Phase structure of exactly solvable fermionic quantum field theories in 1+1 dimensions

Michael Thies, Theorie III, University of Erlangen

Plan

1) Gross∗-Neveu model
2) Nambu∗–Jona-Lasinio model
3) 't Hooft∗ model

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1) Gross-Neveu model

Lagrangian (Gross, Neveu 1974)

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m_0)\psi + \frac{g^2}{2}\left(\bar{\psi}\psi\right)^2$$

- 1+1 dimensions: $[\psi] = L^{-1/2}, \quad [g^2] = 1$ renormalizable
- U(N) flavor symmetry: 't Hooft limit $N \to \infty, \quad Ng^2 = \text{const.}$
- Z_2 chiral symmetry ($m_0 = 0$) $\psi \to \gamma_5 \psi, \quad \bar{\psi}\psi \to -\bar{\psi}\psi$

How does one solve such models?

Password: Semiclassical methods

- Path integral: Saddle point method
- Canonical approach: Relativistic many-body techniques

LO in $1/N$: Dirac-Hartree-Fock or TDHF
(vacuum, baryons, baryon-baryon scattering, dense matter, phase diagram)

$$\left(-i\gamma_5\partial_x + \gamma^0 S\right)\psi_\alpha = \epsilon_\alpha \psi_\alpha$$

Mean field

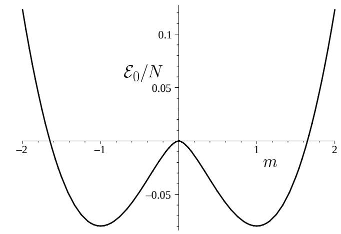
$$S - m_0 = -Ng^2\langle\bar{\psi}\psi\rangle$$

Self-consistency condition

$$S - m_0 = -Ng^2 \sum_\alpha \bar{\psi}_\alpha \psi_\alpha \frac{1}{\mathrm{e}^{\beta(\epsilon_\alpha-\mu)}+1} \underset{T\to 0}{=} -Ng^2 \sum_\alpha^{\mathrm{occ}} \bar{\psi}_\alpha \psi_\alpha$$

NLO in $1/N$: relativistic RPA
(mesons, fermion-antifermion scattering)

Vacuum ($m_0 = 0$): dynamical fermion mass $S(x) = m$



Dimensional transmutation (Coleman, Weinberg 1973)

$$m = \Lambda e^{-\frac{\pi}{Ng^2}}$$

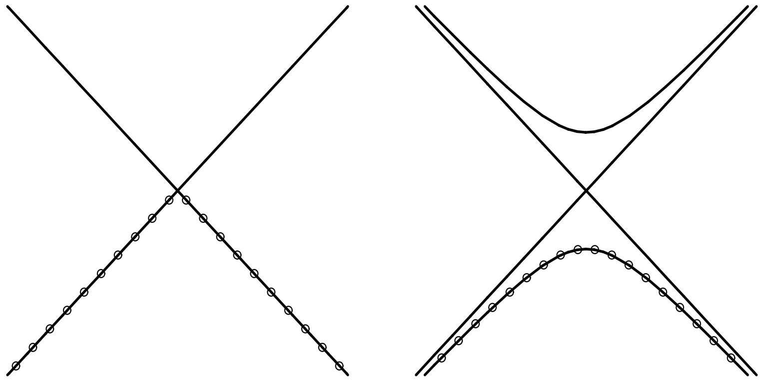
Asymptotic freedom (Gross, Wilczek, Politzer 1974)

$$\beta(g) = \Lambda \frac{dg}{d\Lambda} = -\frac{Ng^3}{2\pi} < 0$$

Why SSB of Z_2 chiral symmetry? Peierls instability (Peierls 1955)

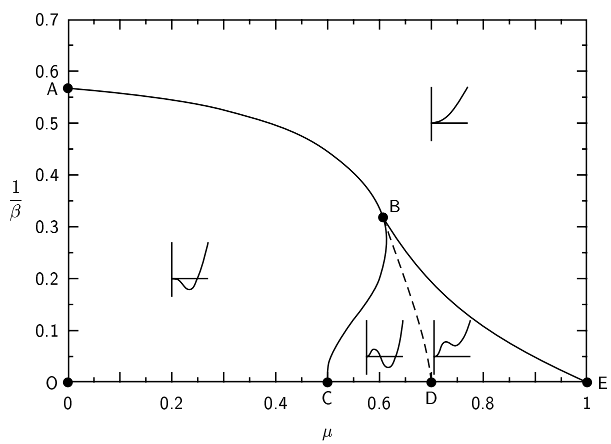
$$\mathcal{E}_0(m) - \mathcal{E}_0(0) = -\frac{Nm^2}{4\pi}$$

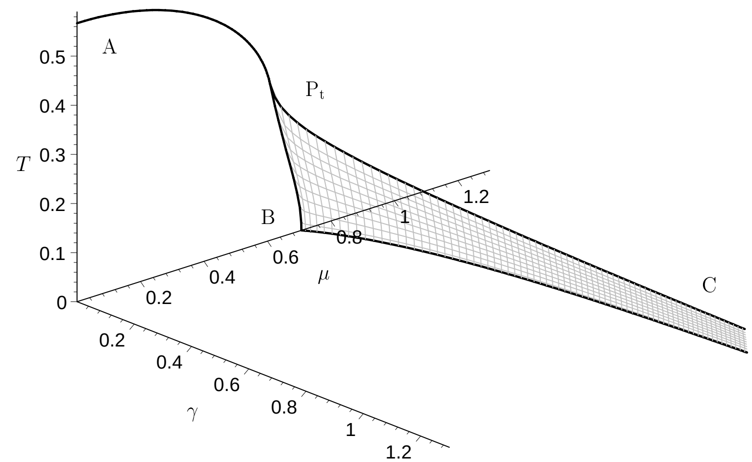
$$\mathcal{E}_0 = -N \int \frac{dk}{2\pi} \sqrt{k^2 + m^2} + \frac{m^2}{2g^2}$$



“Old” Gross-Neveu phase diagram in the (μ, T) plane ($\leq$ 2000)

Thermal Hartree-Fock, assuming $S = m(\mu, T)$ (homogeneous)





$m_0 = 0$ (Wolff 1985)

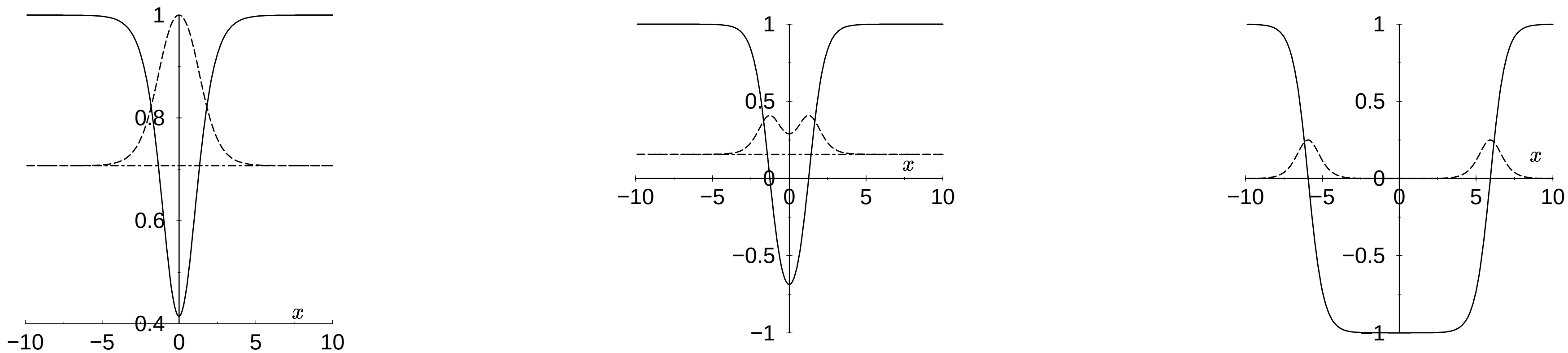
$\gamma = \frac{\pi m_0}{N g^2 m}$ (Barducci et al. 1995)

