Hamiltonian Bigravity and Cosmology

Vladimir O. Soloviev

*Institute for High Energy Physics named after A. A. Logunov
of National Research Center "Kurchatov Institute",
Protvino (in the past – Serpukhov), Russia*

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Kyoto

Many thanks to the Organizers of this Workshop for the invitation and for the opportunity to give a talk today!

My emotions are strong because Kyoto has been my second foreign site to visit (Marcel Grossmann, 1991).

Next, Professor Noboru Nakanishi was our guest in Protvino, and I was his guest in RIMS (1994).

Trieste

The first and most visited cite for me was ICTP, Trieste.
Let me remind one important event

Workshop on Infrared Modifications of Gravity
26 – 30 September 2011, ICTP, Trieste

It was like a new baby was born, and we met him at the doors of the Maternity Hospital. A lot of dreams and hopes arose at this moment.
Now this baby looks like a teenager and sometimes behaves himself as an unsociable person, but his parents and friends are still believing in his future.

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INTERNAL REPORT
(Limited Distribution)

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

BIMETRIC GRAVITY: BRST–INVARIANCE AND SPACE–TIME DIFFEOMORPHISMS *

V.O. Soloviev **
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

From the work by Isham and Kuchař it is well known that in General Relativity space–time diffeomorphism algebra cannot be realized canonically without an additional structure. Here we show that the background Minkowski metric together with the covariant De Donder–Fock condition can serve as such a structure. For this purpose we use the BRST–invariant formalism of Relativistic Theory of Gravitation.

MIRAMARE – TRIESTE
December 1989

* To be submitted for publication.
** Permanent address: Institute for High Energy Physics, Protvino, Moscow region, 142284, USSR.

Father of bimetric gravity – Nathan Rosen

Rosen worked in USSR (1936 – 1938) supported by letters of recomendation sent from Einstein to Stalin and Molotov.

Pioneers of bimetric gravity

1. Nathan Rosen (USSR in 1936-1938), USA (Phys. Rev. 1940), Israel (from 1953)
2. Kraichnan, Gupta, Feynman and others (about 1950's)
3. Logunov and his collaborators (starting from 1980s)
4. de Rham, Gabadadze, Tolley (2011)

de Rham, Gabadadze, Tolley (dRGT)

Pioneers of bigravity

- C.J. Isham, A. Salam and J. Strathdee (1970)
- J. Wess and B. Zumino (1970)
- T. Damour and J. Kogan (2002)
- F. Hassan and R. Rosen (2011)

Pre-history of gravitational research at IHEP

- In 1967 under supervision of Logunov IHEP becomes accelerator center No. 1 in the world
- In 1969 Vladimir Folomeshkin becomes the first man of IHEP writing a paper on gravitational theory
- In 1977 Folomeshkin involves Logunov in gravitational problems
- In 1979 together they attempt to construct a new theory of gravity
- Death of Folomeshkin as a result of a tragic accident in 1979

Bigravity with de Rham, Gabadadze, Tolley potential

The Lagrangian is as follows

$$\mathcal{L} = \mathcal{L}^{(f)} + \mathcal{L}^{(g)} - \sqrt{-g}\, U(f_{\mu\nu}, g_{\mu\nu}),$$

where

$$\mathcal{L}^{(f)} = \frac{1}{16\pi G^{(f)}} \sqrt{-f} f^{\mu\nu} R^{(f)}_{\mu\nu},$$

and

$$\mathcal{L}^{(g)} = \frac{1}{16\pi G^{(g)}} \sqrt{-g} g^{\mu\nu} R^{(g)}_{\mu\nu} + \mathcal{L}^{(g)}_M(\phi^A, g_{\mu\nu}),$$

