Weyl metrics and wormholes

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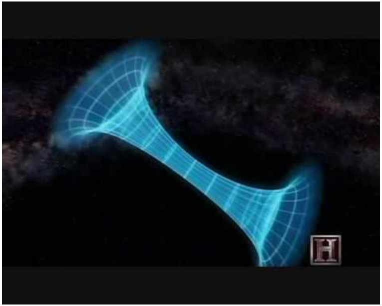
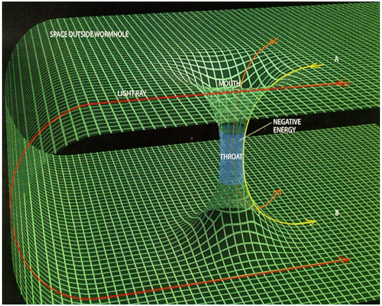
- Introduction
- I. Gravitating scalar field
- II. Vacuum wormholes
- III. Zero mass limit of Kerr spacetime is a wormhole
- IV. Wormholes in massive bigravity

Wormholes – spacetime bridges





Wormholes interpolate between different universes or between different parts of the same universe. Could supposedly be used for interstellar and time travels.

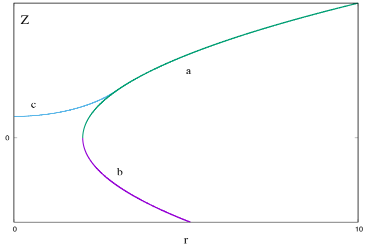
Some history

/Flamm, 1916/ – The spatial part of the Schwarzschild geometry contains a throat

$$dl^2 = \frac{dr^2}{1 - 2M/r} + r^2 d\Omega^2 = dr^2 + r^2 d\Omega^2 + dZ^2$$

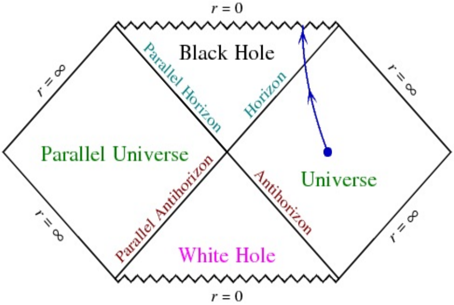
where $$dZ^2 = \frac{dr^2}{r/(2M) - 1} \qquad \Rightarrow \qquad r = r(Z) \equiv 2M + \frac{Z^2}{8M}.$$

Flamm assumed $Z > 0$. Einstein-Rosen considered $Z \in (-\infty, +\infty)$



Some history

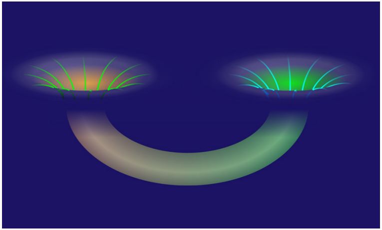
- /Einstein-Rosen, 1935/ – Schwarzschild black hole has two exterior regions connected by a bridge. The ER bridge is spacelike and cannot be traversed by classical objects.



- /Maldacena-Susskind, 2013/ – the ER bridge may connect quantum particles to produce quantum entanglement and the Einstein-Pololsky-Rosen (EPR) effect, hence ER=EPR.

Some history

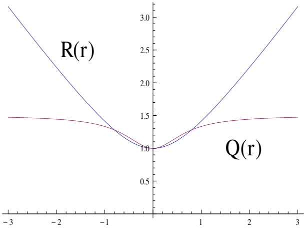
- /Wheeler, 1957/ wormholes may provide geometric models of elementary particles – handles of space trapping inside an electric flux.



- /Misner, 1960/ Wormholes can describe initial data for the Einstein equations. The time evolution of these data corresponds to the black hole collisions of the type observed in the GW events like GW150914.
- /Morris, Thorn, Yurtsever, 1988/ wormholes traversable by classical object may be supported by vacuum polarisation.

Can wormholes be solutions of Einstein equations ?

$$ds^2 = -Q^2(r)dt^2 + dr^2 + R^2(r)(d\vartheta^2 + \sin^2\vartheta d\varphi^2),$$



$G_{\mu\nu} = T_{\mu\nu} \Rightarrow$ energy $\rho = -T_0^0$ and pressure $p = T_r^r$ fulfill

$$\rho + p = -2\frac{R''}{R} < 0, \qquad p = -\frac{1}{R^2} < 0.$$

$\Rightarrow$ the Null Energy Condition (NEC) must be violated.
$/\, T_{\mu\nu} v^\mu v^\nu = R_{\mu\nu} v^\mu v^\nu \geq 0$ for any null $v^\mu /$

NEC violation

The general case without symmetry $\Rightarrow$ topological censorship: compact two-surface of minimal area can exit if only NEC is violated /Friedman, Schleich, Witt, 1993/ $\Rightarrow$
traversable wormholes are possible if only energy is negative.

This may be, for example, due to

- vacuum polarization
- exotic matter: phantom fields, etc.