Problem: Phase diagram inconsistent with known baryon spectrum

$$\mu_{\text{crit}}(T = 0) = \left.\frac{\partial \mathcal{E}_{\text{g.s.}}}{\partial \rho}\right|_{\rho=0} \stackrel{!}{=} M_B$$

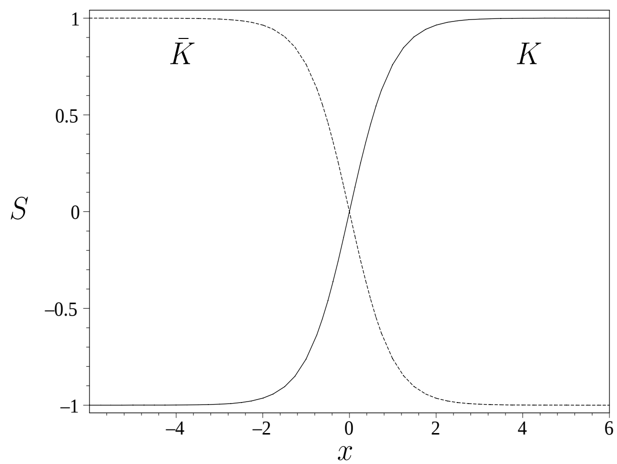
Baryons

- Kink-antikink (Dashen, Hasslacher, Neveu 1975)



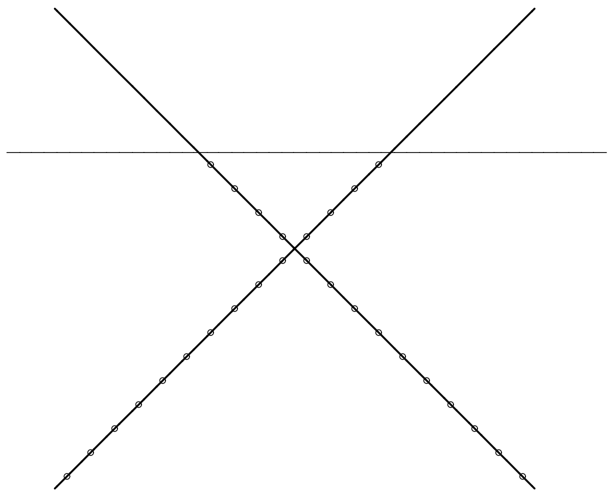
- Kink (Callan, Coleman, Gross, Zee) — reflection of Z_2 chiral symmetry

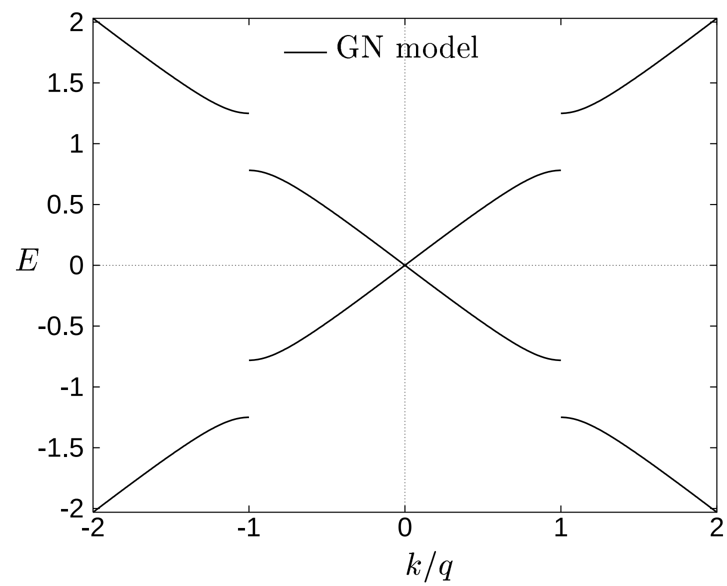
$$S(x) = \pm \tanh x$$



Soliton crystal

- Low density limit $\longrightarrow$ array of isolated baryons
- Peierls instability





How to find self-consistent potential? How to solve Hartree-Fock equations?

$$\left(-i\gamma_5\partial_x + \gamma^0 S\right)\psi_\alpha = \epsilon_\alpha\psi_\alpha, \qquad S - m_0 = -Ng^2\sum_\alpha^{\text{occ}}\bar{\psi}_\alpha\psi_\alpha$$

CCGZ kink

$$S(x) = \tanh x$$

Effective potential in Schrödinger-type equation (Pöschl-Teller)

$$S^2 - S' - 1 = -\frac{2}{\cosh^2 x}$$

- Distinguishing feature: reflectionless

Lattice of such potential wells

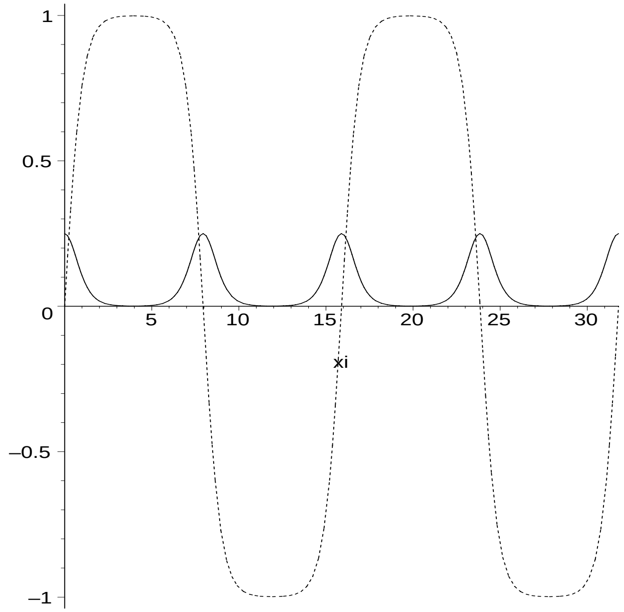
$$\sum_{n=-\infty}^{\infty} \frac{1}{\cosh^2(x - nd)} = a + b\,\mathrm{sn}^2(cx)$$

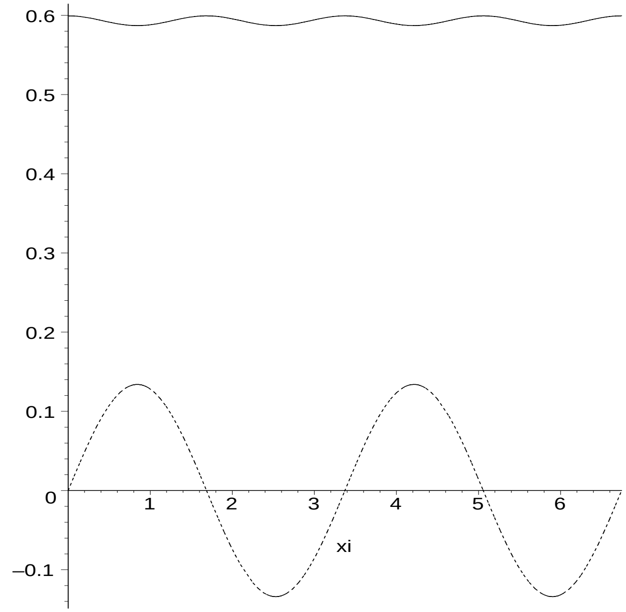
- Distinguishing feature: finite band potential

