Anisotropic Interior Solutions in Hořava Gravity and Einstein-Æther Theory

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based on
DV and S. Carloni, arXiv:1706.06608 [gr-qc]

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Lovelock's Theorem

- *In 4 dimensions the most general 2-covariant divergence-free tensor, which is constructed solely from the metric $g_{\mu\nu}$ and its derivatives up to second differential order, is the Einstein tensor $G_{\mu\nu}$ plus a cosmological constant (CC) term $\Lambda g_{\mu\nu}$.*
- *This result suggests a natural route to Einstein's equations in vacuum:*

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\Lambda g_{\mu\nu}.$$

The Action

With the additional requirement that the eqs. for the gravitational field and the matter fields be derived by a diff.-invariant action, Lovelock's theorem singles out in 4 dimensions the action of GR with a CC term:

$$S_{GR} = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} \left(K_{ij} K^{ij} - K^2 + \mathcal{R} - 2\Lambda \right) + S_M[g_{\mu\nu}, \psi_M] \, .$$

The variation with respect to the metric gives rise to the field equations of GR in presence of matter:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N \, T_{\mu\nu} \, ,$$

where

$$T_{\mu\nu} \equiv \frac{-2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} \, .$$

The Problems

- **GR is not a Renormalizable Theory**
  Renormalization at one-loop demands that GR should be supplemented by higher-order curvature terms, such as R^2 and $R_{\alpha\beta\sigma\gamma}R^{\alpha\beta\sigma\gamma}$ (Utiyama and De Witt '62). However such theories are not viable as they contain ghost degrees of freedom (Stelle '77).

- **The Cosmological Constant**
  The observed cosmological value for the CC is smaller than the value derived from particle physics at best by 60 orders of magnitude.

- **The Dark Side of the Universe**
  The most recent data tell us that about the 95% of the current Universe is made by unknown components, Dark Energy and Dark Matter.

Beyond General Relativity

- **Higher-Dimensional Spacetimes**
  One can expect that for any higher-dimensional theory, a 4-dimensional effective field theory can be derived in the IR, that is what we are interested in.

- **Adding Extra Fields (or Higher-Order Derivatives)**
  One can take into account the possibility to modify the gravitational action by considering more degrees of freedom. This can be achieved by adding extra dynamical fields or equivalently considering theories with higher-order derivatives.

- **Giving Up Diffeomorphism Invariance**
  Lorentz symmetry breaking can lead to a modification of the graviton propagator in the UV, thus rendering the theory renormalizable.

Hořava's Proposal

- *In 2009, Hořava proposed an UV completion to GR modifying the graviton propagator by adding to the gravitational action higher-order spatial derivatives without adding higher-order time derivatives.*
- *This prescription requires a splitting of spacetime into space and time and leads to Lorentz violations.*
- *Lorentz violations in the IR are requested to stay below current experimental constraints.*

P. Hořava, JHEP **0903**, *020 (2009)*
P. Hořava, PRD **79**, *084008 (2009)*

Foundations of the Theory

The theory is constructed using the full ADM metric:

$$ds^2 = N^2 dt^2 - h_{ij}(dx^i + N^i dt)(dx^j + N^j dt),$$

and it is invariant under foliation-preserving diffeomorphysms, i.e.,

$$t \to \tilde{t}(t), \qquad x^i \to \tilde{x}^i(t, x^j).$$

The most general action is:

$$S_H = S_K + S_V.$$

The Kinetic Term

$$S_K = \frac{1}{16\pi G_H} \int dt d^3x \sqrt{h} N \left(K_{ij}K^{ij} - \lambda K^2\right).$$

Foundations of the Theory

The Potential Term

$$S_V = \frac{1}{16\pi G_H} \int dt d^3x \sqrt{h} N \left[L_2 + \frac{1}{M_*^2} L_4 + \frac{1}{M_*^4} L_6 \right].$$

- *Power-counting renormalizability requires as a minimal prescription at least 6th-order spatial derivatives in V.*
- *The most general potential V with operators up to 6th-order in derivatives, contains tens of terms $\sim \mathcal{O}(10^2)$.*
- *The theory propagates both a spin-2 and a spin-0 graviton.*