Wormholes may exist in alternative gravity models:

- Gauss-Bonnet brainworld, etc.
- theories with non-minimally coupled fields (Horndeski)
- massive (bi)gravity

Best known example – phantom-supported wormhole

$$\mathcal{L} = R + 2(\partial\psi)^2$$

Bronnikov-Ellis wormhole:

$$ds^2 = -dt^2 + dr^2 + (r^2 + a^2)(d\vartheta^2 + \sin^2\vartheta d\varphi)^2, \quad \psi = \arctan\left(\frac{r}{a}\right);$$

$$r \in (-\infty, \infty)$$

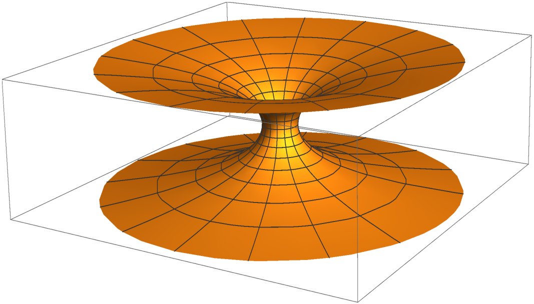


Figure: *Isometric embedding of the equatorial section of the BE wormhole to the 3-dimensional Euclidean space*

Phantom wormholes from the Kaluza-Klein viewpoint

$ds^2 = -dt^2 + dl^2$ where $dl^2 = \gamma_{ik} dx^i dx^k$ fulfills

$$\overset{(3)}{R}_{ik}(\gamma) = -2\partial_i\psi\partial_k\psi, \qquad \Delta\psi = 0 \qquad\qquad (*)$$

The simplest solution is the Bronnikov-Ellis wormhole, more general solutions – superposition of wormholes.
Eqs.($*$) coincide with the vacuum Einstein equations for 5-metric

$$ds_5^2 = \cos(2\psi)[-dx_0^2 + dx_4^2] + 2\sin(2\psi)dx_0 dx_4 + dl^2$$

$\Rightarrow$ wormholes can be interpreted as 5-geometries without invoking phantom fields /Clement/.

4D wormholes without phantom field

Write a phantom field solution in the Weyl form,

$$ds^2 = -e^{2U}dt^2 + e^{2U}\left(e^{2k}(d\rho^2 + dz^2) + \rho^2 d\varphi^2\right), \qquad \psi = \psi$$

A new solution of the same form is obtained by swapping

$$U \leftrightarrow \psi, \qquad k \to -k$$

hence by setting

$$U_{\rm new} = \psi, \qquad \psi_{\rm new} = U, \qquad k_{\rm new} = -k.$$

For the BE wormhole $U = 0$, hence the new solution is vacuum,

$$U_{\rm new} = \psi, \qquad \psi_{\rm new} = 0, \qquad k_{\rm new} = -k$$

but the topology with two asymptotic regions remains – wormhole.
The negative energy is hidden in the singularity.

I. Gravitating scalar field

Ordinary vs. phantom scalar

$$\mathcal{L} = R - 2\epsilon\,(\partial\Phi)^2$$

- $\epsilon = +1$: ordinary scalar $\Phi = \phi$
- $\epsilon = -1$: phantom $\Phi = \psi$.

Static system

$$ds^2 = -e^{2U}dt^2 + e^{-2U}\gamma_{ik}dx^i dx^k,$$

the field equations are

$$\begin{aligned} \frac{1}{2}\overset{(3)}{R}_{ik} &= \partial_i U \partial_k U + \epsilon\, \partial_i \Phi \partial_k \Phi\,, \\ \Delta U &= 0, \qquad \Delta\Phi = 0\,. \end{aligned}$$

Rotational symmetry for real scalar, $\Phi \equiv \phi$,

$$\epsilon = +1: \qquad \begin{aligned} U &\to U\cos\alpha + \phi\sin\alpha\,, \\ \phi &\to \phi\cos\alpha - U\sin\alpha\,, \qquad \gamma_{ik} \to \gamma_{ik}\,, \end{aligned}$$

Boost symmetry for phantom, $\Phi \equiv \psi$

$$\epsilon = -1: \qquad \begin{aligned} U &\to U\cosh\alpha + \psi\sinh\alpha\,, \\ \psi &\to \psi\cosh\alpha + U\sinh\alpha\,, \qquad \gamma_{ik} \to \gamma_{ik}. \end{aligned}$$

Solutions from Schwarzschild

$$ds^2 = -\frac{r-m}{r+m}\,dt^2 + \frac{r+m}{r-m}\,dr^2 + (r+m)^2 d\Omega^2, \qquad \Phi = 0.$$

Rotations with $\cos\alpha = 1/s$ give Fisher-Janis-Robinson-Winicour solutions for ordinary scalar

$$ds^2 = -\left(\frac{r-m}{r+m}\right)^{1/s} dt^2 + \left(\frac{r+m}{x-m}\right)^{1/s} \left[dx^2 + (r^2-m^2)d\Omega^2\right],$$

$$\phi = \pm\frac{\sqrt{s^2-1}}{2s}\ln\left(\frac{r-m}{r+m}\right), \qquad |s| \geq 1,$$

Boosts with $\cosh\alpha = 1/s$ give solutions for phantom

$$ds^2 = -\left(\frac{r-m}{r+m}\right)^{1/s} dt^2 + \left(\frac{r+m}{r-m}\right)^{1/s} \left[dx^2 + (r^2-m^2)d\Omega^2\right],$$

$$\psi = \pm\frac{\sqrt{1-s^2}}{2s}\ln\left(\frac{r-m}{r+m}\right) \qquad |s| \leq 1.$$

Upon analytic continuation

$$m \to i\mu, \qquad s \to -is.$$

one obtains

$$\begin{aligned} ds^2 &= -e^{2\Psi/s}dt^2 + e^{-2\Psi/s}[dx^2 + (x^2+a^2)d\Omega^2]\,, \\ \psi &= \pm\frac{\sqrt{s^2+1}}{s}\,\Psi, \qquad\qquad \Psi = \arctan\left(r/a\right). \end{aligned}$$

Taking $s \to \infty$ gives ultrastatic wormhole of Bronnikov-Ellis.

$$ds^2 = -dt^2 + dx^2 + (x^2+a^2)d\Omega^2\,, \qquad \psi = \Psi.$$

Axial symmetry

$$\mathcal{L} = R - 2\epsilon \left(\partial\Phi\right)^2$$

- $\epsilon = +1$: ordinary scalar $\Phi \equiv \phi$
- $\epsilon = -1$: phantom $\Phi \equiv \psi$

Weyl parametrization

$$ds^2 = -e^{2U}dt^2 + e^{-2U}\left\{e^{2k}\left(d\rho^2 + dz^2\right) + \rho^2 d\varphi^2\right\}$$

where U, k, Φ depend on ρ, z.

