Positivity Constraints on Effective Field Theories for Cosmology

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REVIEWS OF MODERN PHYSICS VOLUME 23, NUMBER 4 OCTOBER, 1951

The Renormalization of Meson Theories*

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The recent developments in quantum electrodynamics (the interaction of photons and the electron-positron field) associated with the names of Tomonaga, Schwinger, and Feynman culminated, as far as the theory of the renormalization of mass and charge is concerned, in the work of Dyson[1] published in 1949. Combining Feynman's technique[2] of depicting field events graphically and Schwinger's invariant procedure of subtracting divergences,[3] Dyson proved two very important results. He showed first that if calculations are made to any arbitrarily high order in the charge in a perturbation expansion, three and only three types of integrals can diverge; and, secondly, that a renormalization of mass and charge would suffice completely to absorb these divergences. This theory has proved to be in very close agreement with experiment.[4]

$S_F(p)$ and $D_F(p)$ for the electron and the photon lines[5] and the factor $e\gamma_\mu$ (charge times a Dirac matrix) for the vertices of the graph. By considering the integrals thus obtained, Dyson showed that the over-all[6] degree of divergence of a particular graph could be estimated simply by counting its external lines. Let E_f denote the number of external fermion (we use the term fermion for any spin half particle) and E_p the number of external photon lines. The integral corresponding to a graph can diverge only if

$$\tfrac{3}{2}E_f + E_p < 5. \quad (1)$$

This basic inequality shows that there are only a finite number of *types* of graph that can introduce divergences in the theory. These are the electron and photon self-energy graphs and vertex parts, simple examples of which are given in Fig. 1 (a, b, and c). Another possible

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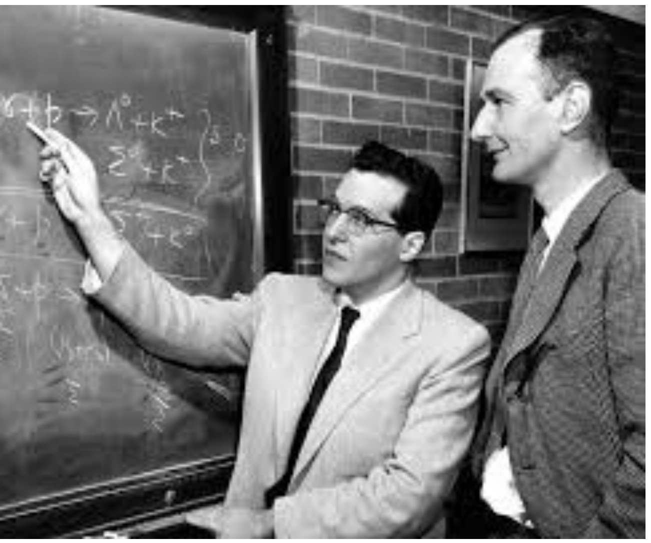

The Renormalization of Meson Theories*

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Salam criterion

A general proof is found ....The procedure looks complicated but the idea is essentially simple. The difficulty, as in all this work, is to find a notation which is both concise and intelligible to at least two people of whom one may be the author

To which Salam later clarified...

We left it unsaid that the other could be the **co-author**

My co-authors

Claudia de Rham

Scott Melville

Shuang Yong Zhou

Positive Bounds for Scalar Theories 1702.06134
Massive Galileon Positivity Bounds 1702.08577
Positivity Bounds for Particles with Spin 1706.02712
Positivity Bounds for Spin 1 and Spin 2 particles 1705.???

Overview

Cosmological Theories, in particular those for inflation/dark energy/modified gravity are

Wilsonian Effective Field Theories

Non-renormalizable interactions - break down at some energy scale which is often **below** Planck scale

$$e^{\frac{i}{\hbar} S_W[\text{Light}]} = \int D\text{Heavy} \quad e^{\frac{i}{\hbar} S_{UV}[\text{Light},\text{Heavy}]}$$

Wilson: Heavy loops already included,
Light loops not yet included

In Cosmology always true because we must have gravity

GR itself should be understood as an EFT with a Planck scale physics - no problem quantizing gravity as a LEEFT,

Perturbative scattering breaks unitarity at Planck scale, known irrelevant operators can renormalize UV divergence

see e.g. reviews by Donoghue, Burgess

$$S = \int d^4x \sqrt{-g} \left[\frac{M_P^2}{2} R + \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} + \cdots + M_P^4 \left(\frac{\nabla}{M_P} \right)^{2N} \left(\frac{\text{Riemann}}{M_P^2} \right)^M \right]$$

For example, we have no trouble computing loop corrections to scalar and tensor fluctuations produced during inflation

Weak or Strong

The 'unknown' UV completion may be weakly coupled, meaning new classical/tree level physics kicks in at some scale, and resolves/improves perturbative unitarity

example: String Theory

$$S = \int d^D x \frac{\alpha'^{-(D-2)/2}}{g_s^2} \left(R + \alpha' R^2 + \alpha' R^3 + \ldots \right) + g_s^0(\ldots) + g_s^2(\ldots) + \ldots$$

string scale below the Planck scale, massive string states kick in before we reach quantum gravity scale

or it may be strongly coupled and the cutoff is the scale at which loops become order unity

$$S = \int d^4 x \sqrt{-g} \left[\frac{M_P^2}{2} R + \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} + \cdots + M_P^4 \left(\frac{\nabla}{M_P} \right)^{2N} \left(\frac{\text{Riemann}}{M_P^2} \right)^{M}$$

Explosion of models beyond GR+SM+standard extensions

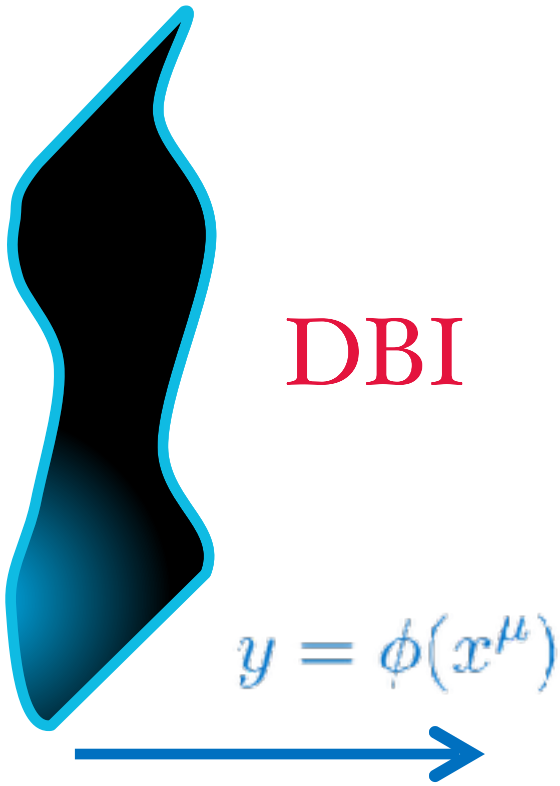
Many models are non-traditional, in the sense that naive non-renormalizable operators play a significant role for cosmology:

$$\mathcal{L} \sim \Lambda^4 \frac{\nabla^N \phi^M}{\Lambda^{N+M}}$$

Non-renormalizable/irrelevant if $N + M > 4$

Despite large irrelevant operators, EFT for fluctuations remains under control!

e.g. DBI-inflation



DBI

$$\mathcal{L}_{\rm DBI} \sim -\sqrt{1 + (\partial\phi)^2} \sim \gamma^{-1}$$

Silverstein, Tong, PRD70, 2004
Alishahiha, Silverstein, Tong, PRD70, 2004

Poincare invariance in 5d implies global symmetry for DBI in 4d:

$$\phi \to \phi + c + v_\mu \left(x^\mu + \phi \partial^\mu \phi\right)$$

Galileon

$$L_{\rm Galileon} = -\frac{1}{2}(\partial\phi)^2 + \frac{\Box\phi}{\Lambda^3}(\partial\phi)^2 + \ldots$$

$$\phi \to \phi + c + v_\mu x^\mu$$

other examples: K-inflation, DGP, Massive Gravity, Generalized galileon, Horndeski, beyond Horndeski, beyond beyond Horndeski, Chameleon, Symmetron, , superfluid dark matter + ...