and

$$U = \frac{m^2}{2\kappa} \sum_{n=0}^{4} \beta_n e_n(X).$$

dRGT terms expressed in eigenvalues of X

$$
\begin{aligned}
e_0 &= 1, \\
e_1 &= \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, \\
e_2 &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_4 + \lambda_4\lambda_1 + \lambda_1\lambda_3 + \lambda_2\lambda_4, \\
e_3 &= \lambda_1\lambda_2\lambda_3 + \lambda_2\lambda_3\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_1\lambda_2\lambda_4, \\
e_4 &= \lambda_1\lambda_2\lambda_3\lambda_4,
\end{aligned}
$$

where λ_i are eigenvalues of matrix

$$X^\mu_\nu = \left(\sqrt{g^{-1}f}\right)^\mu_\nu .$$

dRGT terms expressed in traces of $X,\dots,X^n$

$$
\begin{aligned}
e_1 &= \mathrm{Tr}X,\\
e_2 &= \frac{1}{2}\left((\mathrm{Tr}X)^2 - \mathrm{Tr}X^2\right),\\
e_3 &= \frac{1}{6}\left((\mathrm{Tr}X)^3 - 3\mathrm{Tr}X\mathrm{Tr}X^2 + 2\mathrm{Tr}X^3\right),\\
e_4 &= \frac{1}{24}\left((\mathrm{Tr}X)^4 - 6(\mathrm{Tr}X)^2\mathrm{Tr}X^2 + 3(\mathrm{Tr}X^2)^2 + \right.\\
&+ \left. 8\mathrm{Tr}X\mathrm{Tr}X^3 - 6\mathrm{Tr}X^4\right) = \det X.
\end{aligned}
$$

ADM formulas for General Relativity

We have the canonical variables

$$\gamma_{ij} = g_{\mu\nu} \frac{\partial X^\mu}{\partial x^i} \frac{\partial X^\nu}{\partial x^j}, \qquad \pi^{ij} = -\sqrt{\gamma}(K^{ij} - \gamma^{ij} K),$$

the Hamiltonian

$$H = \int \left(NR + N^i R_i \right),$$

the constraints $R = 0$, $R_i = 0$, and their algebra

$$\begin{aligned}
\{R_i(x), R_j(y)\} &= R_i(y)\delta_{,j}(x-y) + R_j(x)\delta_{,i}(x-y), \\
\{R_i(x), R(y)\} &= R(x)\delta_{,i}(x-y), \\
\{R(x), R(y)\} &= \left[\gamma^{ij}(x)R_j(x) + \gamma^{ij}(y)R_j(y)\right]\delta_{,i}(x-y).
\end{aligned}$$

An axiomatic Hamiltonian approach to bigravity

The Lagrangian of bigravity is taken as a sum of two GR-like Lagrangians plus an ultralocal potential $U(g_{\mu\nu}, f_{\mu\nu})$.

Let us suppose that a potential exists with the following properties:

- it is free of Boulware-Deser ghost
- it is invariant under general transformations of spacetime coordinates
- it admits isotropic metrics

and will try to construct Hamiltonian formalism for it.

A scheme of the method

$$N^\mu \equiv \frac{\partial X^\mu}{\partial t} = N n^\mu + N^i \frac{\partial X^\mu}{\partial x^i} = \bar{N} \bar{n}^\mu + \bar{N}^i \frac{\partial X^\mu}{\partial x^i}.$$

1. Applying Kuchař's method of decomposition for spacetime covariant tensors.
2. Finding new constraints and enforcing them to obey the same algebra.
3. Demanding functional dependence of 4 constraints, this leads to Monge-Ampére equation.
4. Applying the Fairlie-Leznov method for solving the Monge-Ampére equation

Details of the decomposition

By introducing two sets of spacetime coordinates X^μ and (t, x^i) and notations

$$N^\mu \equiv \frac{\partial X^\mu}{\partial t}, \qquad e_i^\mu = \frac{\partial X^\mu}{\partial x^i},$$

we obtain

$$\begin{aligned}
N^\mu &= Nn^\mu + N^i e_i^\mu \bar{N}\bar{n}^\mu + \bar{N}^i e_i^\mu . \\
g^{\mu\nu} &= g^{\perp\perp} n^\mu n^\nu + g^{\perp j} n^\mu e_j^\nu + g^{i\perp} e_i^\mu n^\nu + g^{ij} e_i^\mu e_j^\nu = \\
&= -\bar{n}^\mu \bar{n}^\nu + \gamma^{ij} e_i^\mu e_j^\nu , \\
f_{\mu\nu} &= -n_\mu n_\nu + \eta^{ij} f_{\mu\alpha} f_{\nu\beta} e_i^\alpha e_j^\beta .
\end{aligned}$$