Field equations

$$
\begin{aligned}
\frac{\partial^2 U}{\partial \rho^2} &+ \frac{1}{\rho}\frac{\partial U}{\partial \rho} + \frac{\partial^2 U}{\partial z^2} = 0, \\
\frac{\partial^2 \Phi}{\partial \rho^2} &+ \frac{1}{\rho}\frac{\partial \Phi}{\partial \rho} + \frac{\partial^2 \Phi}{\partial z^2} = 0, \\
\frac{\partial k}{\partial \rho} &= \rho\left[\left(\frac{\partial U}{\partial \rho}\right)^2 - \left(\frac{\partial U}{\partial z}\right)^2 + \epsilon\left(\frac{\partial \Phi}{\partial \rho}\right)^2 - \epsilon\left(\frac{\partial \Phi}{\partial z}\right)^2\right], \\
\frac{\partial k}{\partial z} &= 2\rho\left[\frac{\partial U}{\partial \rho}\frac{\partial U}{\partial z} + \epsilon\,\frac{\partial \Phi}{\partial \rho}\frac{\partial \Phi}{\partial z}\right].
\end{aligned}
$$

Target space symmetries

preserve spherical symmetry:

rotations $$\begin{pmatrix} U \\ \phi \end{pmatrix} \to \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} U \\ \phi \end{pmatrix}, \qquad k \to k$$

boosts $$\begin{pmatrix} U \\ \psi \end{pmatrix} \to \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix} \begin{pmatrix} U \\ \psi \end{pmatrix}, \qquad k \to k$$

interchange BE wormhole and ring wormhole:

swap $$U \leftrightarrow \psi, \qquad k \to -k$$

do not intermix scalar field and gravity amplitudes:

scaling $$U \to \lambda U, \qquad k \to \lambda^2 k, \qquad \Phi \to \lambda\Phi$$

tachyon: $$U \to \ln\rho - U, \qquad k \to k - 2U + \ln\rho, \qquad \Phi \to \Phi$$

Acting with this on vacuum metrics yields new solutions.

Simplest vacuum Weyl metrics

One rod – Schwarzschild

$$ds^2 = -e^{2U}dt^2 + e^{-2U}\left\{e^{2k}\left(d\rho^2 + dz^2\right) + \rho^2 d\varphi^2\right\}$$

with

$$\begin{aligned} U(\rho, z) &= \frac{1}{2}\ln\left(\frac{R-m}{R+m}\right) = -\frac{1}{2}\int_{-m}^{m}\frac{d\zeta}{\sqrt{\rho^2 + (z-\zeta)^2}} \\ k(\rho, z) &= \frac{1}{2}\ln\left(\frac{R^2 - m^2}{R_+ R_-}\right) \end{aligned}$$

where

$$R = \frac{1}{2}(R_+ + R_-), \qquad R_\pm = \sqrt{\rho^2 + (z \pm m)^2}.$$

U is the Newtonian potential of a massive rod of length $2m$ along the z-axis with mass density $1/2$.

Two rods along z-axis

$$U = U_1 + U_2, \qquad k = k_1 + k_2 + k_{12}\,,$$

where (with $a = 1, 2$)

$$U_a = \frac{1}{2}\ln\left(\frac{R_a - m_a}{R_a + m_a}\right), \qquad k_a = \frac{1}{2}\ln\left(\frac{(R_a)^2 - (m_a)^2}{R_{a+}R_{a-}}\right),$$

$$k_{12} = \frac{1}{2}\ln\left(\frac{(R_{1+}R_{2-} + z_{1+}z_{2-} + \rho^2)(R_{1-}R_{2+} + z_{1-}z_{2+} + \rho^2)}{(R_{1+}R_{2+} + z_{1+}z_{2+} + \rho^2)(R_{1-}R_{2-} + z_{1-}z_{2-} + \rho^2)}\right),$$

with

$$z_{a\pm} = z - z_a \pm m_a, \quad R_{a\pm} = \sqrt{\rho^2 + (z_{a\pm})^2}, \quad R_a = \frac{1}{2}(R_{a+} + R_{a-})$$

$k \neq 0$ on the part of symmetry axis between the rods – strut /Israel and Khan 1964/ Similarly for many rods.

Point masses

$$U = -\frac{m}{R}\,, \qquad k = -\frac{m^2\rho^2}{2R^4}\,,$$

with $R = \sqrt{\rho^2 + z^2}$. For two masses $m_\pm$ at $z = \pm m$ one has

$$\begin{aligned} U &= -\frac{m_+}{R_+} - \frac{m_-}{R_-}, \\ k &= -\frac{m_+^2\rho^2}{2(R_+)^4} - \frac{m_-^2\rho^2}{2(R_-)^4} + \frac{m_+ m_-}{2m^2}\left(\frac{\rho^2 + z^2 - m^2}{R_+ R_-} - 1\right), \end{aligned}$$

with $R_\pm = \sqrt{\rho^2 + (z \pm m)^2}$ /Chazy, Curzon 1924/ .