Schrödinger equation with Lamé potential can be solved exactly in terms of Jacobi elliptic functions (Whittaker, Watson)

Principal result: The most general Dirac potential $S(x)$ leading to Lamé equation (3 parameter family) yields self-consistency for all T, μ, γ

Examples of shapes of $S(x)$ and fermion density ($m_0 = 0$)

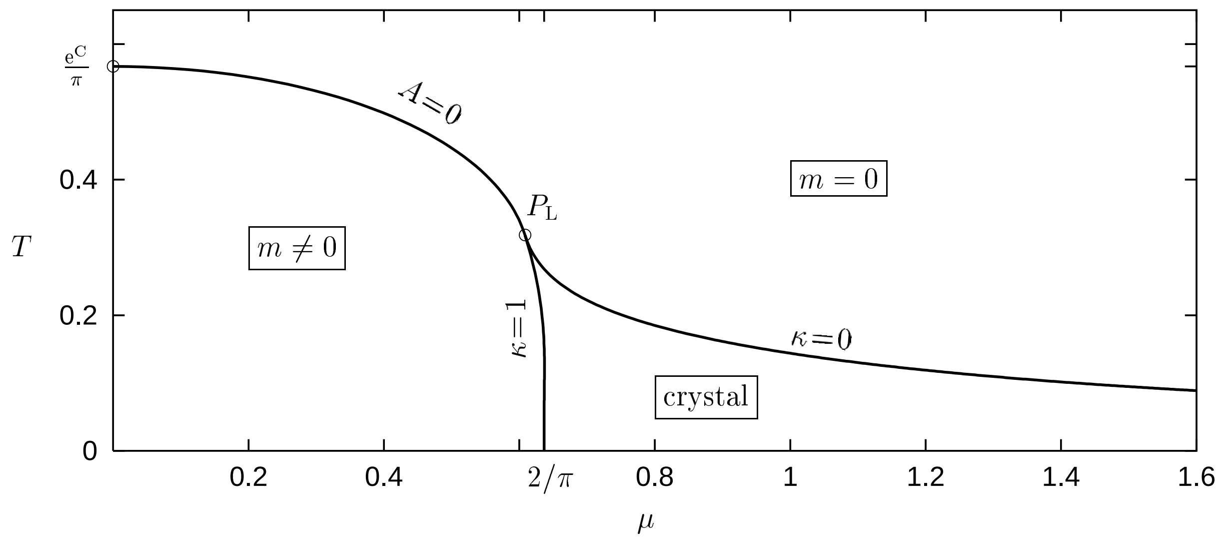


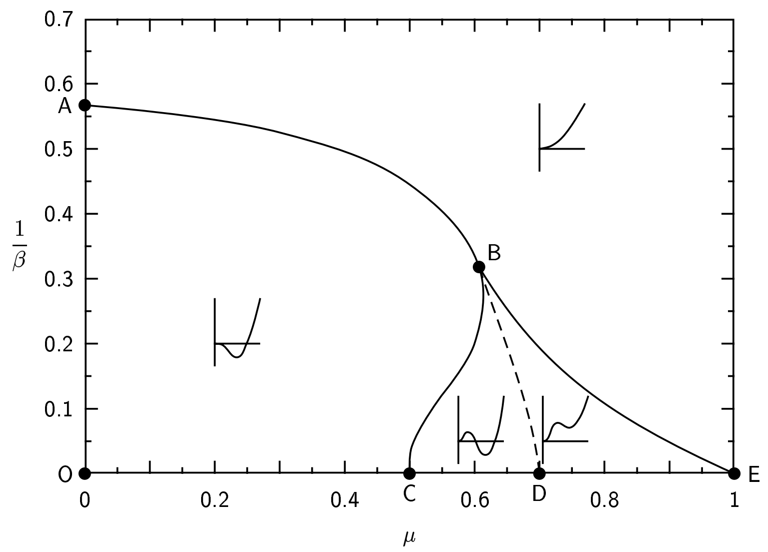


Phase diagram:

Expect 3 different phases at $m_0 = 0$: massless Fermi gas, massive Fermi gas, soliton crystal. No massless phase at $m_0 \neq 0$

Revised phase diagram of the Gross-Neveu model in the chiral limit (Schnetz, Urlichs 2003/04)





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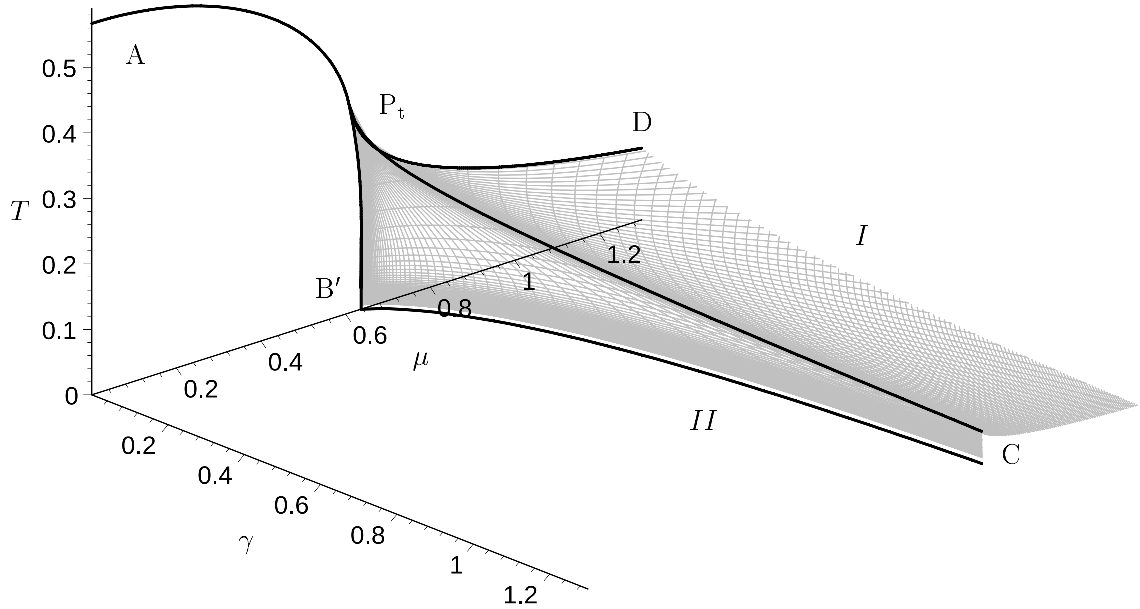
Revised phase diagram of the massive Gross-Neveu model (Schnetz, Urlichs 2006)



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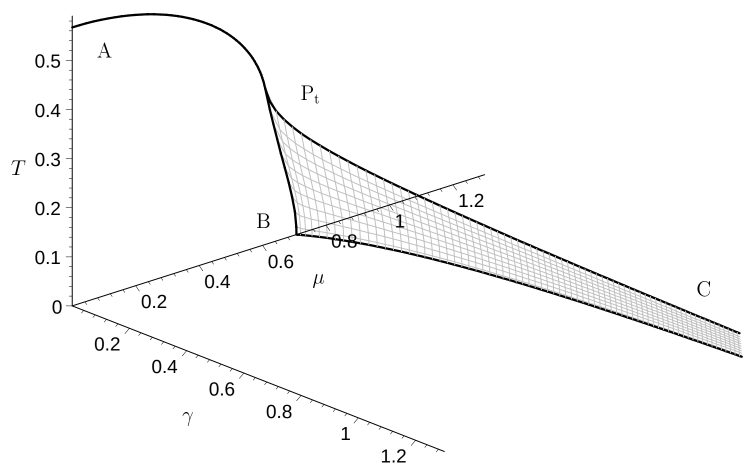


