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Motivic representations in positive characteristic

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Abstract. This is a brief report on the research achievements I obtained jointly with Akio Tamagawa during my stay at R.I.M.S. as a research fellow funded by the International Research Unit of Advanced Future Studies, Kyoto University.
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For about 10 years, my research has been focussing on representations of the étale fundamental group of schemes and their applications in arithmetic geometry. This thematic is at the crossroad of arithmetic geometry, number theory and group theory.

Let k be a of characteristic $p \geq 0$, let S be a variety[1] over k with generic point η. For an integer $d \geq 1$, let $S^{\leq d}$ denote the set of $s \in S$ with residue degree $[k(s) : k] \leq d$ and let $|S|$ denote the set of closed points.

Given a smooth, proper morphism $f : X \to S$ write S_{ex} for the set of $s \in S$ where the motive of the fiber X_s is 'simpler' than the motive of the generic fiber X_η. Here 'simpler' means that the powers X_s^n, $n \geq 1$ contain more algebraic cycles than the X_η^n, $n \geq 1$.

A general problem is to describe the arithmetico-geometric structure of S_{ex}.

For $k = \mathbb{C}$ and the Hodge-theoretic incarnation of this problem, a celebrated result of Cattani-Deligne-Kaplan states that S_{ex} is a countable union of algebraic sub varieties, called 'special'. When $f : X \to S$ is a universal abelian scheme over a Shimura variety, special sub varieties are described by a body of conjectures which culminate in the Zilber-Pink conjecture. A special case of it - the André-Oort conjecture (whose proof for Shimura varieties of abelian type was achieved in 2015) - asserts that the special sub varieties are exactly those containing a Zariski-dense subset of 0-dimensional special sub varieties.

[1] *I.e.* a smooth, separated, geometrically connected scheme of finite type.

For k of finite type, one can also consider the étale incarnation of this problem. In this setting, the naive analogue of Cattani-Deligne-Kaplan theorem is empty and one is mostly interested in the arithmetic properties of S_{ex}, for instance in describing $S_{ex} \cap S^{\leq d}$, $d \geq 1$. To handle this problem Akio Tamagawa and I have developed an anabelian strategy which relies on the representations

$$\rho_{\ell^\infty} : \pi_1(S) \to \mathrm{GL}(\mathrm{H}^*(X_{\overline{\eta}}, \mathbb{Z}_\ell)) \ \ (\ell \neq p)$$

given by the smooth-proper base change theorem (SGAIV). More precisely, by functoriality of étale fundamental group, every $s \in S$ induces a morphism $\sigma_s : \pi_1(s)(\simeq \mathrm{Gal}(k(s)/k)) \to \pi_1(S)$ hence a 'local' Galois representation

$$\rho_{\ell^\infty,s} := \rho_\ell \circ \sigma_s : \pi_1(s) \to \mathrm{GL}(\mathrm{H}^*(X_{\overline{\eta}}, \mathbb{Z}_\ell)).$$

Modulo the canonical isomorphism $\mathrm{H}^*(X_{\overline{s}}, \mathbb{Z}_\ell) \simeq \mathrm{H}^*(X_{\overline{\eta}}, \mathbb{Z}_\ell)$, $\rho_{\ell^\infty,s}$ identifies with the action by transport of structure of $\pi_1(s)$ on $\mathrm{H}^*(X_{\overline{s}}, \mathbb{Z}_\ell)$. Thus studying how the $\pi_1(s)$-modules $\mathrm{H}^*(X_{\overline{s}}, \mathbb{Z}_\ell)$ vary with $s \in S$ amounts to studying the 'global' representation $\rho_{\ell^\infty} : \pi_1(S) \to \mathrm{GL}(\mathrm{H}^*(X_{\overline{\eta}}, \mathbb{Z}_\ell))$ together with its functorial localizations $\rho_{\ell^\infty,s} : \pi_1(s) \to \mathrm{GL}(\mathrm{H}^*(X_{\overline{\eta}}, \mathbb{Z}_\ell))$, $s \in S$. Write

$$\Pi_{\ell^\infty} := \mathrm{im}(\rho_{\ell^\infty}), \ \ \Pi_{\ell^\infty,s} := \mathrm{im}(\rho_{\ell^\infty,s})$$

and let $S[\ell^\infty]$ denote the set of $s \in S$ where $\Pi_{\ell^\infty,s}$ is not open in Π_{ℓ^∞}. According to the Grothendieck-Serre-Tate conjectures, $S[\ell^\infty]$ should coincide with S_{ex} (in particular, $S[\ell^\infty]$ should be independent of ℓ). This motivates