At last we introduce

$$u = \frac{\bar{N}}{N}, \qquad u^i = \frac{\bar{N}^i - N^i}{N}.$$

Results: requirements for the potential

1. There is a differentiable function $\tilde{U} = \tilde{U}(u, u^i, \eta_{ij}, \gamma_{ij})$.
2. Diffeomorphism invariance requires

$$
\begin{aligned}
2\eta_{ik}\frac{\partial \tilde{U}}{\partial \eta_{jk}} + 2\gamma_{ik}\frac{\partial \tilde{U}}{\partial \gamma_{jk}} - u^j\frac{\partial \tilde{U}}{\partial u^i} - \delta^j_i \tilde{U} &= 0, \\
2u^j\gamma_{jk}\frac{\partial \tilde{U}}{\partial \gamma_{ik}} - u^i u\frac{\partial \tilde{U}}{\partial u} + \left(\eta^{ik} - u^2\gamma^{ik} - u^i u^k\right)\frac{\partial \tilde{U}}{\partial u^k} &= 0.
\end{aligned}
$$

3. The big Hessian matrix must be degenerate

$$
\left|\frac{\partial^2 \tilde{U}}{\partial u^a \partial u^b}\right| = 0, \qquad u^a = (u, u^i).
$$

4. The small Hessian matrix is to be nondegenerate

$$
\left|\frac{\partial^2 \tilde{U}}{\partial u^i \partial u^j}\right| \neq 0, \qquad i = 1, 2, 3.
$$

Publications

1. Bigravity in Kuchar's Hamiltonian formalism.
   1. The general case
   V.O. Soloviev and M.V. Tchichikina
   Theoretical and Mathematical Physics, 2013, vol.176 (3) pp. 393 – 407; arXiv:1211.6530;
2. Bigravity in Kuchar's Hamiltonian formalism.
   2. The special case
   V.O. Soloviev and M.V. Tchichikina
   Physical Review D88 084026 (2013); arXiv:1302.5096 (2nd version - April 2013).

There were also independent parallel research (not on bigravity, but on massive gravity) by Italian group: D. Comelli, M. Crisostomi, F. Nesti, and L. Pilo.

Vierbein (tetrad) approach

The vierbein representation of the two metrics

$$g_{\mu\nu} = E^A_\mu E^B_\nu h_{AB}, \qquad g^{\mu\nu} = E^\mu_A E^\nu_B h^{AB},$$

$$f_{\mu\nu} = F^A_\mu F^B_\nu h_{AB}, \qquad f^{\mu\nu} = F^\mu_A F^\nu_B h^{AB},$$

under symmetry conditions

$$E^\mu_A F^B_\mu - E^{\mu B} F_{\mu A} = 0,$$

allows to get the following expression

$$X^\mu_\nu = \left(\sqrt{g^{-1}f}\right)^\mu_\nu = E^{\mu A} F_{\nu A}.$$

Publications

- K. Hinterbichler, R.A. Rosen. Interacting Spin-2 Fields. *JHEP* **07** (2012) 047; arXiv:1203.5783.
- S. Alexandrov, K. Krasnov, and S. Speziale. Chiral description of ghost-free massive gravity. *JHEP* **1306**, 068 (2013); arXiv:1212.3614.
- J. Kluson. Hamiltonian formalism of bimetric gravity in vierbein formulation. *Eur. Phys. J.* **74**, 2985 (2014); arXiv:1307.1974.
- S. Alexandrov. Canonical structure of Tetrad Bimetric Gravity. *Gen. Rel. Grav.* **46**, 1639 (2014); arXiv:1308.6586.
- V.O. Soloviev. Bigravity in Hamiltonian formalism: the tetrad approach. *Theoretical and Mathematical Physics* **182**, 204–307 (2015); arxiv: 1410.0048 (with supplement).