Summary of part I

Applying the target space dualities to the vacuum Weyl metric gives all known and also many new static solutions.
For example, the Fisher-Janis-Robinson-Winicour solutions and their generalizations to axially symmetric case,

$$\begin{aligned} ds^2 &= -\left(\frac{x-m}{x+m}\right)^{\lambda/s} dt^2 + \left(\frac{x-m}{x+m}\right)^{-\lambda/s} dl^2, \\ dl^2 &= \left(\frac{r^2 - m^2\cos^2\vartheta}{x^2-m^2}\right)^{1-\lambda^2} \left[dr^2 + (r^2-m^2)d\vartheta^2\right] \\ &+ (x^2-m^2)\sin^2\vartheta d\varphi^2, \qquad \phi = \frac{\sqrt{s^2-1}}{s}\,\frac{\lambda}{2}\ln\left(\frac{x-m}{x+m}\right), \end{aligned}$$

BE wormholes and thei axially symmetric generalizations;
many other solutions

II. Vacuum wormholes

Starting point

Take the Schwarzschild metric

$$ds^2 = -e^{2U}dt^2 + e^{-2U}\{e^{2k}(d\rho^2 + dz^2) + \rho^2 d\varphi^2\}$$

$$U(\rho, z) = \frac{1}{2}\ln\left(\frac{R-m}{R+m}\right), \qquad k(\rho, z) = \frac{1}{2}\ln\left(\frac{R^2-m^2}{R_+R_-}\right)$$

$$R = \frac{1}{2}(R_+ + R_-), \qquad R_\pm = \sqrt{\rho^2 + (z \pm m)^2}.$$

and apply the scaling to get prolate vacuum metrics /Zipoy-Voorhees /

$$U \to \lambda U, \qquad k \to \lambda^2 k$$

Next step is the analytic continuation of parameters

$$m \to ia, \qquad \lambda \to i\sigma$$

Oblate vacuum metrics

$$ds^2 = -e^{2U}dt^2 + e^{-2U}\left\{e^{2k}\left(d\rho^2 + dz^2\right) + \rho^2 d\varphi^2\right\}$$

$$U = \sigma \arctan\left(\frac{X}{a}\right), \qquad k = \frac{\sigma^2}{2}\ln\left(\frac{X^2+Y^2}{X^2+a^2}\right)$$

$$X + iY = \sqrt{\rho^2 + (z+ia)^2}$$

The square root has a branching point at $\rho = a$, $z = 0 \Rightarrow$ there are two branches $\Rightarrow$ one need two Weyl charts (ρ_+, z_+) and (ρ_-, z_-) to cover the manifold $\Rightarrow$ double-sheeted topology with two asymptotically flat regions $\Rightarrow$ wormhole.

For $\sigma = 1$ the swap

$$U \leftrightarrow \psi, \qquad k \rightarrow -k$$

gives the Bronnikov-Ellis wormhole.

Wormhole topology

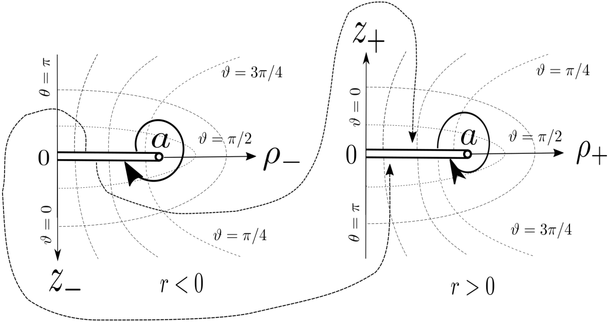


Figure: The two Weyl charts are glued to each other through the cuts.

Global coordinates vs Weyl coordinates

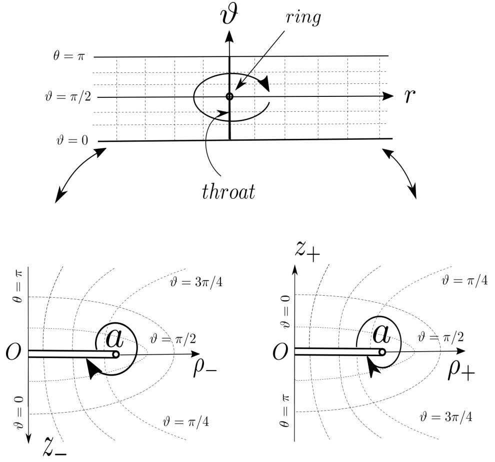


Figure: The r, ϑ coordinates cover the whole of the manifold, each Weyl chart covers only a half. The Weyl charts have branch cuts glued to each other. A winding around the ring in the x, ϑ coordinates corresponds to two windings in Weyl coordinates.

Global coordinates r, ϑ

$$z = r\cos\vartheta, \qquad \rho = \sqrt{r^2 + a^2}\,\sin\vartheta$$

with $r \in (-\infty, \infty)$ (double covering) yields

$$\begin{aligned} ds^2 &= -e^{2U}dt^2 + e^{-2U}dl^2, \qquad\qquad U = \sigma \arctan\left(\frac{r}{a}\right), \\ dl^2 &= \left(\frac{r^2 + a^2\cos^2\vartheta}{r^2 + a^2}\right)^{1+\sigma^2}\left[dr^2 + (r^2 + a^2)d\vartheta^2\right] \\ &+ (r^2 + a^2)\sin^2\vartheta d\varphi^2 \,. \end{aligned}$$

Close to the axis $\cos\vartheta \approx 1$, taking $\sigma \to 0$ gives wormhole metric

$$ds^2 = -dt^2 + dr^2 + (r^2 + a^2)d\Omega^2$$

Wormhole throat is at $r = 0$. The Weyl coordinates (ρ, z) cover either the $r < 0$ part or the $r > 0$ wormhole parts.

