Shell-crossing in quasi-one-dimensional flow

(or: on cosmological pancake formation)

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Outline

- Euler-Poisson equations for 3D pressureless matter fluid (EdS)

$$\partial_\tau \boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} = -\frac{3}{2\tau}\left(\boldsymbol{v} + \nabla\varphi_{\rm g}\right)\,, \quad \partial_\tau\delta + \nabla\cdot[(1+\delta)\boldsymbol{v}] = 0\,, \quad \nabla^2\varphi_{\rm g} = \frac{\delta}{\tau}$$

($\tau \sim a$ is cosmic scale factor; $\boldsymbol{v}$ is peculiar velocity and $\delta = (\rho - \bar\rho)/\bar\rho$ is density contrast)

- For generic initial data, solving these equations (not just for short times) requires multi-time-stepping algorithms

- Known analytical solution for 1D flow since the late 60s
  (either be employing a 1D version of Euler-Poisson eqs., or by embedding a 1D flow in 3D space)

- New exact analytical solution, for quasi-1D flow, embedded in 3D:
  From initial time, in a single time-step, until the break-down of the single-stream description (= "**shell-crossing**", where $\delta \to \infty$)

Initial conditions for the multi-stream computation are set at the time of shell-crossing

Boundary conditions

Let's briefly investigate the fluid equations for arbitrary short times:

$$\partial_\tau \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\frac{3}{2\tau} \left(\boldsymbol{v} + \nabla \varphi_{\mathrm{g}} \right) , \quad \partial_\tau \delta + \nabla \cdot [(1+\delta)\boldsymbol{v}] = 0 , \quad \nabla^2 \varphi_{\mathrm{g}} = \frac{\delta}{\tau}$$

To avoid quasi-singular behaviour as τ becomes small, we need the following *slaving conditions* on the "initial" conditions (at decoupling)

$$\delta^{(\mathrm{init})} = 0 , \qquad \boldsymbol{v}^{(\mathrm{init})} = -\nabla \varphi_{\mathrm{g}}^{(\mathrm{init})}$$

Selects the growing mode solution, and $\nabla \times \boldsymbol{v} = \mathbf{0}$ (until shell-crossing)

Switch to Lagrangian coordinates

Introduce the Lagrangian map $\boldsymbol{q} \mapsto \boldsymbol{x}(\boldsymbol{q}, \tau)$ of the initial position $\boldsymbol{q}$ of a fluid particle to its current position $\boldsymbol{x}(\boldsymbol{q}, \tau) = \boldsymbol{q} + \boldsymbol{\xi}(\boldsymbol{q}, \tau)$, solution of the characteristic equation $\partial_\tau^{\mathrm{L}} \boldsymbol{x} = \boldsymbol{v}(\boldsymbol{x}(\boldsymbol{q}, \tau), \tau)$, and $\boldsymbol{x}(\boldsymbol{q}, \tau = 0) = \boldsymbol{q}$

$\partial_\tau^{\mathrm{L}}$ is the Lagrangian time derivative, and let's denote the Lagrangian derivative w.r.t. component q_i with "$,i$"

The Euler-Poisson equations in Lagrangian coordinates are

$$\varepsilon_{ikl}\varepsilon_{jmn} x_{k,m} x_{l,n} \mathfrak{R}_\tau x_{i,j} = 3(J-1)\,, \quad \varepsilon_{ijk} x_{l,j} \partial_\tau^{\mathrm{L}} x_{l,k} = 0\,,$$

where $J = \det(x_{i,j})$ is the Jacobian, and $\mathfrak{R}_\tau = \tau^2 \left(\partial_\tau^{\mathrm{L}}\right)^2 + (3\tau/2)\partial_\tau^{\mathrm{L}}$

Lagrangian perturbation theory (LPT) as a time-Taylor series

[Zheligovsky & Frisch, 1312.6320]