Foundations of the Theory

In the most general theory some of the terms that one can consider in the potential are:

$$L_2 = \xi\mathcal{R},\ \eta a_i a^i,$$

$$L_4 = \mathcal{R}^2,\ \mathcal{R}_{ij}\mathcal{R}^{ij},\ \mathcal{R}\nabla_i a^i,\ a_i\Delta a^i,\ \left(a_i a^i\right)^2,\ a_i a_j \mathcal{R}^{ij},\ \dots,$$

$$L_6 = (\nabla_i \mathcal{R}_{jk})^2,\ (\nabla_i\mathcal{R})^2,\ \Delta\mathcal{R}\nabla_i a^i,\ a_i\Delta^2 a^i,\ \left(a_i a^i\right)^3,\ \dots,$$

where $a_i = \partial_i lnN$.

D. Blas, O. Pujolas & S. Sibiryakov, PRL **104**, *181302 (2010)*

Hořava Gravity Constraints

- *BBN:*

$$|G_{cosmo}/G_N - 1| < 0.38 \; (99.7\% \, \mathrm{C.L.}) \, ;$$

S. M. Carroll and E. A. Lim, PRD **70***, 123525 (2004)*

- *PPN:*

$$\alpha_1 < 3.0 \cdot 10^{-4} \, , \qquad \alpha_2 < 7.0 \cdot 10^{-7} \; (99.7\% \, \mathrm{C.L.}) \, ;$$

C. M. Will, LRR **17***, 4 (2014)*

- *Cosmological scales:*

$$|G_{cosmo}/G_N - 1| < 6.1 \times 10^{-5} \; (99.7\% \, \mathrm{C.L.}) \, ;$$

N. Frusciante, M. Raveri, DV, B. Hu, A. Silvestri, PDU **13***, 7 (2016)*

- *Astrophysical scales (Binary pulsar)*

K. Yagi et al., PRL **112***, 161101 (2014)*

Æ-Theory

Æ-theory is essentially GR coupled to a timelike, unit-norm vector field, u_α, called the æther. It cannot vanish and thus breaks boost invariance by defining (locally) a preferred frame.
Æ-theory is defined by the action:

$$S_{æ} = \frac{1}{16\pi G_{æ}} \int d^4x \sqrt{-g} \left(-R - 2\Lambda - M^{\alpha\beta\mu\nu} \nabla_\alpha u_\mu \nabla_\beta u_\nu\right) ,$$

where

$$M^{\alpha\beta\mu\nu} = c_1 g^{\alpha\beta} g^{\mu\nu} + c_2 g^{\alpha\mu} g^{\beta\nu} + c_3 g^{\alpha\nu} g^{\beta\mu} + c_4 u^\alpha u^\beta g^{\mu\nu} ,$$

and the æther is assumed to satisfy the unit-constraint:

$$g_{\mu\nu} u^\mu u^\nu = 1 .$$

Hořava Gravity & Æ-Theory

The IR part (L_2) *of Hořava gravity can be formulated in a covariant fashion, and it then becomes equivalent to a restricted version of Æ-theory. Restricting the æther to be orthogonal to the constant-*T *hypersurfaces*, i.e.,

$$u_\alpha = \frac{\partial_\alpha T}{\sqrt{g^{\mu\nu}\partial_\mu T \partial_\nu T}} \,,$$

and choosing T *as the time coordinate, the action of Æ-theory reduces to that of Hořava gravity in the IR, with the correspondence of parameters:*

$$\frac{G_H}{G_{æ}} = \xi = \frac{1}{1 - c_{13}} \,, \qquad \frac{\lambda}{\xi} = 1 + c_2 \,, \qquad \frac{\eta}{\xi} = c_{14} \,,$$

where $c_{ij} = c_i + c_j$.

T. Jacobson, PRD **81**, *101502 (2010)*

Anisotropic Stars in Hořava Gravity

Let us now consider a spherically symmetric spacetime where the metric can be written as:

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right),$$

with the addition of an anisotropic fluid with stress-energy tensor

$$T_\mu{}^\nu = diag\big(\rho, -p_r, -p_t, -p_t\big),$$

where ρ *is the density of the fluid,* p_r *and* p_t *are the radial and transversal pressure respectively. Furthermore let us take into account, for simplicity, a static æther* u^μ

$$u^\mu = \left(\frac{1}{\sqrt{A}}, 0, 0, 0\right),$$

which is always hypersurface-orthogonal.