2) Chiral Gross-Neveu model

Gross-Neveu model with continuous chiral symmetry ($m_0 = 0$)

$$\mathcal{L} = \bar\psi(i\partial\!\!\!/ - m_0)\psi + \frac{g^2}{2}\left[(\bar\psi\psi)^2 + (\bar\psi i\gamma_5\psi)^2\right]$$

$\cong$ Nambu–Jona-Lasinio model (1961) in 1+1 dimensions (NJL$_2$)

U(1)$_R\otimes$ U(1)$_L$ chiral symmetry

$$\mathrm{U}(1)_R: \qquad \psi_R \to e^{i\beta}\psi_R, \qquad \mathrm{U}(1)_L: \qquad \psi_L \to e^{i\alpha}\psi_L$$

Invariance of $\mathcal{L}$

$$\mathcal{L} = \bar\psi_R i\partial\!\!\!/\psi_R + \bar\psi_L i\partial\!\!\!/\psi_L + g^2\left(\bar\psi_R\psi_L\right)\left(\bar\psi_L\psi_R\right)$$

Conserved currents in 1+1 dimensions

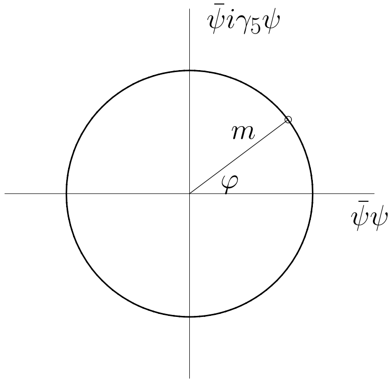
$$j_V = \begin{pmatrix} \psi^\dagger\psi \\ \psi^\dagger\gamma_5\psi \end{pmatrix}, \qquad j_A = \begin{pmatrix} \psi^\dagger\gamma_5\psi \\ \psi^\dagger\psi \end{pmatrix}$$

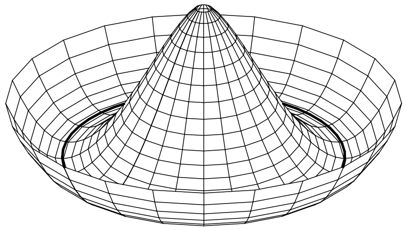
Hartree-Fock vacuum

$$\left(-i\gamma_5\partial_x + \gamma^0 S + i\gamma^1 P\right)\psi_\alpha = \epsilon_\alpha\psi_\alpha$$

Two condensates

$$S = m\cos\varphi = -g^2\langle\bar\psi\psi\rangle$$
$$P = m\sin\varphi = -g^2\langle\bar\psi i\gamma_5\psi\rangle$$





Finite N: SSB of continuous symmetry artefact of mean field approximation
— viable in the limit $N \to \infty$ (Witten 1978)

$$\langle\bar\psi\psi(x)\bar\psi\psi(y)\rangle \sim |x-y|^{-1/N}$$

Mesons: RPA $\to$ scalar σ ($\mathcal{M}_\sigma = 2m$), pseudoscalar π ($\mathcal{M}_\pi = 0$)
Massive model: Gell-Mann, Oakes, Renner relation

$$\mathcal{M}_\pi^2 = -\frac{4\pi}{N} m_0 \langle\bar\psi\psi\rangle_v$$

Global and local chiral rotations

— key for understanding massless baryons and dense matter

Vacuum Dirac-Hartree-Fock equation

$$\left(-i\gamma_5\partial_x + \gamma^0 m e^{i\varphi\gamma_5}\right)\psi = \epsilon\psi$$

- Global chiral rotation

$$\psi = e^{-i\alpha\gamma_5}\tilde{\psi}$$

$$\left(-i\gamma_5\partial_x + \gamma^0 m e^{i\varphi\gamma_5}\right) e^{-i\alpha\gamma_5}\tilde{\psi} = \epsilon e^{-i\alpha\gamma_5}\tilde{\psi}$$

Multiply by $e^{i\alpha\gamma_5}$ from the left

$$\left(-i\gamma_5\partial_x + \gamma^0 m e^{i(\varphi-2\alpha)\gamma_5}\right)\tilde{\psi} = \epsilon\tilde{\psi}$$

Chiral vacuum angle φ is irrelevant

- Local chiral rotation $\alpha \to \alpha(x)$

$$\left(-i\gamma_5\partial_x - \alpha'(x) + \gamma^0 m e^{i(\varphi-2\alpha(x))\gamma_5}\right)\tilde{\psi} = \epsilon\tilde{\psi}$$

Break translational invariance and generate unwanted vector potential

Interesting special case

$$\alpha = q(x - x_0), \qquad \alpha' = q$$

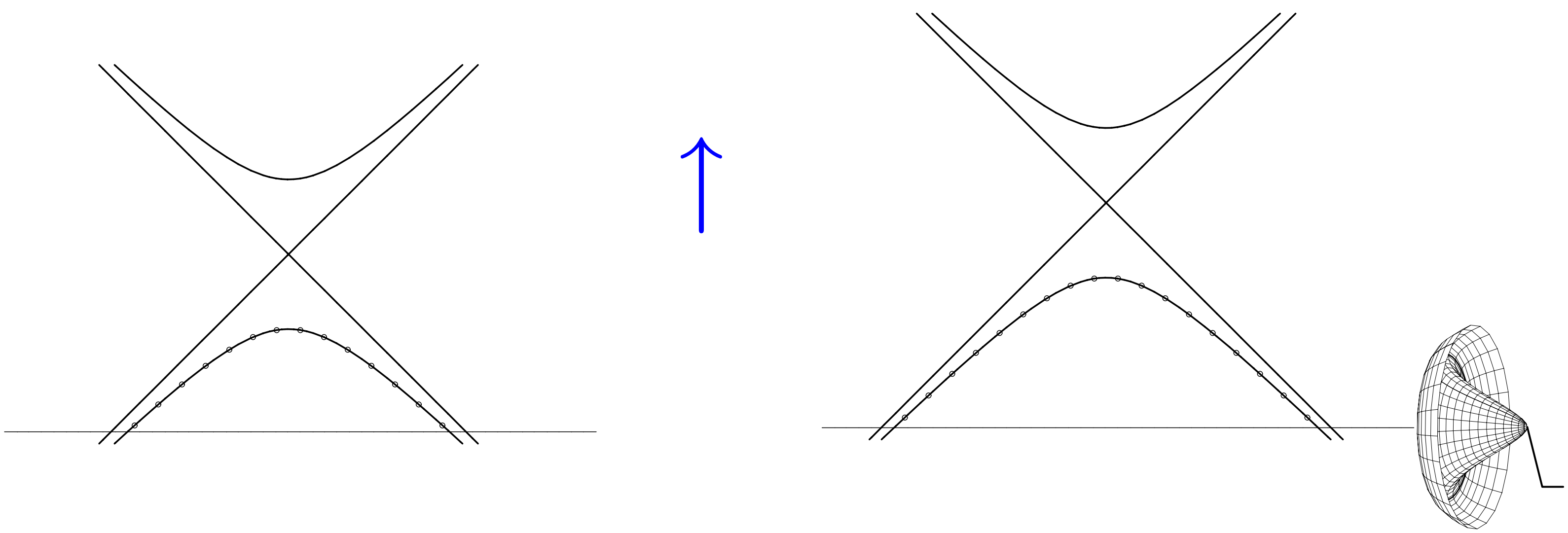
Condensate assumes helical structure — spectrum shifted rigidly

$$\left(-i\gamma_5 \partial_x + \gamma^0 m e^{i(\varphi - 2q(x-x_0))\gamma_5}\right)\tilde{\psi} = (\epsilon + q)\tilde{\psi}$$

$$S(x) = m \cos 2qx, \qquad P(x) = -m \sin 2qx$$

New Hartree-Fock solution of the chiral Gross-Neveu model

Role of Dirac sea: Chiral anomaly



Evaluate fermion density and energy density, using cutoff $E > -\Lambda/2$

• Vacuum: Momentum cutoff $\pm\Lambda/2$

Fermion density

$$\frac{\rho_0}{N} = \int^{\Lambda} \frac{dk}{2\pi} = \frac{\Lambda}{2\pi}$$

Energy density

$$\frac{\mathcal{E}_0}{N} = -\int^{\Lambda} \frac{dk}{2\pi}\sqrt{k^2+m^2} = -\frac{\Lambda^2}{8\pi} - \frac{m^2}{4\pi} - \frac{m^2}{2\pi}\ln\frac{\Lambda}{m}$$