Generic solution scheme to all-orders:

seek a Taylor series representation for the displacement $\boldsymbol{\xi} \equiv \boldsymbol{x} - \boldsymbol{q}$

$$\boxed{\textit{Ansatz:} \quad \boldsymbol{\xi}(\boldsymbol{q}, \tau) = \sum_{n=1}^{\infty} \boldsymbol{\xi}^{(n)}(\boldsymbol{q}) (\tau - 0)^n}$$

expansion parameter

At first order, $n = 1$, the boundary conditions imply

$$\boldsymbol{\xi}^{(1)}(\boldsymbol{q}) = \boldsymbol{v}^{(\mathrm{init})}(\boldsymbol{q}). \qquad \text{("Zel'dovich approximation")}$$

initial velocity

For $n > 1$: Plugging the *Ansatz* in the Euler-Poisson equations, and identifying powers in τ^n, one obtains all-order recursion relations for the displacement coefficients $\boldsymbol{\xi}^{(n)}(\boldsymbol{q}) \ldots$:

From the Lagrangian scalar equation, one gets $(\nabla^{\mathrm{L}} \equiv \nabla_{\boldsymbol{q}})$

$$\nabla^{\mathrm{L}} \cdot \boldsymbol{\xi}^{(n)} = \nabla^{\mathrm{L}} \cdot \boldsymbol{v}^{(\mathrm{init})} \delta_1^n + \sum_{0<s<n} \frac{s^2 + (s-n)^2 + (n-3)/2}{2n^2 + n - 3} \left(\xi_{i,j}^{(n-s)} \xi_{j,i}^{(s)} - \xi_{i,i}^{(n-s)} \xi_{j,j}^{(s)} \right)$$

$$- \frac{1}{6} \sum_{s_1+s_2+s_3=n} \frac{s_1^2 + s_2^2 + s_3^2 + (n-3)/2}{n^2 + (n-3)/2} \varepsilon_{ikl} \varepsilon_{jmn} \xi_{i,j}^{(n_1)} \xi_{k,m}^{(n_2)} \xi_{l,n}^{(n_3)}$$

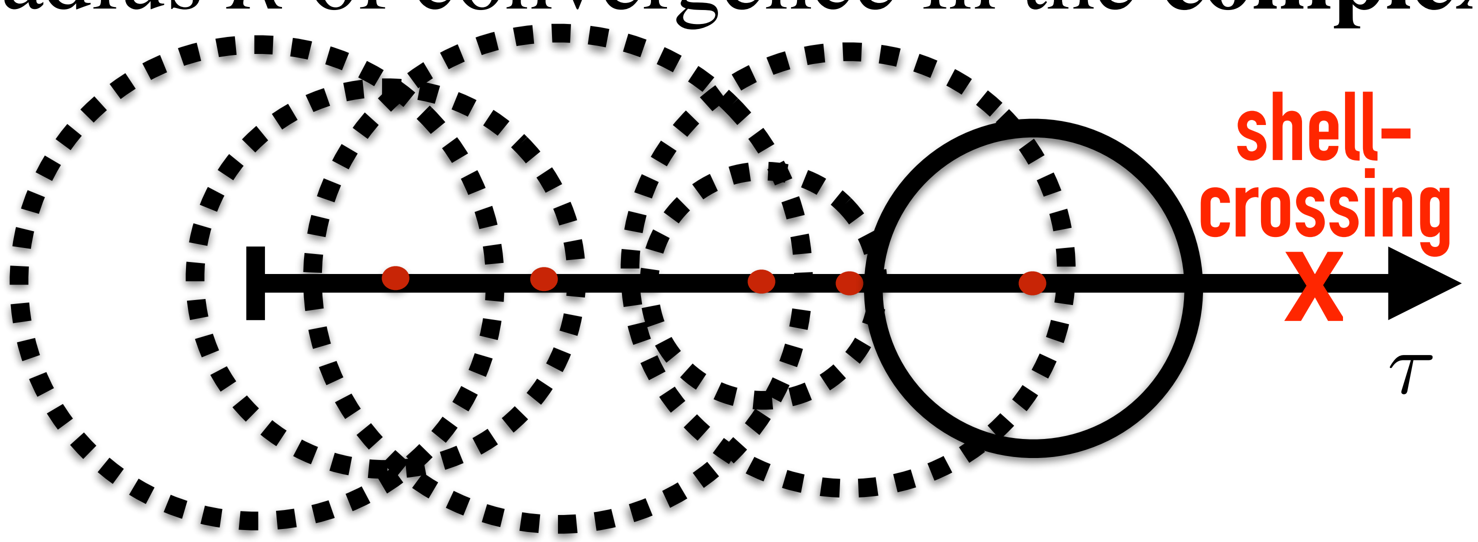
And from the vector equation:

$$\nabla^{\mathrm{L}} \times \boldsymbol{\xi}^{(n)} = \sum_{0<s<n} \frac{n-2s}{2n} \nabla^{\mathrm{L}} \xi_k^{(s)} \times \nabla^{\mathrm{L}} \xi_k^{(n-s)}$$