UV Completion?

$$e^{\frac{i}{\hbar}S_W[\text{Light}]} = \int D\text{Heavy} \quad e^{\frac{i}{\hbar}S_{UV}[\text{Light},\text{Heavy}]}$$

Typical assumption that
UV completion is Local, Causal, Poincare Invariant and Unitary

Locality and Poincare invariance can be question for Quantum Gravity????

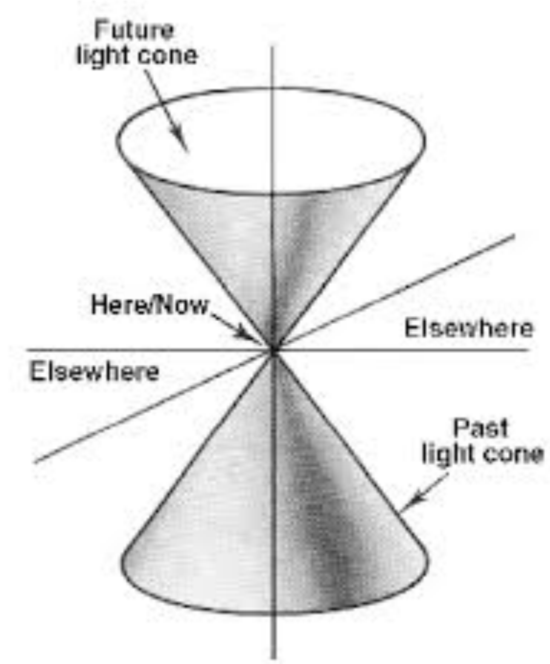
Recent Recognition: Requirement that a given low energy theory admits such a UV completion imposes an (infinite) number of constraints on the form of the low energy effective theory

Positivity Constraints!

Positivity Constraints

Signs of UV completion

Lets Start Simple:
Two-point function of a scalar field



Suppose we have a scalar operator $\hat{O}(x)$

Relativistic Locality tells us that ......

$$[\hat{O}(x), \hat{O}(y)] = 0 \qquad \text{if } (x-y)^2 > 0$$

Unitarity (positivity) tells us that

$$\langle\psi|\hat{O}(f)^2|\psi\rangle > 0 \quad \text{where} \qquad \hat{O}(f) = \int d^4x\, f(x)\hat{O}(x)$$

Kallen-Lehmann Spectral Representation

Together with Poincare invariance these imply:

$$\langle 0|\hat{T}\hat{O}(x)\hat{O}(y)|0\rangle = \int \frac{d^4k}{(2\pi)^4} e^{ik.(x-y)} G_O(k)$$

$$G_O(k) = \frac{Z}{k^2+m^2-i\epsilon} + \int_{4m^2}^{\infty} d\mu \frac{\rho(\mu)}{k^2+\mu-i\epsilon}$$

Positive Spectral Density as a result of Unitarity

$$\rho(\mu) \geq 0$$

Kallen-Lehmann Spectral Representation (minor correction)

In general spectral density grows so integral doesn't converge, but less than a polynomial (so that Wightman function exists)

$$G_O(k) = \frac{Z}{k^2+m^2-i\epsilon} + S(-k^2) + (-k^2)^N \int_{4m^2}^{\infty} d\mu \frac{\rho(\mu)}{\mu^N(k^2+\mu-i\epsilon)}$$

Subtraction Polynomial $\quad S(-k^2) = \sum_{m=0}^{m=N-1} c_m(-k^2)^m$

$$\frac{1}{N!}\frac{d^N G'_O(-k^2)}{d(-k^2)^N} = \int_{4m^2}^{\infty} \frac{\rho(\mu)}{(k^2+\mu-i\epsilon)^{N+1}}$$

Introduce the Complex Plane

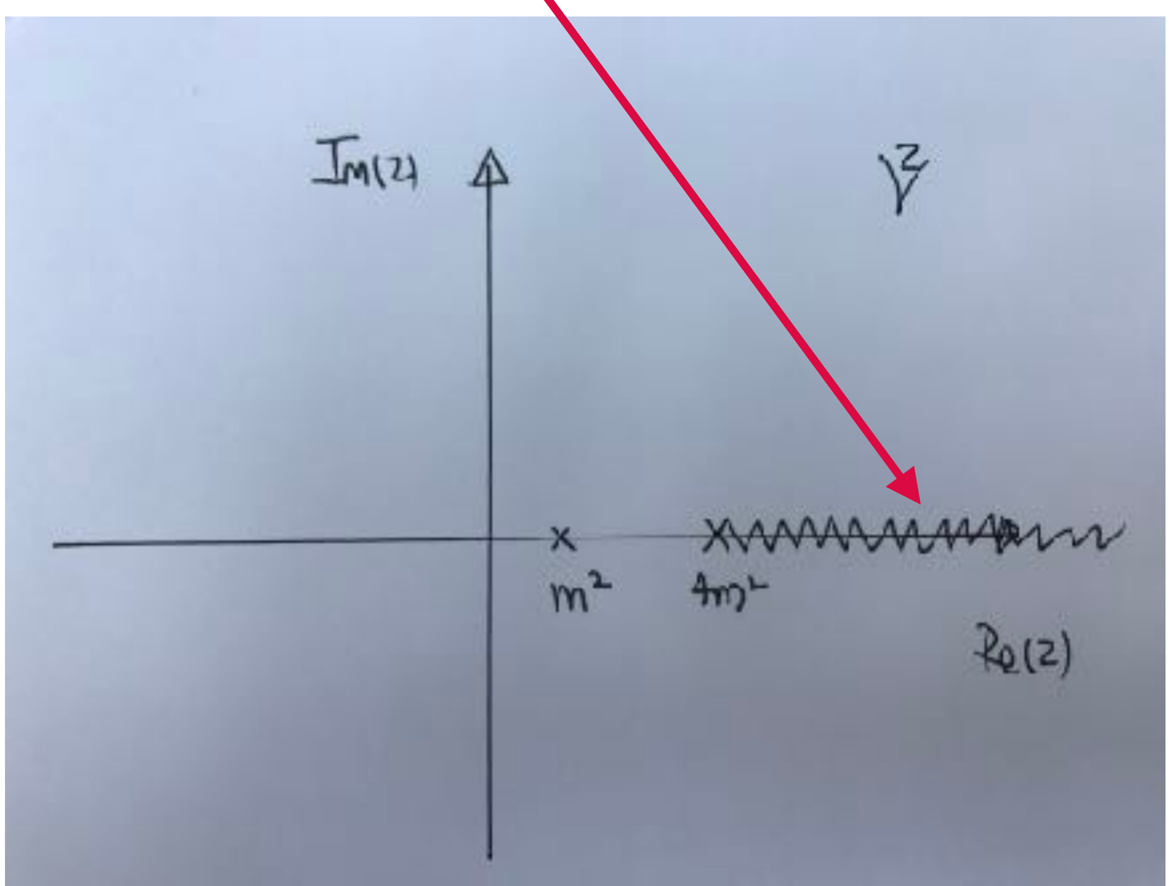
To simplify a problem you should make it complex