Triads instead of induced metrics)

- In metric approach: two induced metrics γ_{ij}, η_{ij}

$$\gamma_{ij} = g_{\mu\nu}\frac{\partial X^\mu}{\partial x^i}\frac{\partial X^\nu}{\partial x^j}, \qquad \eta_{ij} = f_{\mu\nu}\frac{\partial X^\mu}{\partial x^i}\frac{\partial X^\nu}{\partial x^j},$$

  and their conjugate momenta

$$\pi^{ij}, \qquad \Pi^{ij}.$$

- In vierbein approach: two triads e^a_i, f^a_i

$$\gamma_{ij} = e^a_i e^b_j \delta_{ab}, \qquad \eta_{ij} = f^a_i f^b_j \delta_{ab},$$

  and their conjugate momenta

$$\pi^i_a, \qquad \Pi^i_a.$$

Optimal and general vierbeins (tetrads)

One vierbein may be taken in suitable form: let E^μ_A be a unit normal $\bar{n}^\mu$ + lift to spacetime of triad e^i_a:

$$E^\mu_0 = \bar{n}^\mu, \qquad E^\mu_a = \frac{\partial X^\mu}{\partial x^i} e^i_a,$$

but then for F^A_μ we take an arbitrary boost of analog $\mathcal{F}^B_\mu$:

$$F^A_\mu = \Lambda^A{}_B \mathcal{F}^B_\mu$$

this boost is determined as

$$\Lambda^A{}_B = \begin{pmatrix} \varepsilon & p_b \\ p^a & \delta^a{}_b + \frac{1}{\varepsilon+1} p^a p_b \end{pmatrix},$$

where a new arbitrary parameter p_a is taken into play, and

$$p^a = \delta^{ab} p_b, \qquad p^2 = p_a p^a, \qquad \varepsilon = \sqrt{1 + p^2}.$$

The minimal potential ($\beta_i = 0$ and $i \neq 1$) in its explicit form:

$$U = \beta_1 U_1,$$

where

$$U_1 = \sqrt{1 + p^2} + u \left(f_{ia} e^{ia} + \frac{p_a f_i^{\ a} e_b^i p^b}{\sqrt{1 + p^2} + 1} \right) - u^i f_i^{\ a} p_a.$$

The potential is linear in auxiliary variables u, u^i and nonlinear in auxiliary variable p_a. Of course, U is also a function of the canonical variables f_{ia}, e^{ia}.

Auxiliary variables and constraints

GR: N, N^i. Bigravity: N, N^i, u, u^i.
GR in vierbeins: N, N^i, λ_{ab}.
Bigravity in vierbeins: N, N^i, u, u^i, λ^+_{ab}, λ^-_{ab}, Λ^a, Λ^{ab}, p_a.

variable	equation	$\rightarrow$	result 1	$\rightarrow$	result 2	$\rightarrow$	result 3
N	$\mathcal{R} \approx 0$						
N^i	$\mathcal{R}_i \approx 0$						
λ^+_{ab}	$L^+_{ab} \approx 0$						
u	$\mathcal{S} = 0$	$\rightarrow$	$\Omega = 0$	$\rightarrow$	$\{\mathcal{S}, \Omega\} \neq 0$	$\rightarrow$	u
u^i	$\mathcal{S}_i = 0$	$\rightarrow$	p_a				
Λ^a	$G_a = 0$	$\rightarrow$	u^i				
λ^-_{ab}	$L^-_{ab} = 0$	$\rightarrow$	$\{L^-_{ab}, G_{cd}\} \neq 0$	$\rightarrow$	$\Lambda_{cd} = 0$		
Λ^{ab}	$G_{ab} = 0$	$\rightarrow$	$\{G_{ab}, L^-_{cd}\} \neq 0$	$\rightarrow$	$\lambda^-_{cd} = 0$		