Summing up, the usual Helmholtz decomposition then gives

$$\boldsymbol{\xi}(\boldsymbol{q}, \tau) = \sum_{n=1}^{\infty} \boldsymbol{\xi}^{(n)}(\boldsymbol{q})\, \tau^n \qquad \Rightarrow \qquad \boldsymbol{x} = \boldsymbol{q} + \boldsymbol{\xi}$$

Typically, the Lagrangian map is analytic but not entire in time;
it has a finite radius *R* of convergence in the **complex time plane**:



[CR, Villone & Frisch, 1504.00032: bounds on radius of convergence in ΛCDM]

The quasi-one dimensional problem

- In 1D, the particle trajectory is exactly $x(q,\tau) = q + \tau\, v^{(\mathrm{init})}(q)$, and is entire in the time variable. [Novikov '69, Zel'dovich '69]

- Departing from 1D just slightly (perturbatively) generally populates higher-order time-Taylor coefficients.

- Our strategy:
  1. **refine** the perturbation problem
  2. solve the Euler-Poisson equations in Lagrangian coordinates with refined perturbation *Ansatz* for the particle trajectory
  3. the result is an infinite time-Taylor series for the particle trajectory, which is shown to be entire in the time variable

Initial data and solution Ansatz for the perturbed problem

- specify the initial gravitational potential:

$$\varphi_{\rm g}^{(\rm init)} = -\cos q_1 + \epsilon\,\phi^{(\rm init)}(q_1, q_2, q_3)\,,$$

where the first term on the r.h.s. is the 1D problem, $\epsilon > 0$ a small perturbation parameter, and $\phi^{(\rm init)}(q_1, q_2, q_3)$ a generic perturbation

- Perturbation Ansatz: $\boldsymbol{x}(\boldsymbol{q},\tau) = \boldsymbol{q} + \boldsymbol{\xi}^{(0)}(\boldsymbol{q},\tau) + \epsilon\,\boldsymbol{\xi}^{(1)}(\boldsymbol{q},\tau) + \epsilon^2\boldsymbol{\xi}^{(2)}(\boldsymbol{q},\tau) + \cdots$

where $\xi_i^{(0)}(\boldsymbol{q},\tau) = \delta_{i1} F(q_1,\tau)$ is the 1D displacement.

We only go to order ϵ, henceforth we write $\boldsymbol{\xi}^{(1)}(\boldsymbol{q},\tau) = \boldsymbol{\xi}(\boldsymbol{q},\tau)$,

and thus for the Jacobian matrix:

$$x_{i,j} = \delta_{ij} + \delta_{i1}\delta_{j1}F_{,1} + \epsilon\xi_{i,j}$$

Solution to zeroth order in ϵ: the unperturbed 1D flow

Plugging $x_{i,j} = \delta_{ij} + \delta_{i1}\delta_{j1}F_{,1}$ into

$$\varepsilon_{ikl}\varepsilon_{jmn}x_{k,m}x_{l,n}\mathfrak{R}_\tau x_{i,j} = 3(J-1)\,, \qquad \varepsilon_{ijk}x_{l,j}\partial_\tau^{\mathrm{L}} x_{l,k} = 0\,,$$

and using the boundary conditions, one obtains

$$x_i = q_i - \delta_{i1}\tau\,\sin q_1\,,$$

and for the Jacobian $(J = \det(x_{i,j}))$

$$J = 1 - \tau\,\cos q_1$$

Solution to order ϵ^0

Shell-crossing occurs when J vanishes for the first time.

For the unperturbed 1D flow, this occurs at the time $\tau_\star^{(0)} = 1$

The fluid density ($\delta = 1/J - 1$) blows-up at shell-crossing and thus marks the break-down of the fluid description

Solution to first order in ϵ: the perturbed 1D flow

Set $x_{i,j} = \delta_{ij} - \delta_{i1}\delta_{j1}\tau\, \cos q_1 + \epsilon\, \xi_{i,j}$,

and impose the nested Ansatz $\boldsymbol{\xi}(\boldsymbol{q},\tau) = \sum_{n=1}^{\infty} \boldsymbol{\xi}^{(n)}(\boldsymbol{q})\, \tau^n$.

