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On fermionic ghosts and the removal from scalar-fermion systems

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ref. Rampei Kimura, YS, Masahide Yamaguchi
Phys. Rev. D96 (2017) 044015,
another paper in prep.

WHY FERMIONS?

- How are inflaton and SM particles coupled with each other?
  - Interactions between inflation and SM particles … unknown
    e.g. Reheating process depends on the details of the interactions
  - Generalization of interactions including only bosonic fields has been more frequently discussed.
    e.g. Horndeski, Beyond Horndeski,Vector-tensor theories, …
- How can fermions be coupled to gravity?
  - Examination of the interaction between scalar fields and fermions can be a first step to investigate gravity-fermion system
- Can we have quadratic terms of derivatives of fermion?
  - Usually it is difficult because of the appearance of ghosts.
    It may be possible by the introduction of scalar fields.

WHAT KINDS OF INTERACTIONS ARE ALLOWED IN PRINCIPLE?

- Avoidance of negative norm states is crucial
- Fermions can easily have negative norm states (because of extra dofs)
- Let us consider systems both with bosons and fermions and how we can avoid such the negative norm states
- In the sense that we should eliminate “extra degrees of freedom”, the analysis is technically similar to that of higher derivative theories of scalar fields
  ref. H. Motohashi, et.al., JCAP 1607 (2016) 033.
- Extra dofs become explicit in Hamiltonian formulation

CLASSICAL SETUP

bosons: commuting(Grassmann-even) property $q^i q^j - q^j q^i = 0$

fermions: anti-commuting (Grassmann-odd) property

$$\theta^\alpha \theta^\beta + \theta^\beta \theta^\alpha = 0, \quad \theta^\alpha q^i - q^i \theta^\alpha = 0$$

Let us consider simplest models

- Grassmann-even real Lagrangian
- No spatial derivatives (not field theories)
- Up to first time derivatives (no higher derivatives)
- Real variables (complex variables can be decomposed)

$$S = \int_{t_1}^{t_2} L(q^i, \dot{q}^i, \theta^\alpha, \dot{\theta}^\alpha) dt$$

FERMIONIC GHOST

Purely fermionic non-degenerate system $S = \int_{t_1}^{t_2} L(\theta^\alpha, \dot{\theta}^\alpha) dt$

$$\det \left(\frac{\partial^2 L}{\partial \dot{\theta}^\beta \partial \dot{\theta}^\alpha} \right)^{(0)} \neq 0 \ , \qquad \text{where} \quad \left(\frac{\partial^2 L}{\partial \dot{\theta}^\beta \partial \dot{\theta}^\alpha} \right)^{(0)} = \left. \frac{\partial^2 L}{\partial \dot{\theta}^\beta \partial \dot{\theta}^\alpha} \right|_{\theta, \dot{\theta}=0}$$

$\dot{\theta}$ *and* π *have the one-to-one correspondence. (No primary constraints)*

$$\{\theta^\alpha, \pi_\beta\} = -\delta^\alpha_{\ \beta} \ , \qquad \{\theta^\alpha, \theta^\beta\} = \{\pi_\alpha, \pi_\beta\} = 0 \ .$$

$$\xrightarrow{\text{quantization}}$$

$$\{\hat{\theta}^\alpha, \hat{\pi}_\beta\}_+ = -i\delta^\alpha_{\ \beta} \ , \qquad \{\hat{\theta}^\alpha, \hat{\theta}^\beta\}_+ = \{\hat{\pi}_\alpha, \hat{\pi}_\beta\}_+ = 0 \ .$$

orthogonal Hermitian operators $\hat{A}_\alpha = \frac{1}{\sqrt{2}}(\hat{\theta}_\alpha - i\hat{\pi}_\alpha) \ , \quad \hat{B}_\alpha = \frac{1}{\sqrt{2}}(\hat{\theta}_\alpha + i\hat{\pi}_\alpha) \ ,$

$$\boxed{\{\hat{A}_\alpha, \hat{A}_\beta\}_+ = -\delta_{\alpha\beta} \ , \quad \{\hat{A}_\alpha, \hat{B}_\beta\}_+ = 0 \ , \quad \{\hat{B}_\alpha, \hat{B}_\beta\}_+ = \delta_{\alpha\beta}}$$