Define complex momenta squared $z = -k^2 + i\epsilon$

Pole — Branch cut

$$G_O(z) = \frac{Z}{m^2 - z} + S(z) + z^N \int_{4m^2}^{\infty} d\mu \frac{\rho(\mu)}{\mu^N(\mu - z)}$$

Physical region



Two point function is an analytic function with a pole and a branch cut

Discontinuity across branch cut is positive definite

$$\frac{1}{M!} \frac{d^M G'_O(0)}{dz^M} = \int_{4m^2}^{\infty} d\mu \frac{\rho(\mu)}{\mu^{M+1}} > 0$$

$$M \geq N$$

What does this tell us about EFT?

e.g. Suppose scalar field in EFT with tree level action ....

$$S = \int d^4x \hat{O}(x)[\Box + a_1 \frac{\Box^2}{M^2} + a_2 \frac{\Box^3}{M^4} + \ldots]\hat{O}(x)$$

Feynman propagator is

$$G_O = -\frac{1}{z + a_1 \frac{z^2}{M^2} + a_2 \frac{z^3}{M^4} + \ldots}$$

$$G'_O(z) = \frac{a_1}{M^2} + \frac{(a_2 - a_1^2)}{M^4} z + \mathcal{O}(z^2)$$

assuming no subtractions

Positivity Bounds: $a_1 > 0$ and $a_2 > a_1^2$

What about gravity?

If we repeat the same argument for gravity:

spin 2 part — spin 0 part

$$S = \int d^4x \frac{M_P^2}{2} \left[R + \frac{a}{\Lambda^2} (R_{\mu\nu} R^{\mu\nu} - (5/3) R^2) + \frac{b}{\Lambda^2} R^2 \right)\right]$$

$$+ \frac{1}{\Lambda^4} R_{....} \nabla_{.} \nabla_{.} R_{....}$$ corrections

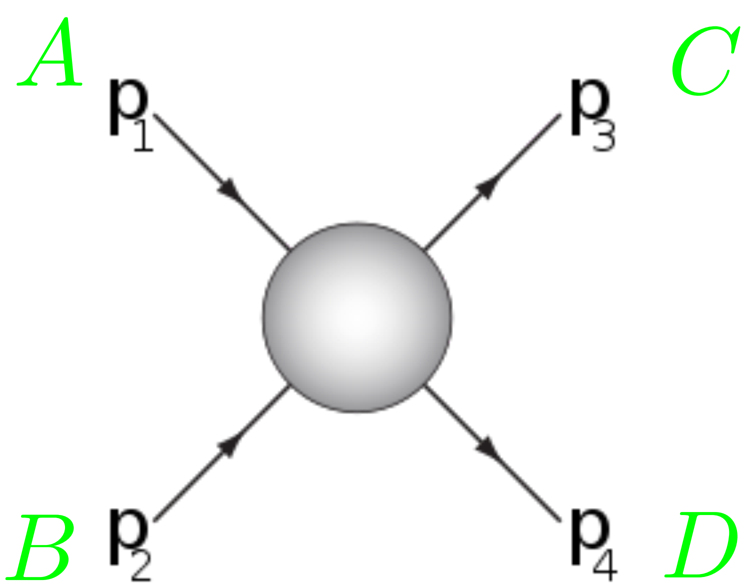
Positivity Bounds: $a \geq 0$ $\quad$ $b \geq 0$

S-matrix Positivity Constraints

Signs of UV completion

2 - 2 Scattering Amplitude

Identical scattering amplitudes for s and u channel interactions (up to analytic continuation)



s-channel

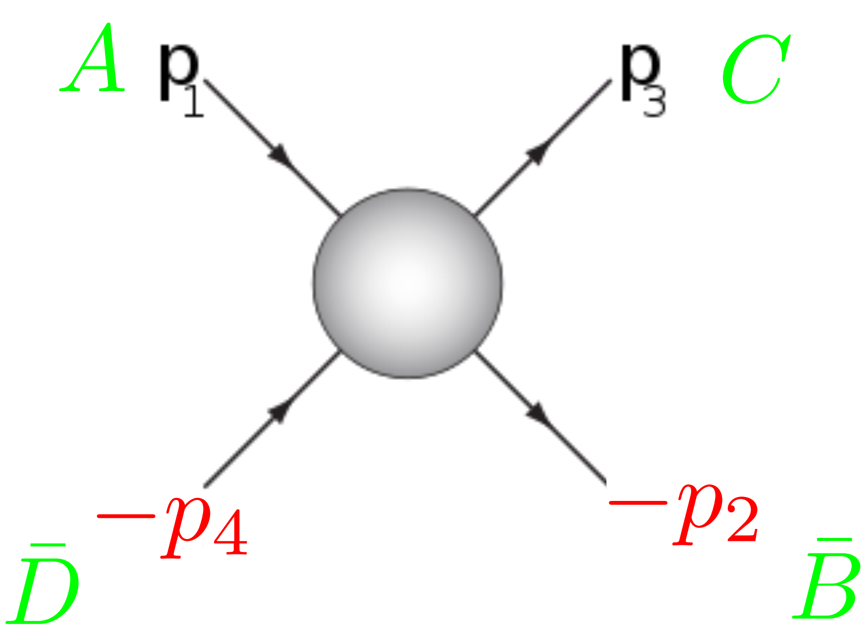
$$A + B \rightarrow C + D$$

$$s + t + u = 4m^2$$

$$s = (p_1 + p_2)^2$$

$$t = (p_1 - p_3)^2$$

$$u = (p_1 - p_4)^2$$



u-channel

$$A + \bar{D} \rightarrow C + \bar{B}$$

$$s \leftrightarrow u$$

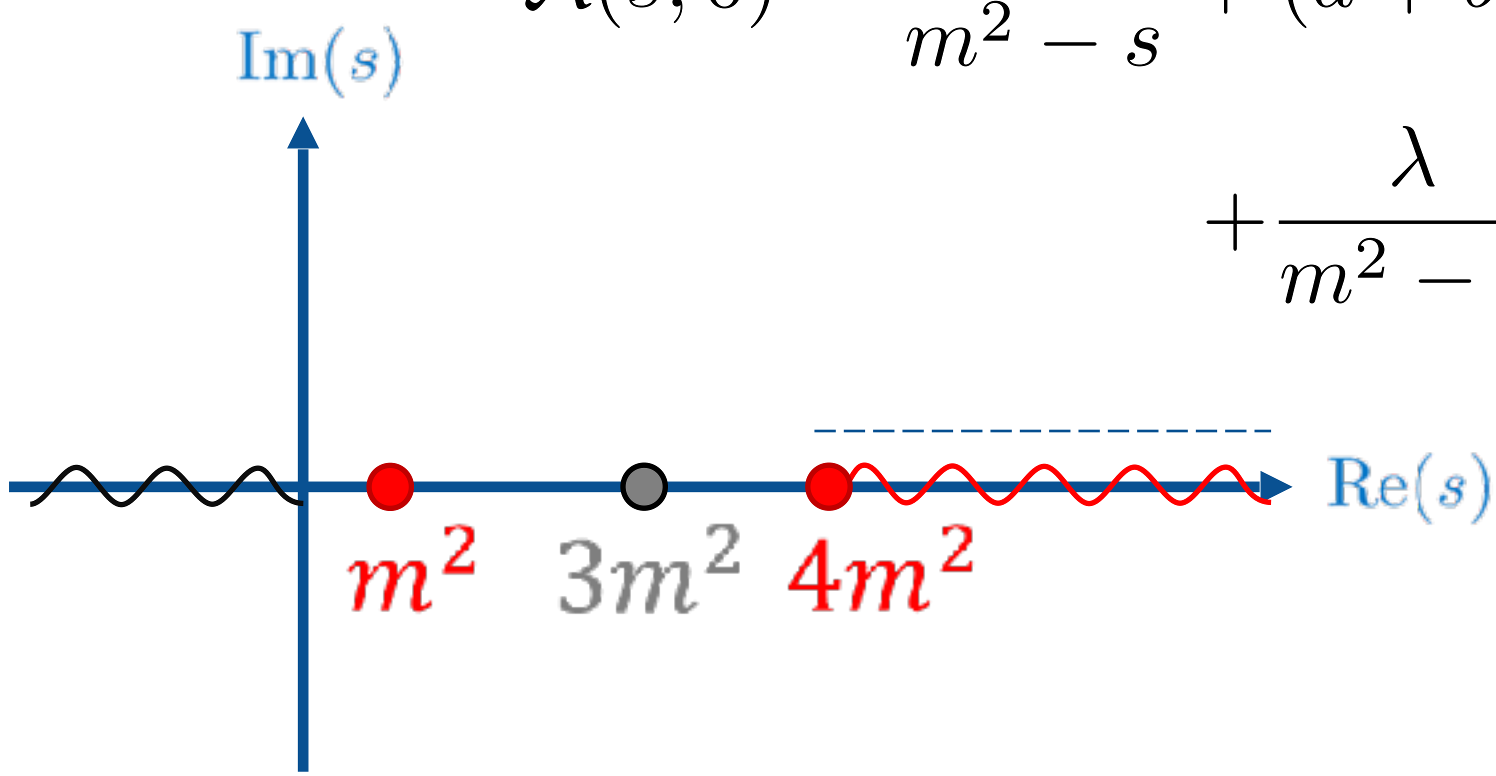
Analyticity of Forward Scattering Limit
Amplitude = analogue of K-L spectral rep