Problem: *What is the arithmetico-geometric structure of $S[\ell^\infty]$?*

We investigated thoroughly this question when S is a curve. In this setting, we basically showed that for every integer $d \geq 1$ (resp. $d = 1$) the set $S[\ell^\infty] \cap S^{\leq d}$ is finite if $p = 0$ (resp. $p > 0$). We obtained similar results for $\mathbb{F}_\ell$-coefficients provided $\ell \gg 0$. I refer in particular to our papers [Inventiones12], [Duke12], [Duke13] ($\mathbb{Q}_\ell$-coefficients) and [Crelle16], [Compositio16] ($\mathbb{F}_\ell$-coefficients). A key ingredient to make our method work is to understand the images $\overline{\Pi}_{\ell^\infty} := \rho_{\ell^\infty}(\pi_1(S_{\overline{k}}))$ and $\overline{\Pi}_\ell := \rho_\ell(\pi_1(S_{\overline{k}}))$ of the geometric étale fundamental group, where $\rho_\ell : \pi_1(S) \to \mathrm{GL}(\mathrm{H}^*(X_{\overline{\eta}}, \mathbb{Z}/\ell))$ $(\ell \neq p)$ is the modulo-ℓ representation (for $\ell \gg 0$, one knows that $\mathrm{H}^*(X_{\overline{\eta}}, \mathbb{Z}_\ell)$ is torsion-free).

During my stay at R.I.M.S. this summer we investigated two aspects of this question when k is finite.

Semisimplicity of $\overline{\Pi}_\ell$ acting on $\mathrm{H}_\ell := \mathrm{H}^*(X_{\overline{\eta}}, \mathbb{Z}/\ell)$. A celebrated consequence of the Weil conjectures observed by Deligne is that up to replacing S by a connected étale cover the Zariski closure G_{ℓ^∞} of $\overline{\Pi}_{\ell^\infty}$ acting on $\mathrm{H}^*(X_{\overline{\eta}}, \mathbb{Q}_\ell)$ is connected semi simple; this ingredients plays a crucial part in our series of papers [Inventiones12], [Duke12], [Duke13]. To make our machinery work for $\mathbb{F}_\ell$-coefficients, a similar result is needed for $\overline{\Pi}_\ell$. We showed that up to replacing S by a connected étale cover and for $\ell \gg 0$

1. $\overline{\Pi}_\ell$ is generated by its order-ℓ-elements ([Preprint15a]);
2. $\overline{\Pi}_\ell$ has trivial abelianization ([Crelle16]).

These results were enough for our purposes in [Crelle16], [Compositio16] but, together with Deligne's results for $\mathbb{Q}_\ell$-coefficients, they also provided evidence for the stronger guess that $\overline{\Pi}_\ell$ should act semisimply on H_ℓ for $\ell \gg 0$. We started investigating this question about two years ago without success. After Chun-Yin Hui wrote to us about it last summer, we focussed again on it and obtained by the spring of 2016 the first version of the joint preprint with Chun-Yin Hui [Prep16a], where we could prove that $\overline{\Pi}_\ell$ acts semisimply on H_ℓ for a set of primes ℓ of density one. This result was based on the following facts

1. the tensor-invariants of $\overline{\Pi}_\ell$ acting on H_ℓ are the reduction modulo-ℓ of those of $\overline{\Pi}_{\ell^\infty}$ acting on $\mathrm{H}^*(X_{\overline{\eta}}, \mathbb{Z}_\ell)$;

2. $\overline{\Pi}_\ell$ acts semisimply on H_ℓ if and only if $\overline{\Pi}_{\ell^\infty} \subset \mathfrak{G}_{\ell^\infty}(\mathbb{Z}_\ell)$ is almost hyper special (that is, roughly, as large as possible), where $\mathfrak{G}_{\ell^\infty}$ denotes the Zariski-closure of G_{ℓ^∞} in $\mathrm{GL}(\mathrm{H}^*(X_{\overline{\eta}}, \mathbb{Z}_\ell))$ (a group scheme over $\mathbb{Z}_\ell$).