Ring singularity

Geometry is singular at the ring in the equatorial plane at $r = 0$ $\vartheta = \pi/2$: Weyl tensor shows a power-low divergence while the Ricci tensor shows a distributional singularity. Introducing polar coordinates (R, α) in the region close to the ring,

$$ds^2 = -dt^2 + dR^2 + R^2 d\alpha^2 + a^2 d\varphi^2 + \ldots$$

with $\alpha \in [0, (2 + \sigma^2)2\pi)$ $\Rightarrow$ a negative angle deficit

$$\delta = -(\sigma^2 + 1)2\pi$$

$\Rightarrow$ a conical singularity generated by an infinitely thin ring of radius a and of *negative* tension (energy per unit length)

$$T = -\frac{(1 + \sigma^2)c^4}{4G}$$

$\Rightarrow$ the wormhole is supported by a negative tension ring.

Ring wormhole with locally flat geometry

In the limit $\sigma \to 0$ one has

$$ds^2 = -dt^2 + \left(\frac{r^2 + a^2 \cos^2 \vartheta}{r^2 + a^2}\right) \left[dr^2 + (r^2 + a^2) d\vartheta^2\right] + (r^2 + a^2) \sin^2 \vartheta d\varphi^2$$

Weyl tensor vanishes and passing to the Weyl coordinates the metric becomes manifestly flat

$$ds^2 = -dt^2 + d\rho^2 + dz^2 + \rho^2 d\varphi^2.$$

The topology is non-trivial since $r \in (-\infty, \infty)$ and one needs two (ρ, z) patches, one for $r > 0$ and the other for $r < 0$, to cover the manifold. The winding angle around the ring core ranges from zero to 4π hence the ring is still there and has the tension

$$T = -c^4/4G$$

$\Rightarrow$ the distributional singularity of the Ricci remains.

Geodesics

Geodesics are straight lines. Those which miss the ring always stay at the same chart. Those threading the ring pass to the other chart and become invisible – the “magic ring” literally creates a hole in flat space.

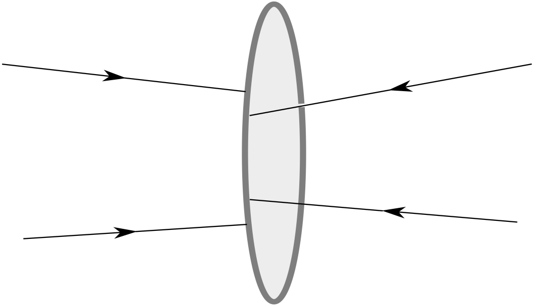


Figure: Particles entering the ring are not seen coming out from the other side

Alice through the looking glass

Energy

To create a ring of radius R one needs the negative energy

$$E = 2\pi RT = -2\pi R \frac{c^4}{4G}$$

To create a ring of radius $R = 1$ metre one needs a negative energy equivalent to the mass of Jupiter.

Small rings can probably appear and disappear in quantum fluctuations. Particles passing through the ring during its existence will disappear – loss of quantum coherence.

The ring can probably be replaced by a thin torus with negative energy associated to quantum fluctuations inside the torus.

Two-ring wormholes

$$U = \sigma_1 U_1 + \sigma_2 U_2, \qquad k = \sigma_1^2\, k_1 + \sigma_2^2\, k_2 + \sigma_1 \sigma_2\, k_{12},$$

with

$$U_s = \arctan\left(\frac{X_s}{a_s}\right), \qquad k_s = \frac{1}{2}\ln\left(\frac{X_s^2 + Y_s^2}{X_s^2 + a_s^2}\right),$$

$$k_{12} \quad = \quad \frac{1}{2}\ln\left(\left|\frac{(X_1 + iY_1)(X_2 + iY_2) + z_1^+ z_2^+ + \rho^2}{(X_1 + iY_1)(X_2 - iY_2) + z_1^+ z_2^- + \rho^2}\right|^2\right)$$

$$X_s \pm iY_s = \sqrt{\rho^2 + (z - z_s \pm i a_s)^2}$$

where σ_s, z_s, a_s are free parameters. Locally flat for $\sigma_s \to 0$.

Two wormholes – four Weyl charts

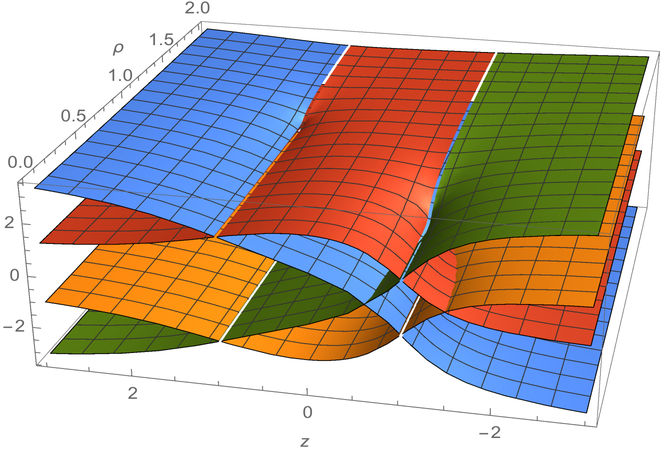


Figure: $U(\rho, z)$ for $\sigma_1 = 1$, $\sigma_2 = 1.5$, $\mu_1 = 1.2$, $\mu_2 = 0.5$, $z_1 = -z_2 = 1$, and for $\rho \in [0, 2]$, $z \in [-4, 4]$. The four different branch sheets correspond to values of U in four different spacetime regions. There are four symmetry axes.