Scattering amplitude $\mathcal{A} = \langle \text{final} \,|\, \hat{T} \,|\, \text{initial} \rangle$

Complex s plane

Physical scattering region is
$s \geq 4m^2$

$$\mathcal{A}(s,0) = \frac{\lambda}{m^2 - s} + (a + bs) + s^2 \int_{4m^2}^{\infty} d\mu \frac{\rho(\mu)}{\mu^2(\mu - s)} + \frac{\lambda}{m^2 - u} + u^2 \int_{4m^2}^{\infty} d\mu \frac{\rho(\mu)}{u^2(\mu - u)}$$



crossing

$u = 4m^2 - s$

Optical Theorem (forward limit)

$$\sigma(s) = \frac{\mathrm{Im}\mathcal{A}(s,0)}{\sqrt{s(s-4m^2)}} > 0$$

Physical scattering for $s \geq 4\ m^2$
In the forward scattering limit, ie. $t = 0$

Number of subtractions determined by Froissart bound:

$$\sigma(s) < \frac{c}{m^2}(\log(s/s_0))^2$$

Positivity Constraint

Recipe: Subtract pole, differentiate to remove subtraction constants

$$\mathcal{B}(s,0) = \mathcal{A}(s,0) - \frac{\lambda}{m^2 - s} - \frac{\lambda}{m^2 - u}$$

$$= (a + bs) + s^2 \int_{4m^2}^{\infty} d\mu \frac{\rho(\mu)}{\mu^2(\mu - s)} + u^2 \int_{4m^2}^{\infty} d\mu \frac{\rho(\mu)}{u^2(\mu - u)}$$

$$\frac{d^M}{ds^M} B(2m^2, 0) = 2M! \int_{4m^2}^{\infty} d\mu \frac{\rho(\mu)}{(\mu - 2m^2)^{M+1}} > 0$$

$$M \geq 2$$

If weakly coupled

$$\frac{d^M}{ds^M} B^{\text{tree}}(2m^2, 0) = 2M! \int_{\Lambda^2_{\text{th}}}^{\infty} d\mu \frac{\rho(\mu)}{\mu - 2m^2}$$

Adams et. al. 2006

Positivity Bounds for P(X)

$P(X)$ $$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 + \frac{c}{\Lambda^4}(\partial\phi)^4 + \cdots$$

$$\mathcal{A}^{\text{tree}}_{2\to 2} = \frac{c}{\Lambda^4}\left(s^2 + t^2 + u^2 - 4m^2\right)$$

Positivity bounds requires: $c > 0$

P(X) models with $c \leq 0$ cannot admit a local/Poincare invariant UV completion

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Galileon $$L = -\frac{1}{2}(\partial\phi)^2 + \frac{1}{\Lambda^3}\Box\phi(\partial\phi)^2 + \dots$$

$c = 0$ Galileons cannot admit a local/Poincare invariant UV completion

Adams et. al. 2006

Positivity Bounds for Massive Galileon (in the forward scattering limit)

$$\mathcal{L}_{\mathrm{mGal}} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{g_3}{\Lambda^3}(\partial\phi)^2\Box\phi - \frac{g_4}{\Lambda^6}(\partial\phi)^2\left((\Box\phi)^2 - (\partial_\mu\partial_\nu\phi)^2\right)$$

$$+\mathcal{L}_{\text{higher derivatives}}$$

Galileon operators protected by a non-renormalization theorem even in presence of mass

Pole subtracted amplitude:

$$B(s, t=0) = \bar{a} + \bar{b}s + \frac{m^2 g_3^2}{8\Lambda^6}s^2 + \cdots$$

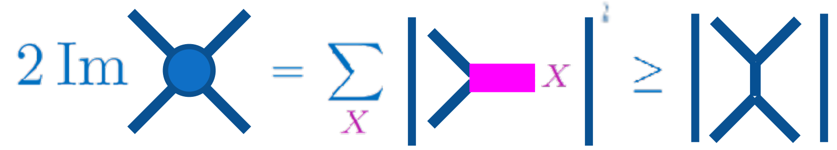
$$> 0$$

No obvious obstruction to UV completion at this level, so long as $m \neq 0$

Any *quartic Galileon* ($g_4 \neq 0$) should necessarily also have a *cubic Galileon* ($g_3 \neq 0$) (to have any chance of potential UV completion existence)

de Rham, Melville, AJT, Zhou, 1702.08577

Away from forward scattering limit



$$\mathcal{A}(s,t) = 16\pi\sqrt{\frac{s}{s-4m^2}}\sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(\cos\theta)a_\ell(s)$$

$$\mathrm{Im}\, a_\ell(s) = |a_\ell(s)|^2 + \cdots$$

$$\mathrm{Im}\, a_l(s) > 0\,, \quad s \geq 4m^2$$

$$0 \leq |a_\ell(s)^2| \leq \mathrm{Im}\, a_\ell(s) \leq 1 \quad \text{for} \quad s \geq 4m^2$$

$$\frac{d^n}{dt^n}\mathrm{Im} A(s,t)\Big|_{t=0} > 0$$

using $\frac{d^n}{dx^n}P_l(x)\Big|_{x=1} > 0$

$$\mathrm{Im}\, A(s,t) > 0\,, \quad 0 \leq t < 4m^2\,,\ s \geq 4m^2$$

$$M \geq 2$$

$$\frac{d^M}{ds^M}B(2m^2 - t/2, t) = \frac{2M!}{\pi}\int_{4m^2}^{\infty} d\mu\, \frac{\mathrm{Im} A(\mu,t)}{(\mu - 2m^2 + t/2)^{M+1}} > 0$$