and a deep group-theoretical result of Larsen which says that, thanks to the Weil conjectures, the set of primes where $\overline{\Pi}_{\ell^\infty} \subset \mathfrak{G}_{\ell^\infty}(\mathbb{Z}_\ell)$ is almost hyper special has density 1. The proof of (1) relies on specialization arguments (alterations, smooth proper base change, Bertini, tame étale fundamental group), the machinery of ℓ-adic cohomology (Kunneth, Poincaré, localization exact sequence, Leray spectral sequence) and weights arguments (Weil conjectures). Once (1) is granted, the proof of (2) is technical but purely group-theoretical; it involves results about algebraic groups (in particular the algebraic envelope in the sense of Nori), ℓ-adic groups, Bruhat-Tits theory. The core argument is Tannakian; the key point is to show that the algebraic enveloppe of $\overline{\Pi}_\ell$ coincides with the special fiber $\mathfrak{G}_\ell$ of $\mathfrak{G}_{\ell^\infty}$ for $\ell \gg 0$.

During the summer 2015, Akio Tamagawa and I elaborated a simple strategy (inspired from Deligne's argument for $\mathbb{Q}_\ell$-coefficients) to remove the density 1 assumption in [Prep16a]. However, to make it work, a strengthening of (1) was required (for certain arithmetic submodules and quotients). In July 2016, Chun-Yin Hui found a beautiful argument to remove the density 1 assumption. His idea is to apply (1) to the adjoint representation and show that, in that case, combinatorial lie-theoretic arguments shows that (1) is enough to ensure the semi simplicity of $\mathfrak{G}_\ell$. Chun-Yin Hui's announcement prompted us to try and finalize our own proof. Eventually, we realized very little was missing and could complete it as well. We also realized that a consequence of our work is that the Grothendieck-Serre-Tate conjectures for $\mathbb{Q}_\ell$ coefficients implies the G-S-T conjectures for $\mathbb{F}_\ell$-coefficients. This result does not seem to be known is characteristic 0 (I think I can prove it for the Grothendieck-Serre semi simplicity conjecture however).

We resubmitted a second version of [Prep16a], including both proofs, by the end of July.

Local versus global constant factor. The second topics we worked on generalizes a question of Röessler and Szamuely for abelian schemes. Precisely, assume that the $\rho_{\ell^\infty,s}$, $s \in |S|$ all have a constant geometric factors (by this, we just mean that there exists $P \in \mathbb{Q}[T]$ such that, after possibly replacing $k(s)$ by a finite field extension $P(T^n)$ divides the characteristic polynomial of the image of Frobenius by $\rho_{\ell^\infty,s}$). Then does this imply that ρ_{ℓ^∞} has an isotrivial factor? We could show that this problem is controlled by the weight-zero part $W_{\ell^\infty,0}$ of the semi simplification of the representation

$$W_{\ell^\infty} := V_{\ell^\infty}/V_{\ell^\infty}^{G_{\ell^\infty}}$$

where we write V_{ℓ^∞} for the ($\pi_1(S)$-semisimplification of) $\mathrm{H}^*(X_{\overline{\eta}}, \mathbb{Q}_\ell)$. More precisely, the dimension of $W_{\ell^\infty,0}$ gives an upper bound for the jump of rank of algebraic cycles modulo homological equivalence under specialization. We also showed that $W_{\ell^\infty,0}$ is equipped with a $\pi_1(k)$-action (which we call the 'core' representation) and that the $W_{\ell^\infty,0}$, ℓ prime form a compatible system of $\mathbb{Q}$-rational ℓ-adic representations. When $X \to S$ is an abelian scheme, it should follow from Zarhin's micro weights conjecture that the core is always trivial. In positive characteristic Zarhin's micro weights conjecture is only known for ordinary abelian varieties. We could prove the triviality of the core representation in a few other cases by *ad-hoc* technics. These results use in particular the characteristic polynomial map introduced by Serre, the Weil conjectures, Cebotarev density theorem and, as a technical ingredient, Chin-Lafforgue theorem about the potential $\mathbb{Q}$-rationality of G_{ℓ^∞} (but this later input is possibly 'too deep' and might be avoided). We hope to finalize this work in 2017. We also expect applications, for instance to the reduction modulo-p of Hecke orbits on Shimura varieties.

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