Negative norm states (ghosts)

Positive norm

WEYL FERMION

$$L = \frac{i}{2}(\bar{\psi}^{\dot{\alpha}}\sigma^{\mu}_{\alpha\dot{\alpha}}\partial_{\mu}\psi^{\alpha} - \partial_{\mu}\bar{\psi}^{\dot{\alpha}}\sigma^{\mu}_{\alpha\dot{\alpha}}\psi^{\alpha}) \qquad \psi^{\alpha} = \psi^{\alpha}_{R} + i\psi^{\alpha}_{I}$$

$$= i\psi^1_R\dot{\psi}^1_R + i\psi^1_I\dot{\psi}^1_I + i\psi^2_R\dot{\psi}^2_R + i\psi^2_I\dot{\psi}^2_I$$ + (without time derivatives)

$$\Rightarrow \quad L = \frac{i}{2}\theta_{\alpha}\dot{\theta}^{\alpha}$$ α = 1, …, N

- N primary constraints $\phi_{\alpha} \equiv \pi_{\alpha} + \frac{i}{2}\theta_{\alpha} = 0 \longrightarrow \{\phi_{\alpha}, \phi_{\beta}\} = -i\delta_{\alpha\beta}$

  π are determined by constraints

- Hamiltonian: H=0, Total Hamiltonian: $H_T = \phi_{\alpha}\mu^{\alpha}$

- No secondary constraints $\dot{\phi}_{\alpha} \approx \{\phi_{\alpha}, \phi_{\beta}\}\mu^{\beta} = -i\mu_{\alpha} \approx 0$

- Dirac brackets $\{\theta_{\alpha}, \theta_{\beta}\}_D = -i\delta_{\alpha\beta} \Rightarrow \{\hat{\theta}_{\alpha}, \hat{\theta}_{\beta}\}_{+} = \delta_{\alpha\beta}$

Positive norm

AVOIDANCE OF NEGATIVE NORM STATES

- A fermionic variable should carry
  1 dof in phase space (1/2 dof in physical sp.) to avoid the fermionic ghost
  - In the system with m bosons and N fermions,
    we need N constraints for eliminating N dofs of fermions in phase sp.

- In addition, another condition, the constraint matrix is invertible, should be satisfied in order that Hamiltonian analysis becomes closed.
  (This condition is actually important since it guarantees definite time evolution of the system. Here we will not discuss this point.)

COEXISTING SYSTEM

n bosons and N fermions

- Variations of momenta

$$L_{XY} = \frac{\partial}{\partial Y}\Big(\frac{\partial L}{\partial X}\Big)$$

$$\begin{pmatrix} \delta p_i \\ \delta \pi_\alpha \end{pmatrix} = K \begin{pmatrix} \delta \dot{q}^j \\ \delta \dot{\theta}^\beta \end{pmatrix} + \begin{pmatrix} L_{\dot{q}^i q^j} & -L_{\dot{q}^i \theta^\beta} \\ L_{\dot{\theta}^\alpha q^j} & L_{\dot{\theta}^\alpha \theta^\beta} \end{pmatrix} \begin{pmatrix} \delta q^j \\ \delta \theta^\beta \end{pmatrix} \qquad - \ *$$

- Kinetic matrix $\quad K = \begin{pmatrix} A_{ij} & \mathcal{B}_{i\beta} \\ \mathcal{C}_{\alpha j} & D_{\alpha\beta} \end{pmatrix}$

$$A_{ij} = \frac{\partial p_i}{\partial \dot{q}^j} = L_{\dot{q}^i \dot{q}^j} \ , \qquad \mathcal{B}_{i\beta} = -\frac{\partial p_i}{\partial \dot{\theta}^\beta} = -L_{\dot{q}^i \dot{\theta}^\beta} \ ,$$

$$\mathcal{C}_{\alpha j} = \frac{\partial \pi_\alpha}{\partial \dot{q}^j} = L_{\dot{\theta}^\alpha \dot{q}^j} \ , \qquad D_{\alpha\beta} = \frac{\partial \pi_\alpha}{\partial \dot{\theta}^\beta} = L_{\dot{\theta}^\alpha \dot{\theta}^\beta} \ \left(= -L_{\dot{\theta}^\beta \dot{\theta}^\alpha}\right) \ .$$