Bounds on t derivatives

$$\partial_s^2\, B(s,t)|_{s=0} = \int_{4m^2}^{\infty} \mathrm{d}\mu \frac{\mathrm{Im}A(\mu,t)}{(\mu+t-4m^2)^3} > 0$$

+ crossing symmetric terms

$$\partial_s^2 \partial_t\, B(s,t)|_{s=0} = \int_{4m^2}^{\infty} \mathrm{d}\mu \left[\underbrace{\frac{\partial_t \mathrm{Im}A(\mu,t)}{(\mu+t-4m^2)^3}}_{>0} - \underbrace{\frac{3\mathrm{Im}A(\mu,t)}{(\mu+t-4m^2)^4}}_{<0} \right]$$

$$> -3\frac{\partial_s^2 B}{\mathcal{M}^2}$$

$$\mathcal{M}^2 = \mu_{\mathrm{min}} + t - 4m^2$$

de Rham, Melville, AJT, Zhou, 1702.06134

Higher order positivity bound

$$\partial_s^2 \, B(s,t)|_{s=0} = \int_{4m^2}^{\infty} \mathrm{d}\mu \frac{\mathrm{Im} A(\mu,t)}{(\mu+t-4m^2)^3} > 0$$

+ crossing symmetric terms

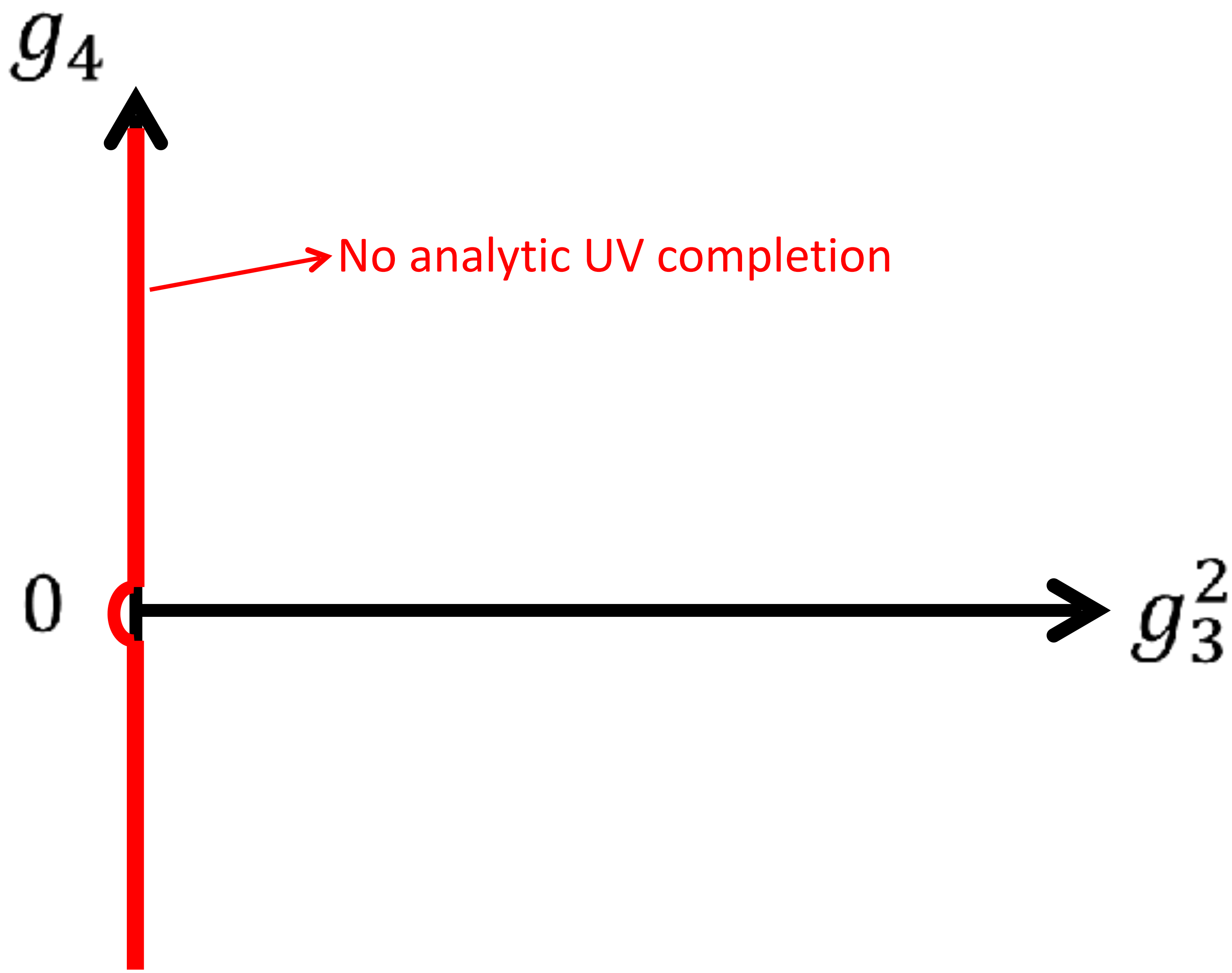
$$\partial_s^2 \partial_t \, B(s,t)|_{s=0} + \frac{3}{\mathcal{M}^2} \partial_s^2 \, B(s,t)|_{s=0} > 0$$

The same logic can be extended to
arbitrary number of s and t derivatives,
valid for all $0 \leq t < 4m^2$

$$\mathcal{M}^2 = \mu_{\rm min} + t - 4m^2$$

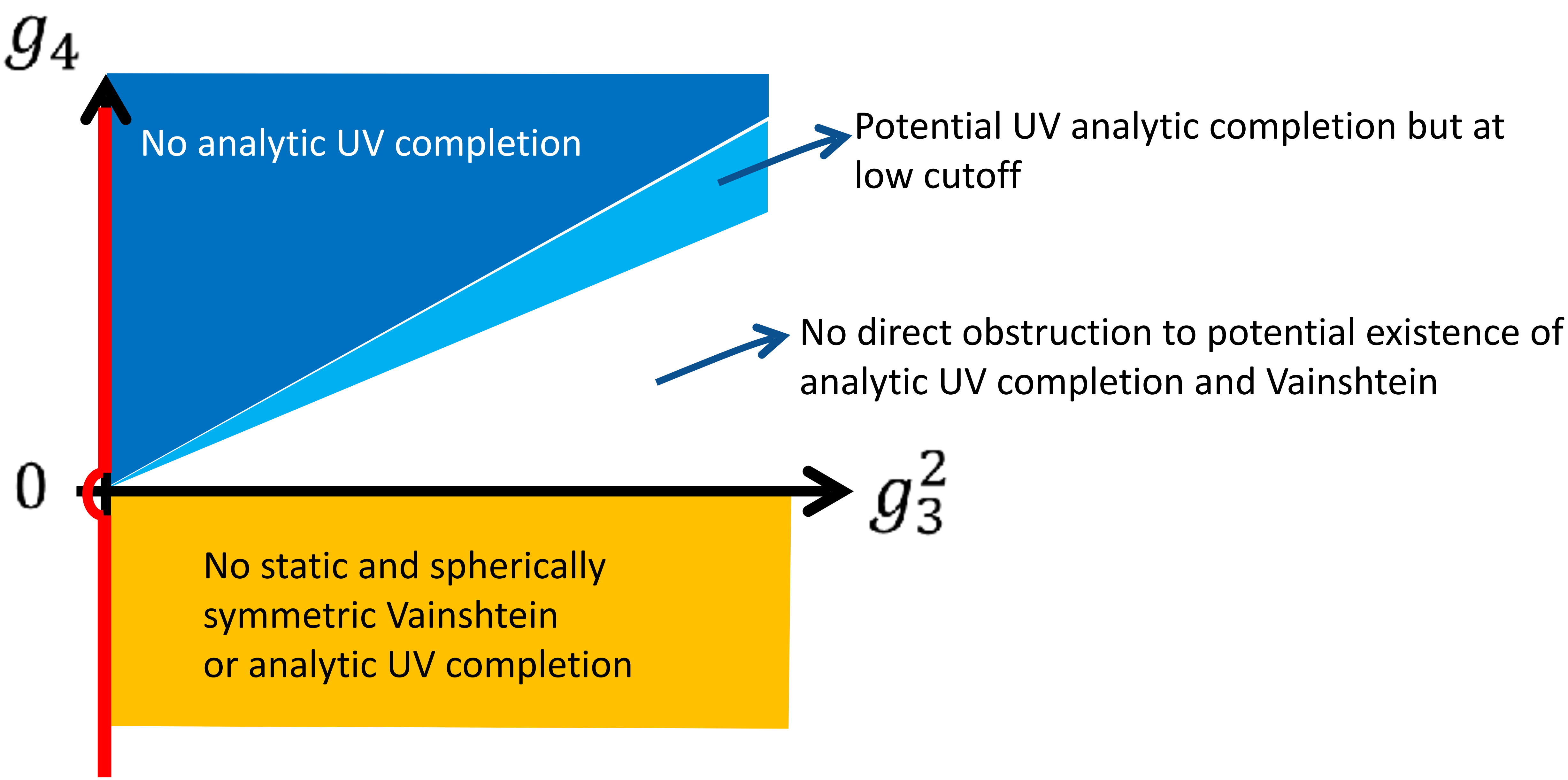
de Rham, Melville, AJT, Zhou, 1702.06134