Degrees of freedom calculation

$$\mathrm{DOF} = \frac{1}{2}\left(n - 2n_{f.c.} - n_{s.c.}\right).$$

	Bigravity (general)	Bigravity (dRGT)	Bigravity (vierbein)
(q,p) n	$(\gamma_{ij},\pi^{ij}),(\eta_{ij},\Pi^{ij})$ 24	$(\gamma_{ij},\pi^{ij}),(\eta_{ij},\Pi^{ij})$ 24	$(e_{ia},\pi^{ia}),(f_{ia},\Pi^{ia})$ 36 (48 - Alex)
1st class $n_{f.c.}$	$\mathcal{R},\mathcal{R}_i$ 4	$\mathcal{R},\mathcal{R}_i$ 4	$\mathcal{R},\mathcal{R}_i,L^{+}_{ab}$ 7 (10 - Alex)
2nd class $n_{s.c.}$	— 0	$\mathcal{S},\Omega$ 2	$\mathcal{S},\Omega,L^{-}_{ab},G_{ab}$ 8 (14 - Alex)
DoF	8	7	7

Alexander Friedmann (1888-1925)

Lacqua chio prendo gi mai non si corse;
Minerva spira, e conducemi Appollo,
e nove Muse mi dimostran l′Orse
(Dante Aligheri, La Divina Commedia)

Hamiltonian Friedmann cosmology in bigravity

At last, the easiest way to get an elementary form of the dRGT potential is to reduce the number of degrees of freedom, and just this trick can be used in the study of a background cosmology.

Let us take the cosmologilcal ansatz for both metrics

$$f_{\mu\nu} = (-N^2(t), {R_f}^2(t)\delta_{ij}), \qquad g_{\mu\nu} = (-\bar{N}^2(t), R^2(t)\delta_{ij}),$$

then new variables appear

$$\boxed{u = \frac{\bar{N}}{N}}, \qquad \boxed{r = \frac{R_f}{R}}.$$

Elementary calculation

$$Y^{\mu}_{\nu} = \left(g^{-1}f\right)^{\mu}_{\nu} = g^{\mu\alpha}f_{\alpha\nu} = \mathrm{diag}\left(u^{-2}, r^2\delta_{ij}\right),$$

The positive square root of this diagonal matrix is here

$$X = \sqrt{Y} = \mathrm{diag}\left(+\sqrt{u^{-2}}, +\sqrt{r^2}\delta_{ij}\right) \equiv \mathrm{diag}\left(u^{-1}, r\delta_{ij}\right),$$

λ_i and e_i are as follows

$$\begin{aligned}
\lambda_1 &= u^{-1}, \qquad \lambda_2 = \lambda_3 = \lambda_4 = r, \\
e_0 &= 1, \\
e_1 &= \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = u^{-1} + 3r, \\
e_2 &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 = 3ru^{-1} + 3r^2, \\
e_3 &= \lambda_1\lambda_2\lambda_3 + \lambda_2\lambda_3\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_1\lambda_2\lambda_4 = 3r^2u^{-1} + r^3, \\
e_4 &= \lambda_1\lambda_2\lambda_3\lambda_4 = r^3u^{-1}.
\end{aligned}$$

dRGT potential in cosmology of bigravity

$$\boxed{U = \frac{2m^2}{\kappa} N \left(uV + W \right)}.$$

The potential is linear in u and

$$\begin{aligned} V &= \frac{1}{N}\frac{\partial U}{\partial u} = R^3 B_0(r), \\ W &= \frac{1}{N}\left(U - u\frac{\partial U}{\partial u} \right) = R^3 B_1(r) \equiv R_f^3 \frac{B_1(r)}{r^3}, \end{aligned}$$