Collecting all terms $O(\epsilon)$ in

$$\varepsilon_{ikl}\varepsilon_{jmn} x_{k,m} x_{l,n} \mathfrak{R}_\tau x_{i,j} = 3(J-1)\,, \quad \varepsilon_{ijk} x_{l,j} \partial_\tau^{\mathrm{L}} x_{l,k} = 0\,,$$

and identifying the powers in τ, we obtain recursion relations for the perturbed displacement, which are most easily formulated in Fourier space…:

Fourier transformation for any scalar or vector $\mathbf{f} \equiv \hat{\mathbf{f}}_{k_1} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{q}}$

The perturbed displacement in Fourier space is

$$\hat{\boldsymbol{\xi}}_{k_1} = -\mathrm{i}\boldsymbol{k}\,\hat{\phi}^{(\mathrm{init})}\delta_{1n}$$
$$- \left(k_1^2+k_2^2+k_3^2\right)^{-2}\left(\mathrm{i}\boldsymbol{k}\frac{n-1/2}{2n+3}\left[\hat{\zeta}_{k_1+1}^{(n-1)}+\hat{\zeta}_{k_1-1}^{(n-1)}\right]+\frac{n-2}{2n}\left(-k_2^2-k_3^2,k_1k_2,k_1k_3\right)^{\mathrm{T}}\left[\hat{\chi}_{k_1+1}^{(n-1)}+\hat{\chi}_{k_1-1}^{(n-1)}\right]\right)$$

where $\hat{\chi}_{k_1}^{(1)} = -\mathrm{i}k_1\hat{\phi}^{(\mathrm{init})}$ and $\hat{\zeta}_{k_1}^{(1)} = k_\perp^2\hat{\phi}^{(\mathrm{init})}$, where $k_\perp^2 = k_2^2+k_3^2$,

and for $n>1$

$$\hat{\zeta}_{k_1}^{(n)} = \left(1+\frac{k_1^2}{k_\perp^2}\right)^{-1}\left[\frac{n-1/2}{2n+3}\left(\hat{\zeta}_{k_1+1}^{(n-1)}+\hat{\zeta}_{k_1-1}^{(n-1)}\right)-\mathrm{i}k_1\frac{n-2}{2n}\left(\hat{\chi}_{k_1+1}^{(n-1)}+\hat{\chi}_{k_1-1}^{(n-1)}\right)\right],$$

$$\hat{\chi}_{k_1}^{(n)} = \left(1+\frac{k_1^2}{k_\perp^2}\right)^{-1}\left[-\mathrm{i}k_1\,k_\perp^{-2}\frac{n-1/2}{2n+3}\left(\hat{\zeta}_{k_1+1}^{(n-1)}+\hat{\zeta}_{k_1-1}^{(n-1)}\right)+\frac{n-2}{2n}\left(\hat{\chi}_{k_1+1}^{(n-1)}+\hat{\chi}_{k_1-1}^{(n-1)}\right)\right].$$

Solution up to order ϵ^1

Trajectory: $x_i(\boldsymbol{q};\tau) = q_i - \delta_{1i}\tau\sin q_1 + \epsilon\sum_{n=1}^{\infty}\xi_i^{(n)}(\boldsymbol{q})\,\tau^n$,

Jacobian: $J = 1-\tau\cos q_1 + \epsilon\sum_{n=1}^{\infty}\left(\xi_{1,1}^{(n)}+[1-\tau\cos q_1]\left\{\xi_{2,2}^{(n)}+\xi_{3,3}^{(n)}\right\}\right)\tau^n$.

Quasi 1D: Entirety in time and its consequences

- Taylor series for the Lagrangian map is an entire function in time.
  (i.e., the radius of convergence of the time-Taylor series is formally infinite)

- Our trick to go beyond 1D: *a linearisation in Lagrangian space!*

- This allows the determination of the time and location of the first shell-crossing, which generically is taking place earlier than for an unperturbed 1D flow: $\tau_\star = 1 - \epsilon C$
  Here, C is a (positive) space constant, which precise value is easily determined for a given $\phi^{(\mathrm{init})}(q_1, q_2, q_3)$

- By contrast, the Zel'dovich approximation, which in our framework amounts to *ignoring all time-Taylor coefficients beyond n=1*, generically predicts the wrong time and location of the collapse.