Positivity Bounds for Massive Galileon AGAIN



de Rham, Melville, AJT, Zhou, 1702.08577

Positivity Bounds for Massive Galileon AGAIN



de Rham, Melville, AJT, Zhou, 1702.08577

All orders Positivity Bounds

Define recursively:
$$B^{2N,M}(t) = \frac{1}{M!}\partial_s^{2N}\left(\partial_t - \frac{1}{2}\partial_s\right)^M B(s,t)\Big|_{s=2m^2-t/2}$$

$$Y^{(2N,M)}(t) = \sum_{r=0}^{M/2} c_r B^{(2N+2r,M-2r)}(t) + \frac{1}{\mathcal{M}^2}\sum_{k \text{ even}}^{(M-1)/2} \beta_{k,N} Y^{(2(N+k),M-2k-1)}(t) > 0$$

$$N > 1$$

$$\mathcal{M}^2 = \mu_{\rm min} + t/2 - 2m^2$$

General Bound is:
$$Y^{2N,M}(t) > 0\,, \quad 0 \le t < 4m^2$$

Punch Line: There is a bound on every s and t derivative of the scattering amplitude

de Rham, Melville, AJT, Zhou, 1702.06134

Implications

- An extension to the positivity bound exist for arbitrary number of s and t derivatives (infinite number of bounds)

- Any interacting EFT (which is not already UV finite) needs to involve higher derivatives operators (at some scale) to have any chance of standard UV completeness
  (even allowing fine-tuning)

Eg. $$\mathcal{L} \supset \mathcal{L}_{\rm Gal} + \frac{\alpha}{\Lambda^5}(\partial_\mu\partial_\nu\phi)^3 + \frac{\beta}{\Lambda^8}(\partial_\mu\partial_\nu\phi)^4$$

Unitarity bounds ⟶ $3g_3\alpha + 2\beta > 0$

de Rham, Melville, AJT, Zhou, 1702.06134 & 1702.08577

S-matrix Positivity Constraints

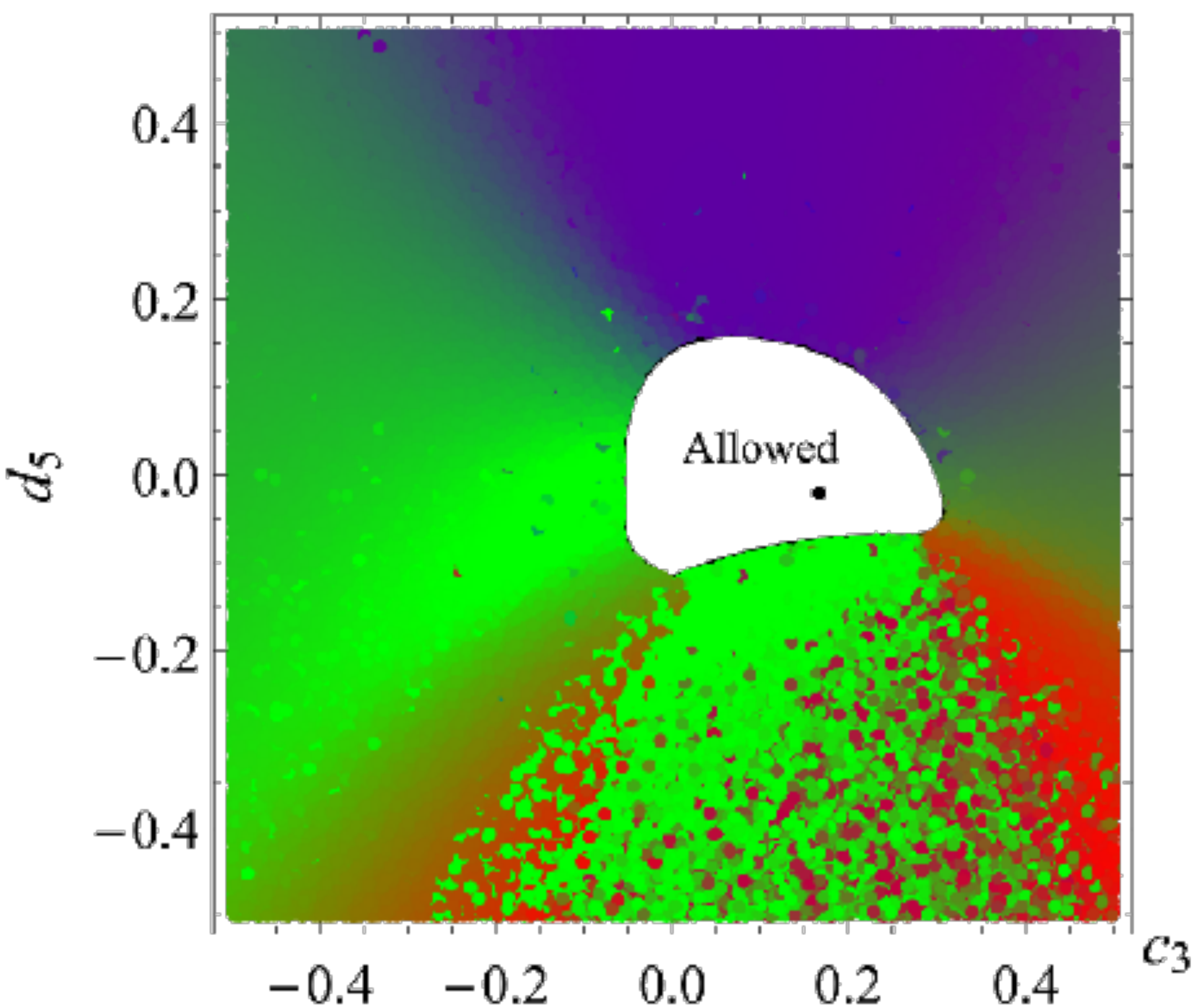
Signs of UV completion

What about gravity?