deformed formulae for $(1+r)^3$ arise above

$$B_i(r) = \beta_i + 3\beta_{i+1} r + 3\beta_{i+2} r^2 + \beta_{i+3} r^3.$$

Coupling to effective metric

$$\mathcal{G}_{\mu\nu} = g_{\mu\nu} + 2\beta g_{\mu\alpha}\sqrt{g^{-1}f}^{\alpha}_{\ \nu} + \beta^2 f_{\mu\nu} = \left(E^A_\mu + \beta F^A_\mu\right)\left(E_{A\nu} + \beta F_{A\nu}\right),$$

$$\mathcal{L}_\phi = \sqrt{-\mathcal{G}}\left(\frac{1}{2}\mathcal{G}^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - U(\phi)\right),$$

Cosmological ansatz is as follows

$$\mathcal{G}_{00} = -\mathcal{N}^2, \qquad \mathcal{G}_{ij} = a^2\delta_{ij},$$

$$\mathcal{N} = N(u+\beta), \qquad a = R + \beta R_f.$$

$$\sqrt{-\mathcal{G}} = \mathcal{N}a^3,$$

$$\mathcal{L}_\phi = \mathcal{N}a^3\left(\frac{1}{2}\left(\frac{\dot{\phi}}{\mathcal{N}}\right)^2 - U(\phi)\right), \qquad \pi_\phi = \frac{a^3}{\mathcal{N}}\dot{\phi}.$$

The constraints

The primary constraints ($\mu = G_f / G_g$)

$$\mathcal{S} = \frac{3R^3}{8\pi G_g}\left[-{H_g}^2 + (1+\beta r)^3\frac{8\pi G_g\rho}{3} + \frac{m^2}{3}B_0(r)\right] = 0,$$

$$\mathcal{R}' = \frac{3R^3}{8\pi G_g}\left[-\frac{r^3 {H_f}^2}{\mu} + \beta(1+\beta r)^3\frac{8\pi G_g\rho}{3} + \frac{m^2}{3}B_1(r)\right] = 0$$

The secondary constraint

$$\Omega = \frac{3R}{8\pi G_g}\Omega_1\Omega_2 = 0,$$

$$\Omega_1 = rH_f - H_g,$$

$$\Omega_2 = \beta_1 + 2\beta_2 r + \beta_3 r^2 - \beta\left(1+\beta r\right)^2 8\pi G_g p = 0.$$

First branch

The Friedmann equation for the observable Hubble constant

$$H = \frac{\dot{a}}{\mathcal{N} a}$$

is as follows

$$\boxed{H^2 = \frac{8\pi \tilde{G} \rho}{3} + \frac{\Lambda(r)}{3}}, \qquad \tilde{G} = (1 + \beta r) G.$$

The cosmological term and matter density become functions of r:

$$\begin{aligned} \Lambda(r) &= m^2 \frac{B_0(r)}{(1+\beta r)^2}, \\ \rho &= \frac{m^2}{8\pi G} \frac{\frac{\mu B_1(r)}{r} - B_0(r)}{(1+\beta r)^3 \left(1 - \frac{\mu\beta}{r}\right)}. \end{aligned} \qquad (1)$$

Cosmology as evolution of “hidden variable” r

The study of cosmological dynamics transforms into a study of dynamics for r (we suppose equation of state $p = w\rho$)

$$\dot{r} = \frac{3NHa(1+w)(1+\beta r)\left(\frac{\mu B_1}{r} - B_0\right)}{B_0 - (B_{-1})' + \frac{\mu B_0'}{r} + \left(\frac{\mu B_1}{r} - B_0\right)\left(\frac{1}{1-\frac{\mu\beta}{r}} + \frac{3w}{1+\beta r}\right)}.$$

Critical points may be

$$r = -\frac{1}{\beta},$$

$$r = \mu\beta,$$

and the roots of quartic equation

$$\frac{\mu B_1(r)}{r} - B_0(r) = 0.$$

Conclusion

We hope that dRGT bigravity or some of its extensions will be able to correspond to the real world. If so, the Hamiltonian formalism will be called for many problems.