Weyl charts

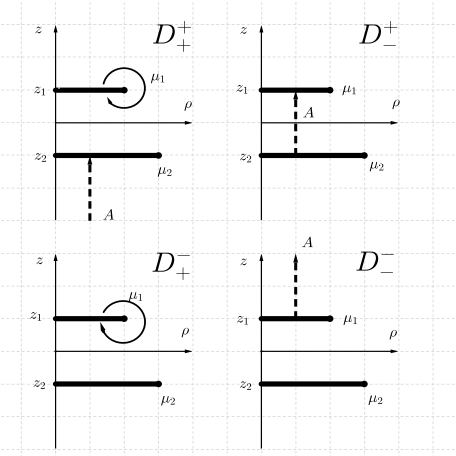


Figure: The two-ring wormhole is covered by four Weyl charts, each having two branch cuts. The upper cuts on $D_\pm^+$ and $D_\pm^-$ are glued to each other such that the upper edge of the one is identified with the lower edge of the other and vice-versa; similarly for the lower cuts on $D_+^\pm$ and $D_-^\pm$. This generalises to N wormholes.

Multi-ring wormholes

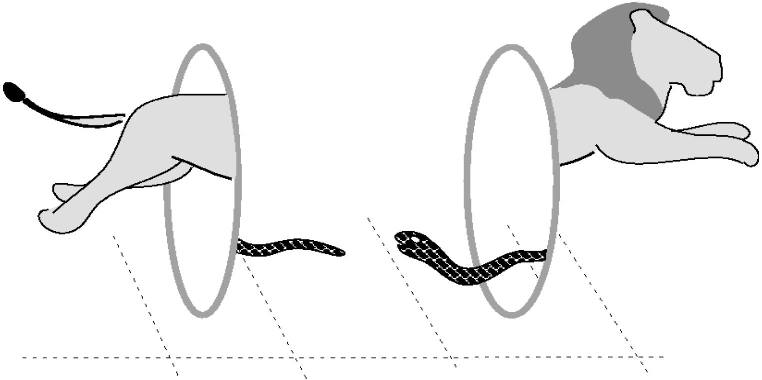
Solution can be generalized to the case of N rings. In the limit

$$\sigma_s \to 0$$

they all have the same tension

$$T = -c^4/4G$$

the geometry is locally flat outside the rings. The rings connect 2^N flat universes.



Appell ring

$$
\begin{aligned}
U &= -\frac{m_+}{R_+} - \frac{m_-}{R_-}, \\
k &= -\frac{m_+^2\rho^2}{2(R_+)^4} - \frac{m_-^2\rho^2}{2(R_-)^4} + \frac{m_+m_-}{2m^2}\left(\frac{\rho^2+z^2-m^2}{R_+R_-} - 1\right)
\end{aligned}
$$

with $R_\pm = \sqrt{\rho^2 + (z \pm m)^2}$ /Chazy, Curzon 1924/ . Upon

$$
m \to ia, \qquad m_\pm \to -\frac{M}{2}\, e^{\pm i\eta}
$$

one obtains

$$
\begin{aligned}
R_\pm &\to \sqrt{\rho^2 + (z \pm ia)^2} \equiv \mathcal{R}e^{\pm i\mathcal{S}}, \\
U &= \frac{M}{\mathcal{R}}\cos(\mathcal{S} - \eta), \qquad \text{(Appell potential)} \\
k &= -\frac{M^2\rho^2}{4\mathcal{R}^4}\cos(4\mathcal{S} - 2\eta) - \frac{M^2}{8a^2}\left(\frac{\rho^2+z^2+a^2}{\mathcal{R}^2} - 1\right).
\end{aligned}
$$

Summary of part II

- In vacuum GR there are wormholes sources by negative tension rings. Each ring encircles the wormhole throat. Solutions depend on a parameter σ.
- For $\sigma \neq 0$ the ring supports a power-law singularity of the Weyl tensor and a conical singularity of the Ricci tensor.
- For $\sigma \to 0$ the Weyl tensor vanishes, the geometry becomes locally flat, but there remains the conical singularity of the Ricci tensor corresponding to the negative energy $T = -c^4/(4G)$ along the ring. The ring "cuts a hole" in flat space.
- Other ring solutions are singular. All of them can be dressed up with the scalar field by applying dualities.

III. Ring wormhole as the $M \to 0$ limit of Kerr spacetime

Minkowski space in spheroidal coordinates

$$ds^2 = -dt^2 + d\rho^2 + dz^2 + \rho^2 d\varphi^2.$$

expressed in oblate spheroidal coordinates $r \in [0, \infty)$, $\vartheta \in [0, \pi)$

$$z = r\cos\vartheta, \qquad \rho = \sqrt{r^2 + a^2}\,\sin\vartheta \quad \Rightarrow \quad \frac{z^2}{r^2} + \frac{\rho^2}{r^2 + a^2} = 1$$

reads

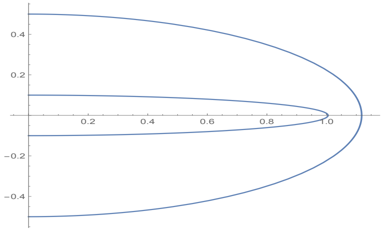
$$ds^2 = -dt^2 + \left(\frac{r^2 + a^2\cos^2\vartheta}{r^2 + a^2}\right)\left[dr^2 + (r^2 + a^2)d\vartheta^2\right] + (r^2 + a^2)\sin^2\vartheta d\varphi^2$$

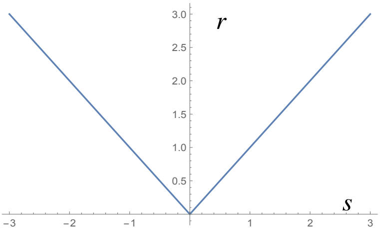
Coordinate singularity at the ring $r = 0$, $\vartheta = \pi/2$. Geodesic

$$\frac{dr}{ds} = \pm\sqrt{\mathcal{E}^2 - \mu^2}$$

is discontinuous since one is bound to chose different signs.