• Chirally twisted case: Momentum cutoff $\pm\Lambda'/2$ with $\Lambda' = \Lambda + 2q$

Fermion density

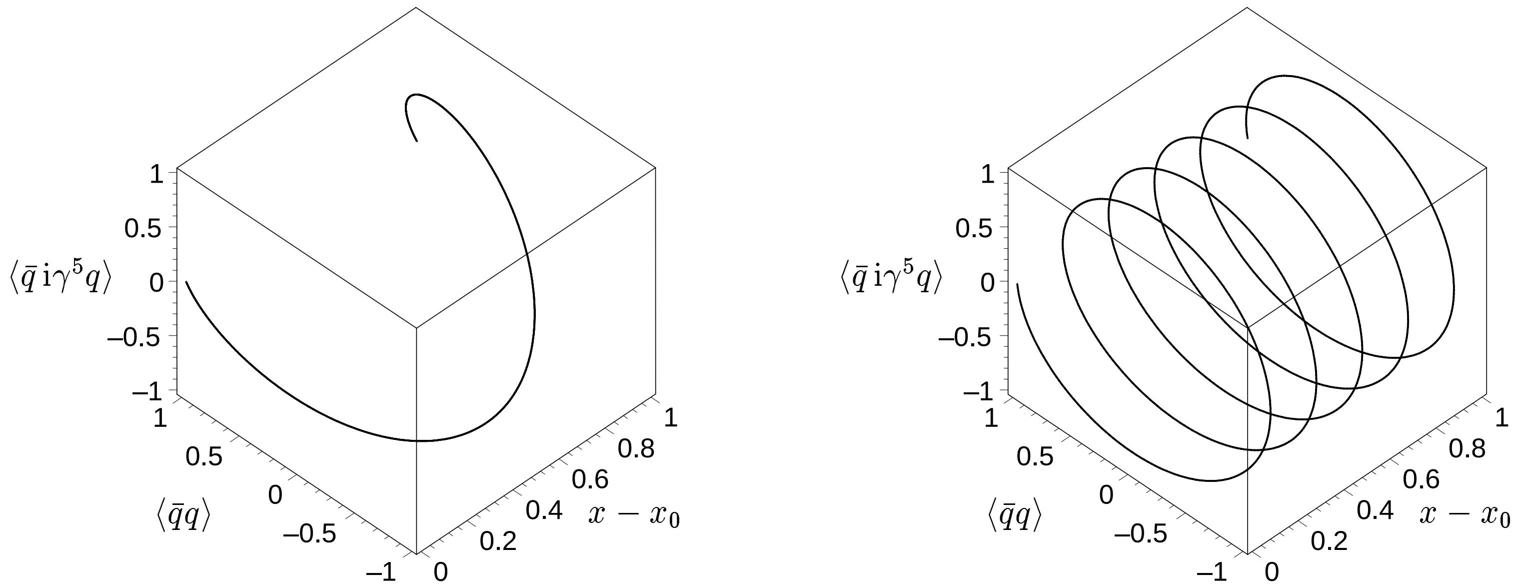
$$\frac{\rho}{N} = \int^{\Lambda'} \frac{dk}{2\pi} = \frac{\rho_0}{N} + \frac{q}{\pi}$$

Energy density

$$\frac{\mathcal{E}}{N} = -\int^{\Lambda'} \frac{dk}{2\pi}\left(\sqrt{k^2+m^2} - q\right) = \frac{\mathcal{E}_0}{N} + \frac{q^2}{2\pi}$$

Mimics massless Fermi gas with $k_f = q$

Resulting Hartree-Fock solution at finite density: Crystal with helical order parameter — chiral spiral (Schön 2000)



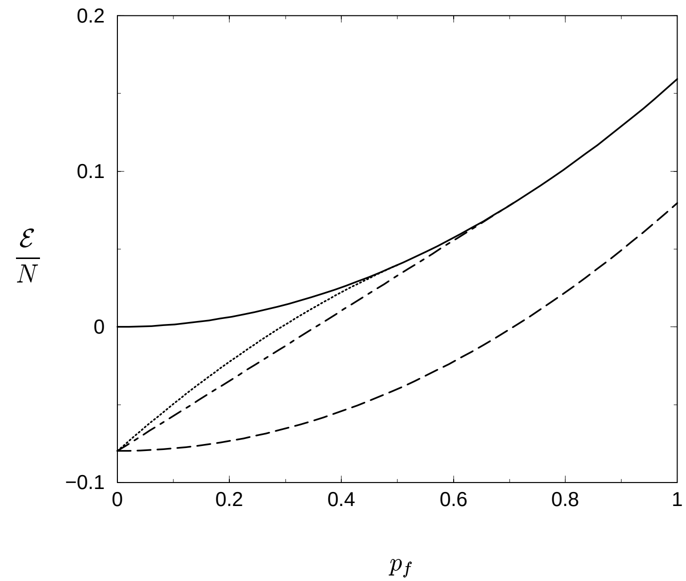
Properties

- Topological baryon number (Skyrmion, Skyrme crystal)
- Massless baryons ($L \to \infty$)
- Chiral spiral is true ground state of dense matter

$$S(x) = m \cos 2k_f x, \qquad P(x) = -m \sin 2k_f x$$

Compare energy density with homogeneous solutions

$$\frac{\rho}{N} = \frac{k_f}{\pi}$$
$$\frac{\mathcal{E}}{N} = -\frac{1}{4\pi} + \frac{k_f^2}{2\pi}$$



- Fermion density spatially constant — result of axial current conservation

$$0 = \partial_\mu \langle j_5^\mu \rangle = \partial_x \langle j_5^1 \rangle = \partial_x \langle \psi^\dagger \psi \rangle$$

- Translational and chiral symmetries broken, screw symmetry unbroken

$$P + k_f Q_5$$

- Mesons in dense matter $\leftrightarrow$ RPA on chiral spiral ground state:
  Only one gapless mode, pion-phonon hybrid (Riedl 2001)

Phase diagram of chiral Gross-Neveu model

Hartree-Fock at finite (T, μ): Start from finite temperature, $\mu = 0$ homogeneous Fermi gas (same as Gross-Neveu model)

$$\left(-i\gamma_5\partial_x + \gamma^0 m(T)\right)\psi_\alpha = \epsilon_\alpha\psi_\alpha, \qquad m(T) = -Ng^2\sum_\alpha \bar{\psi}_\alpha\psi_\alpha\frac{1}{e^{\beta\epsilon_\alpha}+1}$$

Local chiral rotation

$$\psi(x) \to e^{i\mu x\gamma_5}\psi(x)$$

Chiral spiral condensate with radius $m(T)$ and spatial period π/μ. Shift in spectrum by $-\mu$ generates chemical potential