Spinning particles in Forward Limit

Constrains on the mass parameters in massive gravity

Eg. see Cheung & Remmen, JHEP 1604 (2016) for massive gravity



And for Proca field, see Bonifacio, Hinterbichler & Rosen PRD94 (2016)

$$
\begin{aligned}
\mathcal{L} = & -\frac{1}{4}F_{\mu\nu}^2 + m^2 A^2 + \lambda_3 \frac{m}{M_p} A^\mu A^\nu \partial_\mu A_\nu \\
& + \lambda_4 \frac{1}{M_p^2} A^2 (\partial_\mu A^\mu \partial_\nu A^\nu - \partial_\mu A^\nu \partial_\nu A^\mu) \\
& + \lambda_5 \frac{1}{M_p^2} A_\mu A_\nu \partial_\lambda A^\mu (\partial^\lambda A^\nu - \partial^\nu A^\lambda) \\
& + \lambda_6 \frac{1}{M_p^2} A^2 \partial_\mu A_\nu (\partial^\mu A^\nu - \partial^\nu A^\mu) \\
& + \lambda_7 \frac{1}{M_p^2} A_\mu A_\nu (\partial^\mu A_\lambda \partial^\nu A^\lambda - \partial_\lambda A^\mu \partial^\lambda A^\nu) \\
& + \mathcal{O}\left(A^5\right),
\end{aligned}
$$

Obstruction to UV completion unless higher derivative operators are included

$$
\frac{1}{m^2 M_p^2}\left[c_1 F^\mu{}_\nu F^\nu{}_\rho F^\rho{}_\sigma F^\sigma{}_\mu + c_2 (F_{\mu\nu}F^{\mu\nu})^2\right]
$$

both in the forward scattering limit

Can we extend these results away from the forward scattering limit?

Very non-trivial because of 2 things

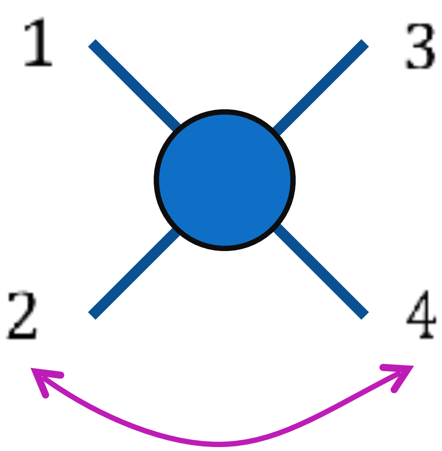
1. Crossing Symmetry is very complicated for general spin scattering
2. Discontinuity along left hand branch cut is **no longer positive** definite
3. Scattering amplitude for general spin have a significantly more complicated analytic structure

Crossing Symmetry for Spins

Away from the forward scattering limit $t \neq 0$,
the $s \leftrightarrow u = 4m^2 - s - t$ crossing symmetry is highly non-trivial

s-channel	u-channel
$A + B \to C + D$	$A + \bar{D} \to C + \bar{B}$

A definite helicity mode transforms non-trivially under crossing



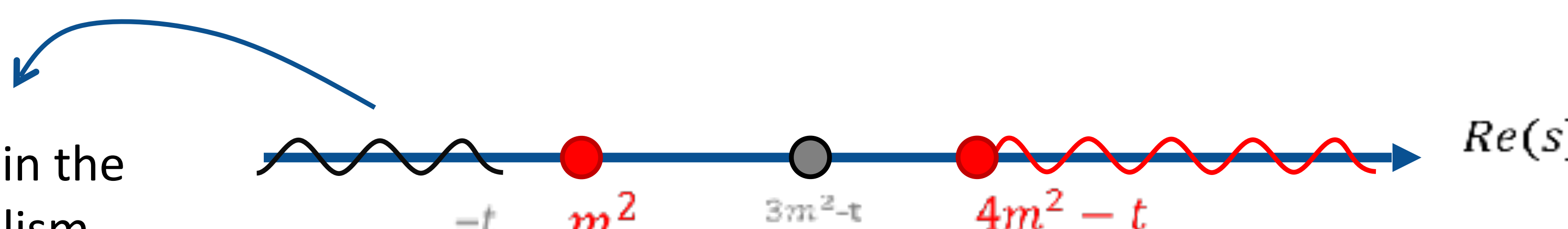
$$\overset{s}{\mathcal{H}}_{\lambda_1\lambda_2\mu_1\mu_2}(s,t) = (-1)^\sigma e^{i\pi(\mu_1-\lambda_1)} \sum_{\lambda_1'\lambda_2'\mu_1'\mu_2'} d^{S_1}_{\lambda_1'\lambda_1}(\pi-\chi) d^{S_2}_{\lambda_2'\lambda_2}(\chi) d^{S_1}_{\mu_1'\mu_1}(\chi-\pi) d^{S_2}_{\mu_2'\mu_2}(-\chi) \overset{u}{\mathcal{H}}_{\lambda_1\mu_2,\mu_1\lambda_2}(u,t)$$