Geodesic





Analytic continuation to $r \in (-\infty, \infty)$

If r is allowed to be negative – no need to change sign in geodesic equation; geodesics analytically continue. The metric is the same

$$ds^2 = -dt^2 + \left(\frac{r^2 + a^2\cos^2\vartheta}{r^2 + a^2}\right)\left[dr^2 + (r^2 + a^2)d\vartheta^2\right] + (r^2 + a^2)\sin^2\vartheta d\varphi^2\,,$$

and close to the ring $r = 0$, $\vartheta = \pi/2$ this reduces to

$$ds^2 = -dt^2 + dR^2 + R^2 d\alpha^2 + a^2 d\varphi^2 + \ldots$$

where $\alpha \in [0, 4\pi]$ hence the negative angle deficit and the distributional conical singularity of the curvature. The relation $\rho, z \Leftrightarrow r, \vartheta$ is no longer bijective, the geometry can be covered by two flat charts (ρ_+, z_+) and (ρ_-, z_-)

$$ds^2 = -dt^2 + d\rho_\pm^2 + dz_\pm^2 + \rho_\pm^2 d\varphi^2$$

Wormhole topology

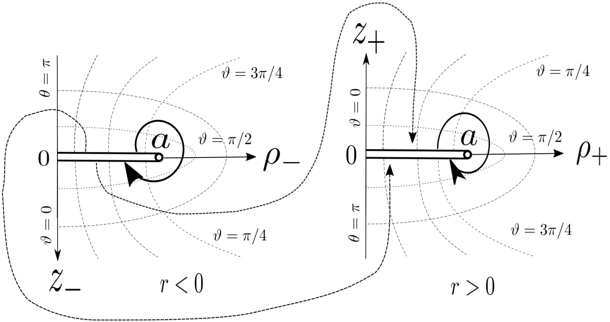


Figure: Analytic continuation from one flat chart to the other. A contour around the string core makes one revolution of 2π, then passes to the other chart, and only after the second revolution of 2π closes – the angle increment is 4π.

Moral

The same metric

$$ds^2 = -dt^2 + \left(\frac{r^2 + a^2\cos^2\vartheta}{r^2+a^2}\right)\left[dr^2 + (r^2+a^2)d\vartheta^2\right] + (r^2+a^2)\sin^2\vartheta d\varphi^2$$

describes flat Minkowski space if $r \in [0,\infty)$ and locally flat wormhole if $r \in (-\infty,\infty)$.
This metric is the $M \to 0$ limit of Kerr

$$\begin{aligned} ds^2 &= -dt^2 + \frac{2Mr}{\Sigma}\left(dt - a\sin^2\vartheta\, d\varphi\right)^2 + \Sigma\left(\frac{dr^2}{\Delta} + d\vartheta^2\right) \\ &+ (r^2+a^2)\sin^2\vartheta d\varphi^2\,; \\ \Delta &= r^2 - 2Mr + a^2, \qquad \Sigma = r^2 + a^2\cos^2\vartheta, \end{aligned}$$

where $r \in (-\infty,\infty)$ since the geodesics pass to the $r<0$ region.

Kerr geodesics

$$\frac{1}{\mu^2}\left(\frac{dr}{ds}\right)^2 + V(r) = E$$

As $M \to 0$ the geodesics freely move in $r \in (-\infty, \infty)$.

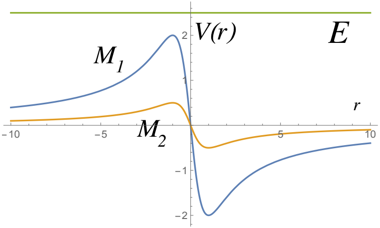


Figure: Potential $V(r) = -2Mr/(r^2 + a^2)$ in the geodesic equation

$\Rightarrow$ zero mass limit of Kerr is the wormhole !

Kerr-Schild: $t, r, \vartheta, \varphi \to T, \rho, z, \varphi$

$$\begin{aligned}
\rho &= \sqrt{r^2+a^2}\sin\vartheta, \quad z = r\cos\vartheta, \\
T &= t + \int \frac{2Mr}{\Delta}\,dr, \quad \phi = \varphi + \int \frac{2Mar}{\Sigma\Delta}\,dr,
\end{aligned}$$

which yields

$$\begin{aligned}
ds^2 = &- dT^2 + d\rho^2 + \rho^2 d\phi^2 + dz^2 \\
&+ \frac{2Mr^3}{r^4+a^2z^2}\left(\frac{r\rho}{r^2+a^2}\,d\rho - ar\sin^2\vartheta d\phi + \frac{z}{r}\,dz + dT\right)^2
\end{aligned}$$

For $M \to 0$ the metric is flat. However, one needs two Kerr-Schild chars: (ρ_+, z_+) for $r > 0$ and (ρ_-, z_-) for $r < 0$. These two charts are glued together precisely as was shown before (Hawking-Ellis), hence for $M \to 0$ one obtains the two-sheeted wormhole topology and a conical singularity.

Fig.27 from Hawking-Ellis

ally incomplete at the ring singularity. However the only timelike and null geodesics which reach this singularity are those in the equatorial plane on the positive r side (Carter (1968a)).