Field Equations

Equation 0-0

$$\frac{\eta}{\xi}\left[-\frac{A''(r)}{2A(r)^2B(r)}+\frac{A'(r)B'(r)}{4A(r)^2B(r)^2}+\frac{3A'(r)^2}{8A(r)^3B(r)}-\frac{A'(r)}{rA(r)^2B(r)}\right]+ \\ +\frac{B'(r)}{rA(r)B(r)^2}-\frac{1}{r^2A(r)B(r)}+\frac{1}{r^2A(r)}=\frac{8\pi G_{æ}\rho(r)}{A(r)}\,,$$

Equation 1-1

$$\frac{\eta A'(r)^2}{8\xi A(r)^2B(r)^2}+\frac{A'(r)}{rA(r)B(r)^2}+\frac{1}{r^2B(r)^2}-\frac{1}{r^2B(r)}=\frac{8\pi G_{æ}p_r(r)}{B(r)}\,,$$

Field Equations

Equation 2-2

$$-\frac{\eta A'(r)^2}{8\xi r^2 A(r)^2 B(r)} + \frac{A''(r)}{2r^2 A(r)B(r)} - \frac{A'(r)B'(r)}{4r^2 A(r)B(r)^2} + \frac{A'(r)}{2r^3 A(r)B(r)} + \\ -\frac{A'(r)^2}{4r^2 A(r)^2 B(r)} - \frac{B'(r)}{2r^3 B(r)^2} = \frac{8\pi G_{æ} p_t(r)}{r^2},$$

Conservation Equation

$$p_r'(r) + [\rho(r) + p_r(r)]\frac{A'(r)}{2A(r)} = \frac{2}{r}[p_t(r) - p_r(r)].$$

A Reconstruction Algorithm

The 3 independent field equations can be written as:

$$\begin{aligned}\rho(r) &= \frac{1}{8\pi G_{æ}}\left[-\frac{\eta A''(r)}{2\xi A(r)B(r)}+\frac{\eta A'(r)B'(r)}{4\xi A(r)B(r)^2}+\frac{3\eta A'(r)^2}{8\xi A(r)^2B(r)}-\frac{\eta A'(r)}{\xi rA(r)B(r)}\right.\\ &\quad\left.+\frac{B'(r)}{rB(r)^2}-\frac{1}{r^2B(r)}+\frac{1}{r^2}\right],\end{aligned}$$

$$p_r(r)=\frac{1}{8\pi G_{æ}}\left[\frac{\eta A'(r)^2}{8\xi A(r)^2B(r)}+\frac{A'(r)}{rA(r)B(r)}+\frac{1}{r^2B(r)}-\frac{1}{r^2}\right],$$

$$\begin{aligned}p_t(r) &= \frac{1}{8\pi G_{æ}}\left[\frac{A''(r)}{2A(r)B(r)}-\frac{A'(r)B'(r)}{4A(r)B(r)^2}+\frac{A'(r)}{2rA(r)B(r)}-\frac{\eta A'(r)^2}{8\xi A(r)^2B(r)}\right.\\ &\quad\left.-\frac{A'(r)^2}{4A(r)^2B(r)}-\frac{B'(r)}{2rB(r)^2}\right].\end{aligned}$$

Physical Conditions

- *The thermodinamical quantities* ρ, p_r *and* p_t *should be finite and positive inside the star;*
- *The gradients* $\frac{d\rho}{dr}$, $\frac{dp_r}{dr}$ *and* $\frac{dp_t}{dr}$ *should be negative;*
- *The anisotropy should be zero in the centre:* $p_r(r=0) = p_t(r=0)$;
- *Stability at the surface and junction to exterior vacuum:* $p_r(r=R) = 0$;
- *Subluminal propagation speeds:* $0 < c_r^2 = \frac{dp_r}{d\rho} < 1$, $0 < c_t^2 = \frac{dp_t}{d\rho} < 1$;
- *Absence of curvature singularities:* R, $R_{\mu\nu}R^{\mu\nu}$ *and* $R_{\mu\nu\gamma\sigma}R^{\mu\nu\gamma\sigma}$ *are finite.*

A Specific Example

Let us consider the following choice of the metric coefficients:

$$A(r) = D_1 + D_2 r^2 \,, \quad B(r) = \frac{D_3 + D_4 r^2}{D_3 + D_5 r^2 + D_6 r^4} \,,$$

which are qualitatively the same as the ones characterizing the Tolmann IV solution in GR for an isotropic fluid.