d: Wigner matrices

$$\sin\chi = \frac{-2m\sqrt{t}}{\sqrt{(s-4m^2)(u-4m^2)}}$$

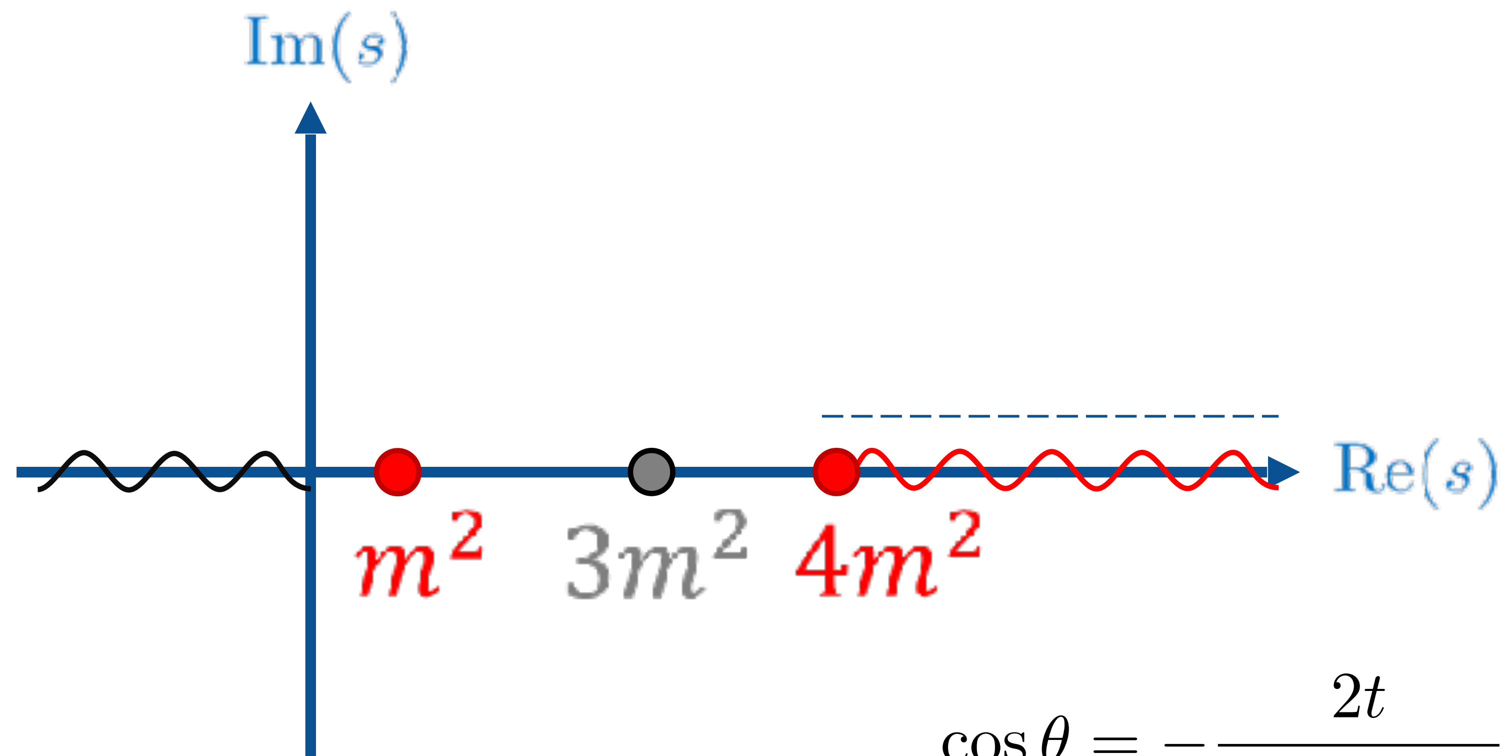
Results from change of c.o.m. frame

No obvious positivity properties in the 2nd branch cut in helicity formalism



Analyticity for Spins

In addition to usual scalar poles and branch cuts we have ……..



$$\cos\theta = -\frac{2t}{(s-4m^2)}$$

1. Kinematic (unphysical) poles at $s = 4m^2$
2. $\sqrt{stu}$ branch cuts
3. For Boson-Fermion scattering $\sqrt{-su}$ branch cuts

Origin: non-analyticities of polarization vectors/spinors

Both Problems Solvable!

Problem 2 Solution

1. Kinematic (unphysical) poles at $s = 4m^2$
2. $\sqrt{stu}$ branch cuts
3. For Boson-Fermion scattering $\sqrt{-su}$ branch cuts

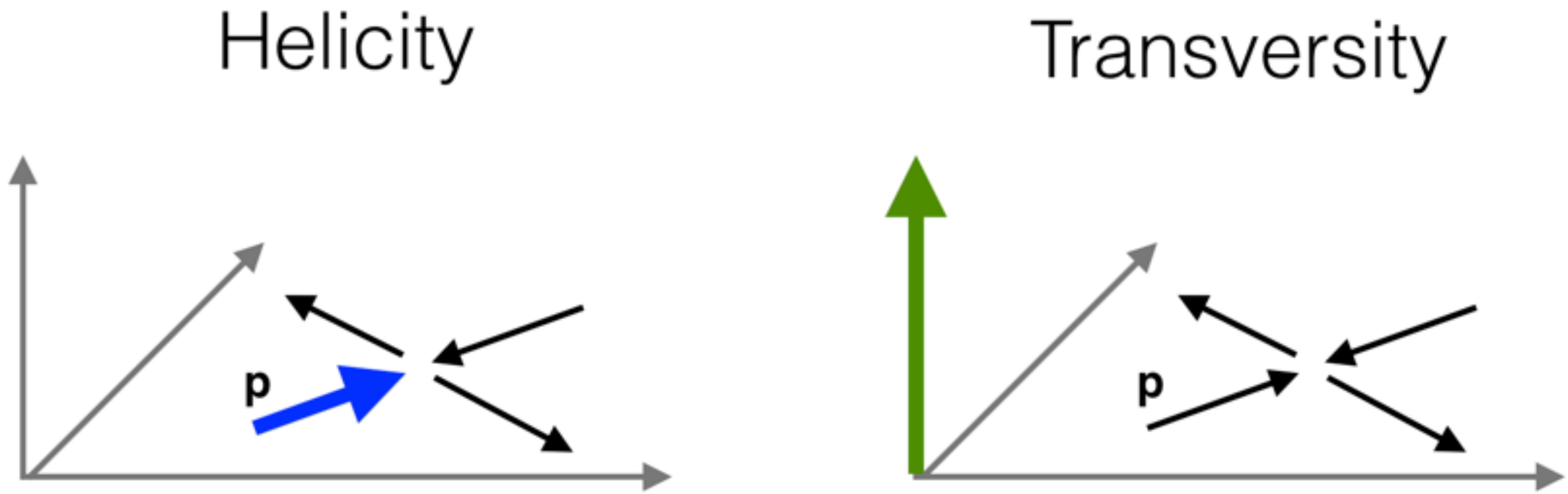
All kinematic singularities are factorizable or removable by taking special linear combinations of helicity amplitudes (known historically as regularized helicity amplitudes)

RESULT: It is possible to find combinations of general helicity scattering amplitudes that have the EXACT same analytic structure as scalar scattering amplitudes

Transversity Formalism

Problem 1 Solution

Need to work instead in the transversity formalism
(i.e. spin projections orthogonal to the scattering plane)
Only makes sense for $2 \to 2$



$$|\mathbf{p}_t, S, \tau\rangle = \sum_\lambda u^S_{\lambda\tau} \, |\mathbf{p}, S, \lambda\rangle$$

Change of Basis

Crossing now 1-1 betwen s and u channel:

$$T^s_{\tau_1\tau_2\tau_3\tau_4}(s,t,u) = e^{-i\sum_i \tau_i \chi} T^u_{-\tau_1-\tau_4-\tau_3-\tau_2}(u,t,s)$$

Problem 2 Solution

Transversity Formalism

Discontinuity along left hand branch cut for transversity amplitudes is now positive definite!!!!! (not obvious but true)

Can derive Dispersion Relations for any spin same analyticity and positivity properties as scalar theories

Derived an infinite number of positivity bounds valid for any spin, applicable to any EFT

Now regularized transversity scattering amplitude

$$B^{2N,M}(t) = \frac{1}{M!}\partial_s^{2N}\left(\partial_t - \frac{1}{2}\partial_s\right)^M B(s,t)\Big|_{s=2m^2-t/2}$$

Only difference is number of subtraction

$$Y^{(2N,M)}(t) = \sum_{r=0}^{M/2} c_r B^{(2N+2r,M-2r)}(t) + \frac{1}{\mathcal{M}^2}\sum_{k\ \text{even}}^{(M-1)/2} \beta_{k,N} Y^{(2(N+k),M-2k-1)}(t) > 0$$

$$N \geq (2 + 2S_A + 2S_B + \xi)/2$$

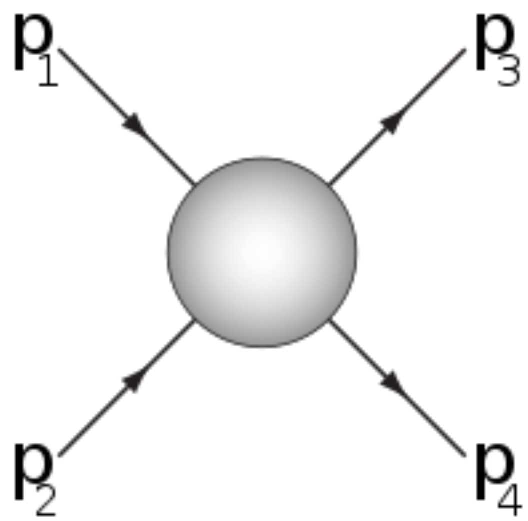
de Rham, Melville, AJT, Zhou

Summary

For the 2-2 scattering amplitude for four particles of different masses and spins (bosons AND fermions)

S_A, m_A

S_B, m_B



S_C, m_C

S_D, m_D

We have been able to derive an infinite number of conditions on s and t derivatives of **transversity scattering amplitude** which impose positive properties on combinations of coefficients in the EFT

Largest set of conditions we know that determine whether a given EFT admits a local UV completion

Currently applying to Massive Gravity and many other cosmological theories