- Assumption: No degeneracy in bosonic sector

$$\det A^{(0)}_{ij} \neq 0 \ , \qquad \text{where} \quad A^{(0)}_{ij} = A_{ij}|_{\theta, \dot{\theta}=0}$$

→ *The first line of* $*$ *can be solved for* $\delta \ddot{q}^i$

MAXIMALLY DEGENERATE CONDITION

- Removing $\delta\dot{q}^i$ dependence from the second line of $*$,

$$\underbrace{(D_{\alpha\beta}-\mathcal{C}_{\alpha i}A^{ij}\mathcal{B}_{j\beta})}_{\text{dependence on } \dot{\theta}}\delta\dot{\theta}^\beta = \underbrace{\delta\pi_\alpha - \mathcal{C}_{\alpha i}A^{ij}\delta p_j + \left(\mathcal{C}_{\alpha i}A^{ij}L_{\dot{q}^j q^k} - L_{\dot{\theta}^\alpha q^k}\right)\delta q^k - \left(\mathcal{C}_{\alpha i}A^{ij}L_{\dot{q}^j\theta^\beta} + L_{\dot{\theta}^\alpha\theta^\beta}\right)\delta\theta^\beta}_{\text{canonical variables}}$$

dependence on $\dot{\theta}$

canonical variables

If $D_{\alpha\beta}-\mathcal{C}_{\alpha i}A^{ij}\mathcal{B}_{j\beta}=0$,

$$\delta\phi_\alpha = \delta\pi_\alpha - \mathcal{C}_{\alpha i}A^{ij}\delta p_j + \left(\mathcal{C}_{\alpha i}A^{ij}L_{\dot{q}^j q^k} - L_{\dot{\theta}^\alpha q^k}\right)\delta q^k - \left(\mathcal{C}_{\alpha i}A^{ij}L_{\dot{q}^j\theta^\beta} + L_{\dot{\theta}^\alpha\theta^\beta}\right)\delta\theta^\beta = 0$$

Integrability condition

$$\Rightarrow \quad \phi_\alpha = \pi_\alpha - F_\alpha(q,p,\theta) = 0$$

N primary constraints are obtained.

AN EXAMPLE OF HEALTHY THEORIES

Example: 1 scalar + N fermions

$$L = \frac{1}{2}\dot{q}^2 + i\big(f_1(q,\theta^\beta) + f_2(q,\theta^\beta)\dot{q}\big)\theta_\alpha\dot{\theta}^\alpha + \boxed{\frac{1}{2}g(q,\theta^\gamma)\theta_\alpha\theta_\beta\dot{\theta}^\alpha\dot{\theta}^\beta}$$

Maximally degenerate condition

$$L_{\dot{\theta}^\alpha\dot{\theta}^\beta} + L_{\dot{\theta}^\alpha\dot{q}}L^{-1}_{\dot{q}\dot{q}}L_{\dot{q}\dot{\theta}^\beta} = \big(g - (f_2)^2\big)\theta_\alpha\theta_\beta = 0 \quad \Rightarrow \quad g = f_2 \text{ or } -f_2$$

Momenta

$$p = \dot{q} + i f_2\theta_\alpha\dot{\theta}^\alpha$$

$$\pi_\alpha = -i(f_1 + f_2\dot{q})\theta_\alpha + g\theta_\alpha\theta_\beta\dot{\theta}^\beta = -i(f_1 + f_2 p)\theta_\alpha$$

Constraints $\quad \phi_\alpha = \pi_\alpha + i(f_1 + f_2 p)\theta_\alpha$

Quadratic terms of the time derivative of fermions are included!

SUMMARY

We have …

- discussed that, when we construct new interactions between bosons and fermions, we need to avoid what we call fermionic ghosts, coming from the extra degrees of freedom in fermionic sector. *Analogy with Ostrogradsky's ghost may be pointed out. (See also J.Phys. A35 (2002) 6169-6182)*

- found ghost free conditions, which eliminate such the ghosts, in general formulation and non-trivial examples. Remarkably, some of them include quadratic terms of the time derivative of fermionic variables.

Thank you for your attention