Radius depends only on T, pitch on μ

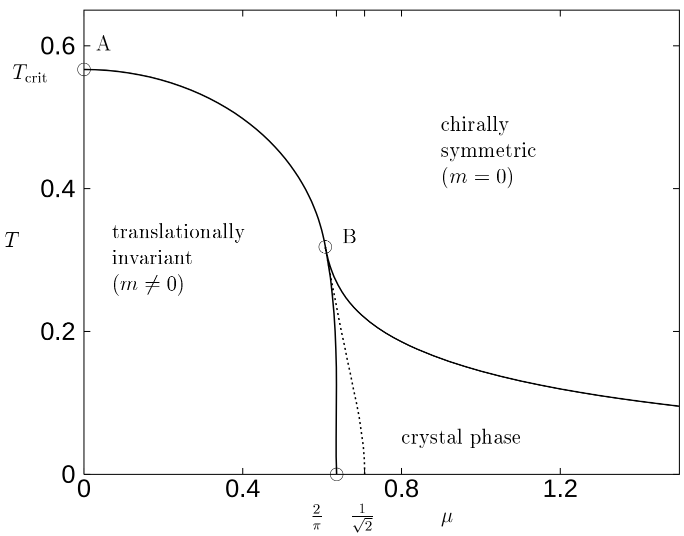
Renormalized grand canonical potential and fermion density

$$\begin{aligned} \psi(T,\mu)|_{\text{spir}} &= \psi(T,0) - N\frac{\mu^2}{2\pi} \\ \rho(T,\mu)|_{\text{spir}} &= N\frac{\mu}{\pi} \end{aligned}$$

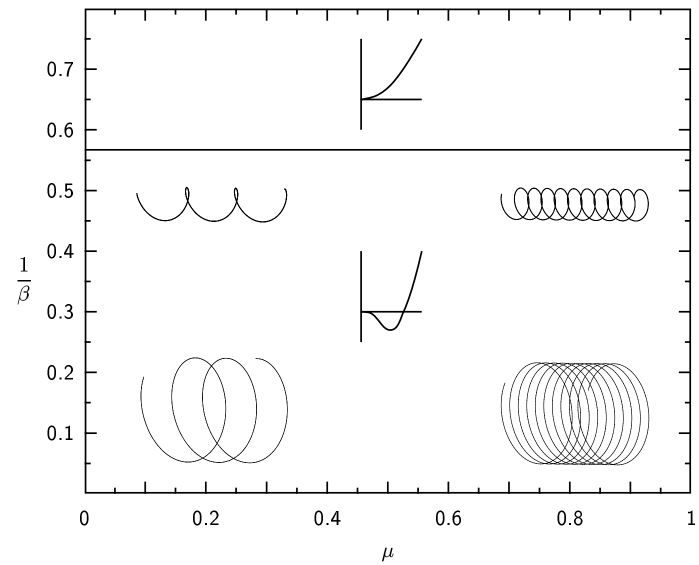
Anomaly is UV effect $\to$ no T-dependence

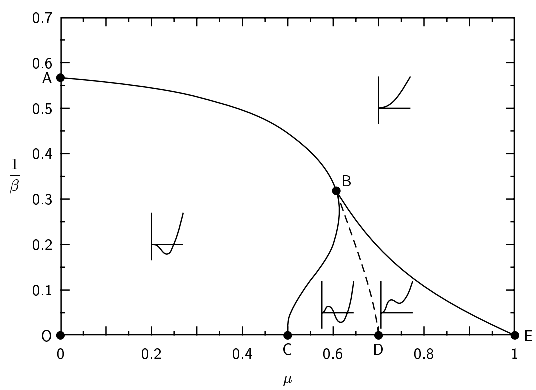
Phase diagram of the NJL_2 model in the chiral limit (Schön 2000)

GN_2



NJL_2





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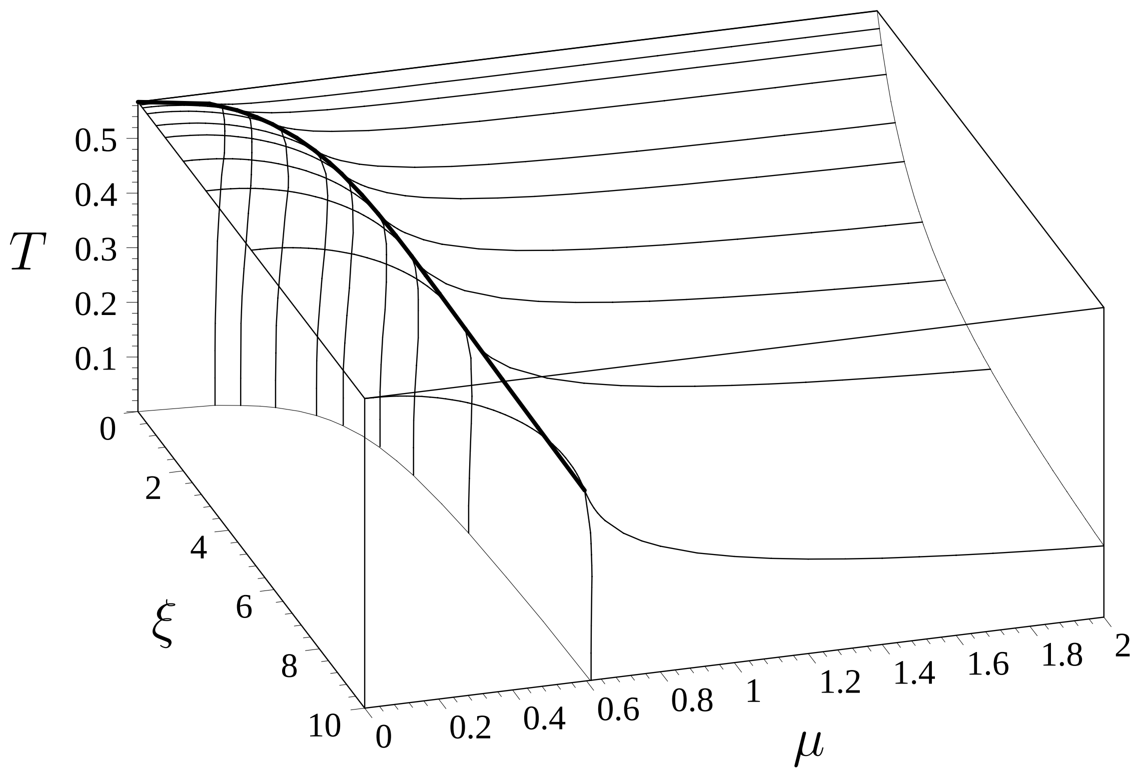
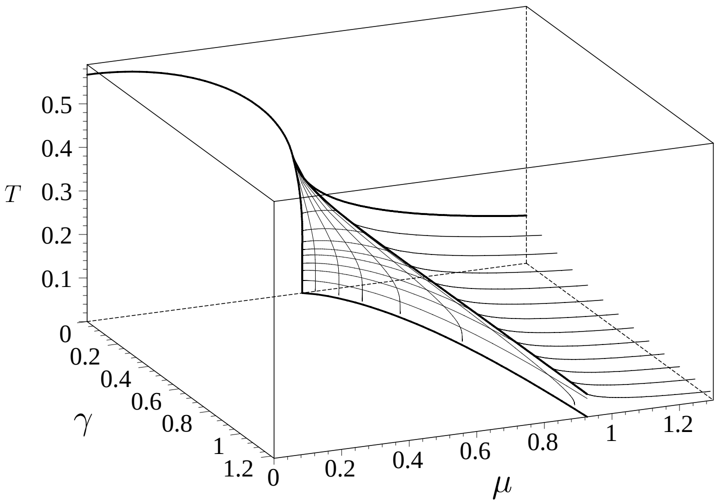
Interpolating between Gross-Neveu and NJL$_2$ models (Boehmer 2009)

$$\mathcal{L} = \bar{\psi} i \partial\!\!\!/ \psi + \frac{g^2}{2} \left(\bar{\psi}\psi \right)^2 + \frac{G^2}{2} \left(\bar{\psi} i \gamma_5 \psi \right)^2$$

