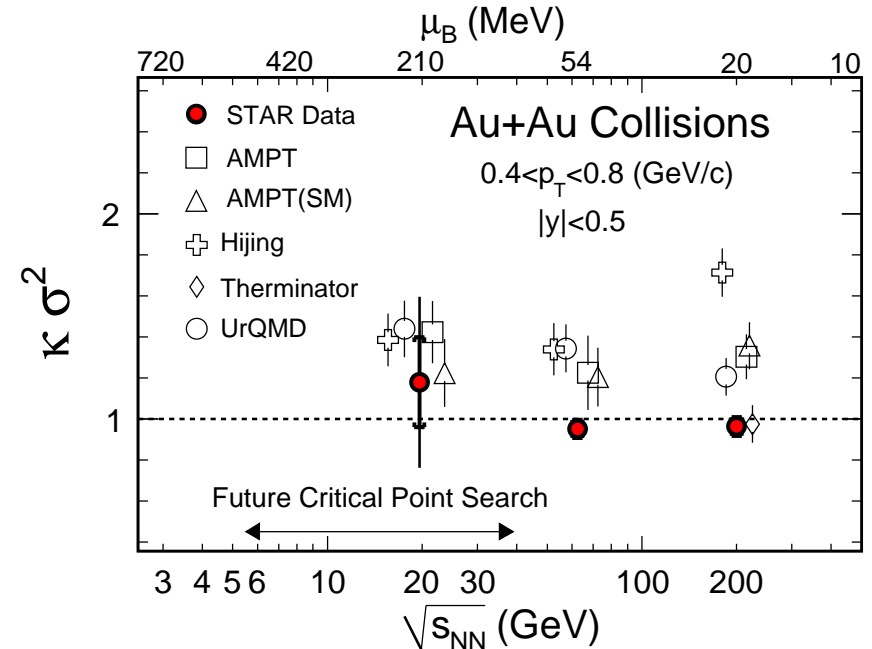
In contrast, for the critical point contribution:

$$\omega_k(N_{\pi+} + N_{\pi-})_\sigma = 2^{k-1} \omega_k(N_{\pi+})_\sigma.$$

# Concluding remarks II

- Measuring  $\omega_3$  maybe be harder (more statistics needed?) than  $\omega_2$ .  
More so for  $\omega_4 \sim \langle (\delta N)^4 \rangle - 3\langle (\delta N)^2 \rangle^2 = \mathcal{O}(N^2) - \mathcal{O}(N^2) \sim \mathcal{O}(N^1)$ .
- Other, non-critical, sources contribute: remnants of initial fluctuations, flow, jets – to name just a few.

- Calculate and subtract background.
- Apply kinematic cuts. (Low pt.)
- Measure bkgnd during the energy scan.  
STAR: background is small  $\omega_4 \approx 1$  (Poisson).



- Non-Gaussian moments have stronger dependence on  $\xi$ , and thus are more sensitive signatures of the critical point.

# Beam Energy Scan