- *The choice of $A(r)$ is motivated by the fact that it reproduces the Newtonian potential for a fluid sphere of constant density while $B(r)$ has been chosen for convenience in the calculations.*
- *Notice that the choice of the constants in $B(r)$ guarantees the avoidance of a divergence in the centre for the curvature invariants, $\rho(r)$ and $p_r(r)$.*

A Specific Example

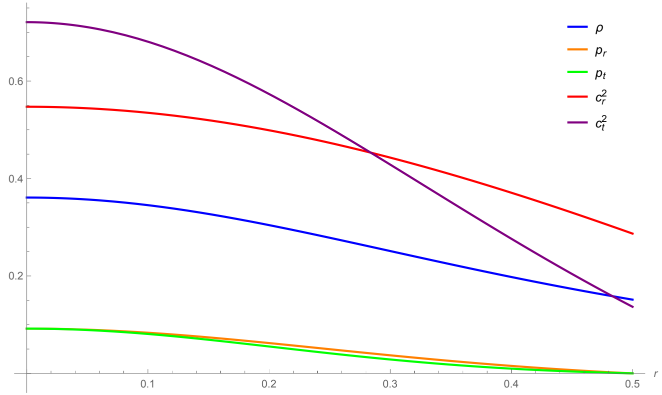

Figure: It is shown the behaviour of the density ρ (blue line), the radial pressure p_r (orange line), the tangential pressure p_t (green line), the squared radial c_r^2 (red line) and the tangential c_t^2 (purple line) speeds of sound as functions of the radius for a wide choice of the constants in units $G_N = 1$.

Conclusions and Future Perspectives

- *Hořava gravity is a quantum renormalizable theory, very well tested at astrophysical and cosmological scales.*
- *It is very hard to find exact analytical solutions because of the intrinsic highly non-linear structure of its field equations.*
- *Considering anisotropic fluids and a static æther in spherical symmetry it is possible to find a double-infinity of interior solutions.*
- *The solutions have to satisfy many physical requirements in order to represent realistic stellar objects, as in the specific case we studied.*
- *A deep understanding of the phase space of solutions is needed, as well as a comprehensive study of the deviations obtained in the case of a non-static æther.*

Lorentz Violations as Field Theory Regulator

We take a scalar field theory whose action has the following form:

$$S_\phi = \int dt dx^d \left[\dot{\phi}^2 - \sum_{m=1}^{z} a_m \phi (-\Delta)^m \phi + \sum_{n=1}^{N} b_n \phi^n \right] .$$

Space and time coordinates have the following dimensions in units of the spatial momentum p:

$$[dt] = [p]^{-z} , \qquad [dx] = [p]^{-1} .$$

A theory is said to be "power-counting renormalizable" if all of its interaction terms scale like momentum to some non-positive power, as in this case Feynman diagrams are expected to be convergent or have at most a logarithmic divergence.

$\Rightarrow$ *$z \geq d$, for $d = 3$ at least 6th-order spatial derivatives in the action.*

Dispersion Relations and Propagators

In general the dispersion relation one gets for such a Lorentz-violating field theory is of the following form:

$$\omega^2 = m^2 + a_1 p^2 + \sum_{n=2}^{z} a_n \frac{p^{2n}}{P^{2n-2}} ,$$

where P is the momentum-scale suppressing the higher-order operators. The resulting Quantum Field Theory (QFT) propagator is then:

$$G(\omega, p) = \frac{1}{\omega^2 - \left[m^2 + a_1 p^2 + \sum_{n=2}^{z} a_n p^{2n}/P^{2n-2}\right]} .$$

The very rapid fall-off as $p \to \infty$ improves the behaviour of the integrals one encounters in the QFT Feynman diagram calculations.