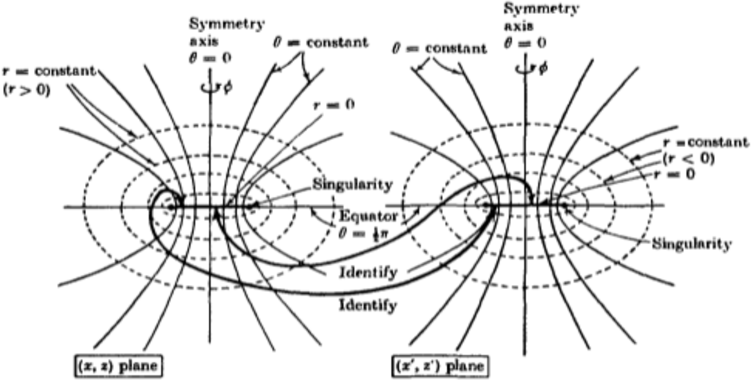


FIGURE 27. The maximal extension of the Kerr solution for $a^2 > m^2$ is obtained by identifying the top of the disc $x^2 + y^2 < a^2$, $z = 0$ in the (x, y, z) plane with the bottom of the corresponding disc in the (x', y', z') plane, and vice versa. The

Summary of part III

- Kerr spacetime has the two-sheeted topology also in the $M \to 0$ limit. The limiting spacetime is locally flat but it cannot be globally flat Minkowski space since it is topologically non-trivial.

- The Kerr ring supports a power-law singularity of the Weyl tensor that vanishes for $M \to 0$, but it also supports a distributional singularity of the Ricci tensor that remains even in the $M \to 0$ limit. Carter '68: *in the special case where M vanishes there must still be a curvature singularity at $\Sigma = 0$, although the metric is then flat everywhere else*.

- It follows that the $M \to 0$ limit of the Kerr spacetime is the wormhole sourced by the negative tension ring – the simplest way to produce wormholes.

IV. Wormholes in massive bigravity

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JCAP 06 (2015) 017

Ghost-free bigravity

$$S = \frac{m^2}{M_{\rm Pl}^2} \int \left(\frac{1}{2\kappa_1} R(g)\sqrt{-g} + \frac{1}{2\kappa_2} R(f)\sqrt{-f} - \mathcal{U}\sqrt{-g} \right) d^4x$$

$$\begin{aligned} \mathcal{U} &= b_0 + b_1 \sum_A \lambda_A + b_2 \sum_{A<B} \lambda_A \lambda_B \\ &+ b_3 \sum_{A<B<C} \lambda_A \lambda_B \lambda_C + b_4\, \lambda_0 \lambda_1 \lambda_2 \lambda_3 \end{aligned}$$

where λ_A are eigenvalues of $\gamma^\mu{}_\nu = \sqrt{g^{\mu\alpha} f_{\alpha\nu}}$.

$$\begin{aligned} G^\mu_\nu(g) &= \kappa_1\, T^\mu{}_\nu(g,f), \\ G^\mu_\nu(f) &= \kappa_2\, \mathcal{T}^\mu{}_\nu(g,f), \end{aligned}$$

The two energy-momentum tensors do not apriori fulfill ant energy conditions.

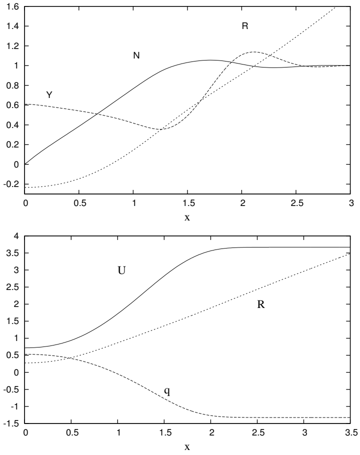
Reduction to the S-sector

$$
\begin{aligned}
ds_g^2 &= -Q^2 dt^2 + \frac{R'^2}{N^2}\, dr^2 + R^2 d\Omega^2 \\
ds_f^2 &= -q^2 dt^2 + \frac{U'^2}{Y^2}\, dr^2 + U^2 d\Omega^2
\end{aligned}
$$

Q, N, R, q, Y, U depend on r, one can impose 1 gauge condition.
5 independent equations

$$
\begin{aligned}
G^0_0(g) &= \kappa_1\, T^0_0, \\
G^r_r(g) &= \kappa_1\, T^r_r, \\
G^0_0(f) &= \kappa_2\, \mathcal{T}^0_0, \\
G^r_r(f) &= \kappa_2\, \mathcal{T}^r_r, \\
T^{r\prime}_r &+ \frac{Q'}{Q}\,(T^r_r - T^0_0) + \frac{2}{r}(T^\vartheta_\vartheta - T^r_r) = 0.
\end{aligned}
$$

Wormholes – global solutions



Solutions for $\kappa_1 = 0.688$, $\kappa_2 = 0.312$, $b_k = b_k(c_3, c_4)$, $c_3 = 3$, $c_4 = -6$, for the neck radius $h = 2.2$. Here $\sigma = 0.444$ and $N = R'$.

Asymptotic behavior

For $R \to \infty$ solutions approach the AdS solution, $ds_f^2 = \lambda^2 ds_g^2$

$$ds_g^2 = -Q^2 dt^2 + \frac{dR^2}{N^2} + R^2 d\Omega^2$$

with $N^2 \to N_0^2 = 1 - \frac{\Lambda r^2}{3}$ and $Q^2 \to const \times N_0^2$. When $R \to \infty$

$$N^2 \to N_0^2 \times \left(1 + \frac{C}{R^3} + \frac{A}{R\sqrt{R}} \cos\left(\omega \ln(R) + \varphi\right)\right)$$

Newtonian tail + oscillations due to the scalar polarization of the massive graviton which becomes a tachyon with

$$m_{FP}^2 = \left(\frac{\kappa_2}{\lambda} + \kappa_1 \lambda\right)(b1 + 2b_2\lambda + b_3\lambda^2) < \frac{3}{4}\Lambda < 0$$

hence the Breitenlohner-Freedman bound is violated.

Summary of part IV

- The ghost-free bigravity theory admits solutions for which the f-metric can be singular, but the g-metric describes globally regular wormholes.
- The wormholes interpolate between two AdS spaces.
- The wormhole throat is cosmologically large (could we live inside it ?)
- Solutions show tachyons in the asymptotics.

Final conclusion

- It seems that traversable wormholes might really exist.