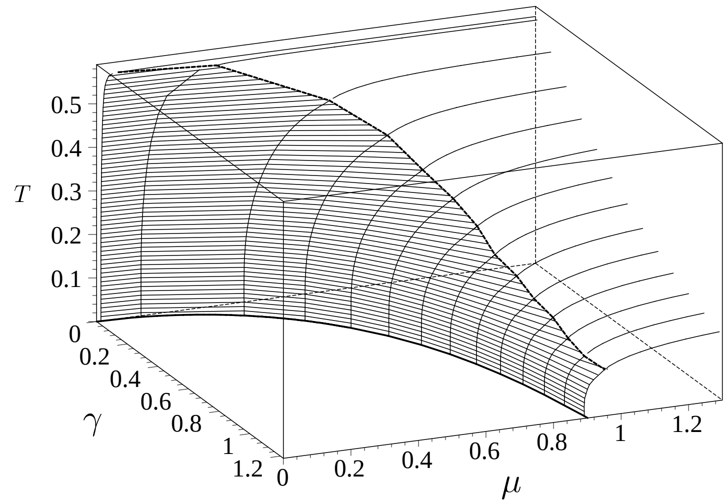


Phase diagram of the massive NJL$_2$ model (Boehmer, Fritsch, Kraus 2009)

GN$_2$



NJL$_2$



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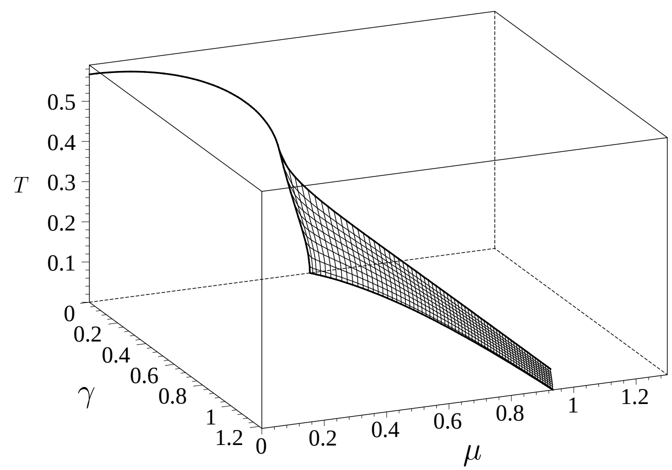
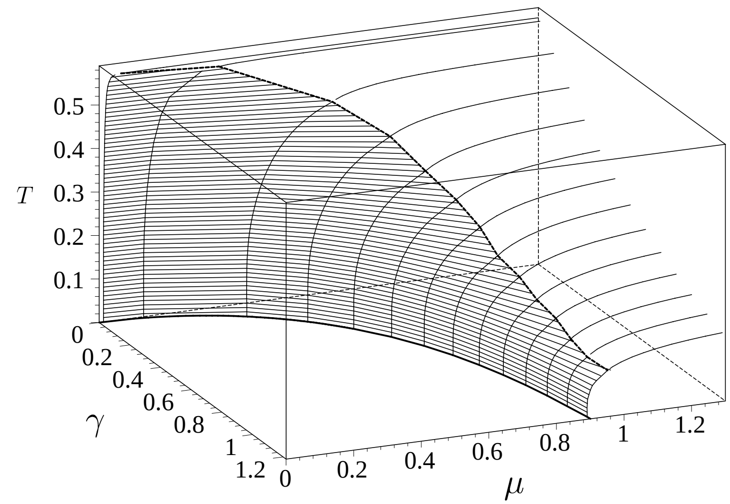
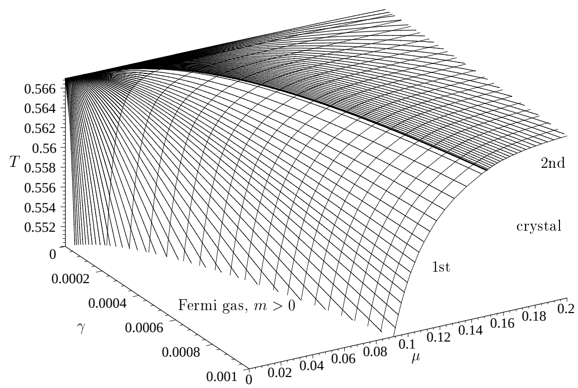


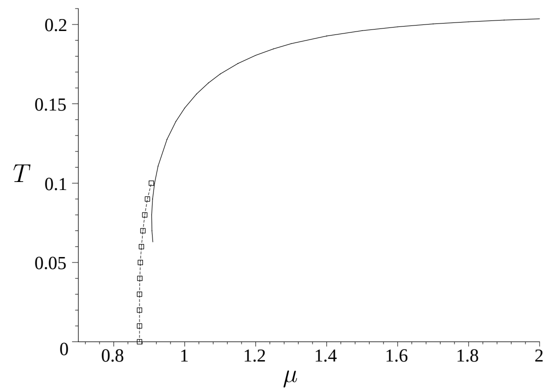
Details of the calculation

NJL$_2$





Tricritical point



1st and 2nd order critical lines

3) 't Hooft model

Lagrangian ('t Hooft 1974)

$$\mathcal{L} = \bar{\psi}(i\not{D} - m_0)\psi - \frac{1}{2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu}$$

$$D_\mu = \partial_\mu + igA_\mu, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig\left[A_\mu, A_\nu\right]$$

- 1+1 dimensions: $[A_\mu] = 1, \quad [g] = M$ super-renormalizable
- SU(N) color: 't Hooft limit $N \to \infty, \quad Ng^2 = \text{const.}$
- U(1)$_R \otimes$U(1)$_L$ chiral symmetry ($m_0 = 0$)

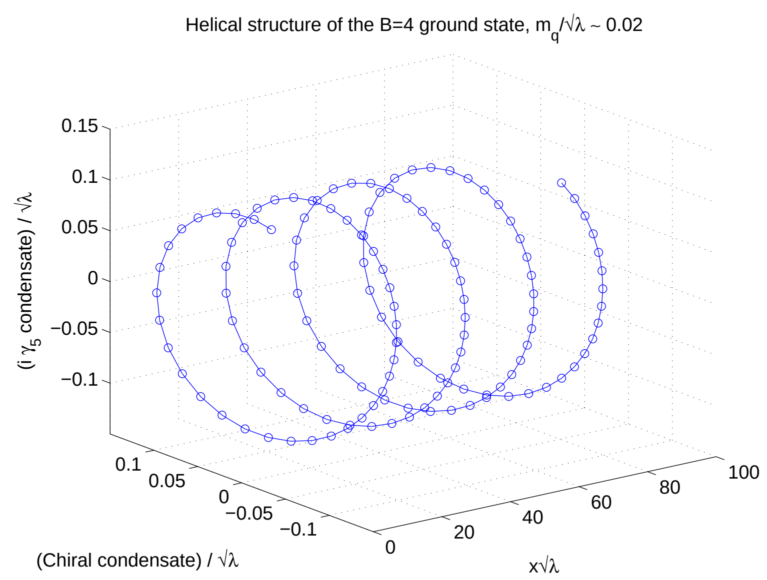
Hartree-Fock approach — what is different?

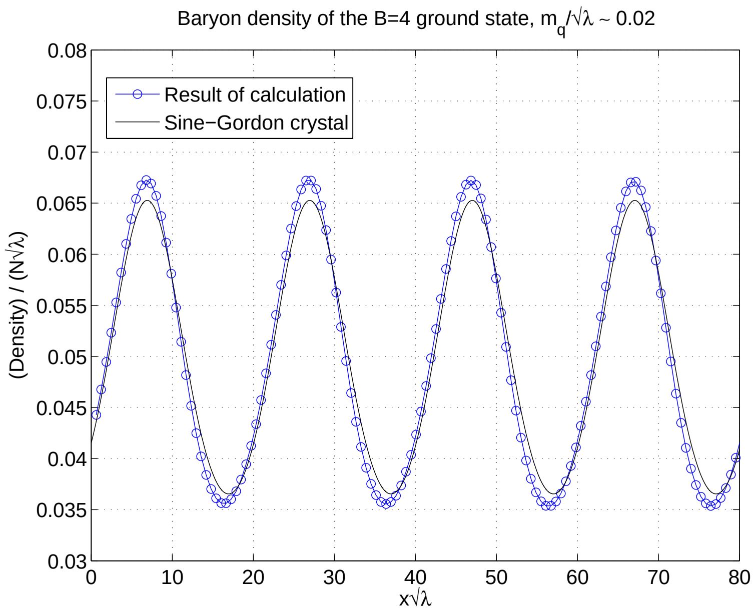
- Linear Coulomb potential, no transverse gluons
- Non-covariant momentum dependence of fermion self-energy (numerical)
- Chiral condensate not related to dynamical mass (Zhitnitskii)

$$\langle \bar{q}q \rangle = -\frac{N}{\sqrt{12}} \left(\frac{Ng^2}{2\pi} \right)^{1/2}$$

- Confinement in independent particle picture: divergent p-independent part of fermion self-energy
- Infinite tower of mesons ($q\bar{q}$ states)
- Goldstone boson, massless baryons and chiral spiral identical to NJL$_2$ model — interaction invariant under local chiral rotations. Universal behavior
- Analytic results confirmed by numerical Hartree-Fock studies on the lattice (Salcedo, Levit, Negele 1990, Bringoltz 2009)

(Bringoltz 2009)

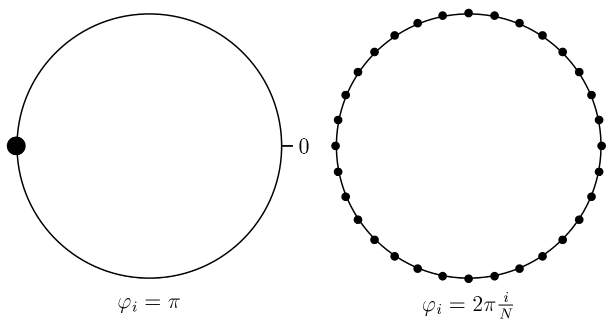


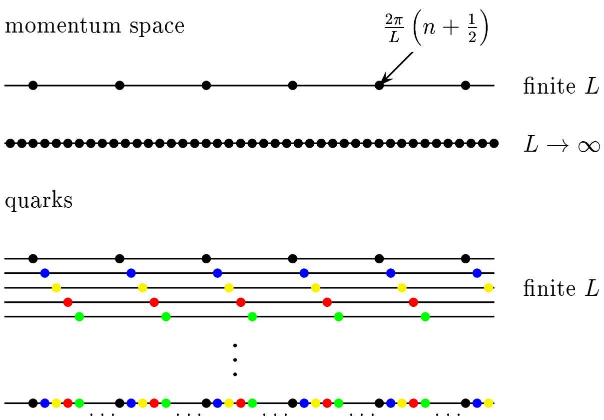


Finite temperature 't Hooft model

Large N limit: No temperature dependence to LO in $1/N$
Reason: Divergent quark self-energy (confinement)

Alternative: Swap space and Euclidean time — spatial circle
Gauge away eigenphases of Polyakov-loop variables $\longrightarrow$ color dependent, quasiperiodic boundary conditions for quarks (Schön 2000)

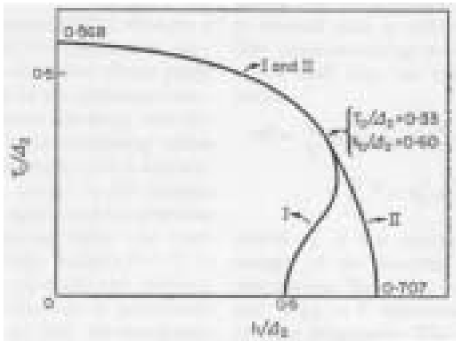




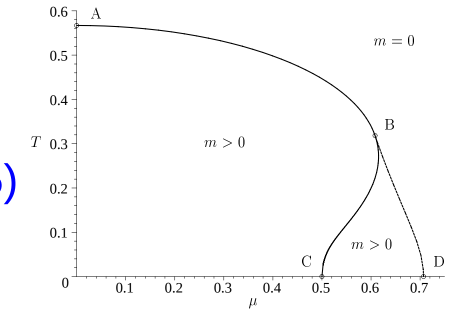
Related to Eguchi-Kawai reduction on the lattice (Bringoltz). No restoration of chiral symmetry — phase diagram in (μ, T) plane not universal

Gross-Neveu model and condensed matter systems

Sarma (1963) BCS

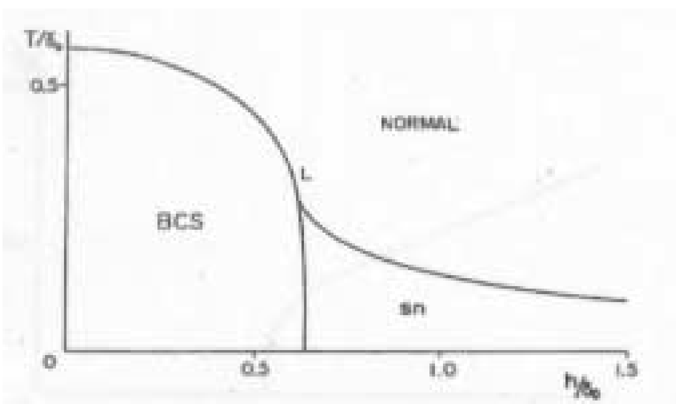


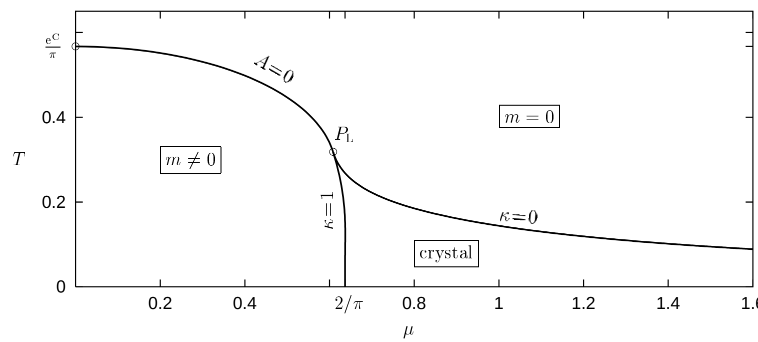
Wolff (1985)



Machida and Nakanishi (1984) ($ErRh_4B_4$)

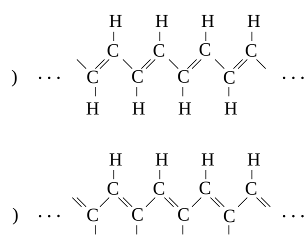
Schnetz et al. (2003)





Conducting polymers (*trans*-polyacetylene, *cis*-polyacetylene)

massless GN_2 model



massive GN_2 model