Extending the Classical Double Copy

Mark Trodden
University of Pennsylvania

GC2018 One-Day Workshop

YITP, Kyoto, February 2018

Overview

- Quick comments on previous work:
  - Amplitudes and the BCJ (Gravity / Yang-Mills) connection
  - The classical double copy.
- Our Motivations - some technical puzzles
- Extending the Double Copy
  - Our main goals - some technical issues
  - **(A)dS-Schwarzschild** and Kerr-(A)dS
  - Charged Black Holes
  - Black Strings and Black Branes
  - **Wave Solutions**
  - **The BTZ Black Hole**

"The Classical Double Copy in Maximally Symmetric Spacetimes"
M. Carrillo González, Riccardo Penco and M.T., arXiv:1711.01296 [hep-th]. (and upcoming papers)

- Open questions and future work

Amplitudes & the BCJ Connection

- Relationship between scattering amplitudes (Bern, Carrasco, Johannson). Simplest example.

$$\text{Gravity} = (\text{Yang} - \text{Mills})^2$$

- Relates Einstein Gravity (the double copy) to two copies of Yang-Mills theory (the single copies).
- Replace kinematic factors in amplitudes of YM theory with color factors.
- Extend to yield amplitudes for a bi-adjoint scalar (the zeroth copy).
- Real massless scalar (bi-adjoint) with cubic interaction

$$f^{abc}\,\tilde{f}^{ijk}\,\phi^{a\,i}\,\phi^{b\,j}\,\phi^{c\,k}$$

$$G \times \tilde{G}$$

- Many generalizations: Born-Infield theory / Special Galileon; Einstein-Maxwell /Einstein-Yang-Mills theories, …

Amplitudes II

- Gluon scattering amplitudes in BCJ form expressed schematically as

Kinematic Factors Color Factors

$$A_{\rm YM} = \sum_i \frac{N_i\, C_i}{D_i}$$

Scalar Propagators

- Double copy: exchange color factors for second instance of kinematic factors (can in general take from different YM theory.
- Replacement gives rise to a *gravitational* scattering amplitude

$$A_{\rm G} = \sum_i \frac{\tilde{N}_i\, N_i}{D_i}$$

- Different choices of kinematic factors yield gravitational amplitudes with same number of external gravitons; different intermediate states

The Classical Double Copy

- Exist perturbative versions of a classical version …

W. Goldberger and A. Ridgway, arXiv:1611.03493.

- … and a classical version on asymptotically flat backgrounds

R. Monteiro, D. O'Connell, and C. D. White, arXiv:1410.0239.

- Consider metric in Kerr-Schild form, with Minkowski base metric:

$$g_{\mu\nu} = \eta_{\mu\nu} + \underbrace{\phi}_{\text{Scalar}}\, \underbrace{k_\mu k_\nu}_{\text{Null, geodetic vector}}$$

$$g^{\mu\nu} k_\mu k_\nu = \eta^{\mu\nu} k_\mu k_\nu = 0$$

$$k^\mu \nabla_\mu k^\nu = k^\mu \partial_\mu k^\nu = 0$$

- Crucial property - Ricci tensor linear in scalar in up-down form.
- Define "single copy" YM field via:

$$A^a_\mu = \underbrace{c^a}_{\text{Color Factors}} k_\mu \phi$$

- If metric solves Einstein equation, then (up to technical issue) YM field guaranteed to satisfy YM equations if replace $8\pi G \to g$
- and gravitational sources replaced by color sources

- Because ansatz for YM field is factorized, field defined without color factors satisfies Maxwell's equations.
- Can think of color charges as electric charges.
- So will refer to $A_\mu \equiv k_\mu \phi$ as the single copy.

- Can combine KS scalar with two copies of color factors to define a bi-adjoint scalar

$$\phi^{a\,b} = c^a c'^b \phi$$

- Satisfies (up to technicality) linearized equations

$$\bar{\nabla}^2 \phi^{a\,b} = c^a c'^b \bar{\nabla}^2 \phi = 0$$

- Again, focus on field stripped of color indices - the "zeroth copy".
- EOMs for single copy and zeroth copy are linear precisely because of the Kerr-Schild ansatz and linearity of Ricci tensor.

Technical Issues - Our Motivations

- Rather modest goals.
- We have been quite puzzled by two questions in the basic setup
  - What is it that determines whether equations that make sense are satisfied by the single and zeroth copies?
  - What determines how we choose the definitions of the single and zeroth copies?
- Worthwhile seeking a broader framework to understand the construction.

Extending to Curved Backgrounds

M. Carrillo González, Riccardo Penco and M.T., arXiv:1711.01296

- Consider generalized Kerr-Schild form of the metric

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \phi\, k_\mu k_\nu \longleftarrow \text{Null, geodetic vector}$$

Base metric - think of as (A)dS

Scalar

- Ricci scalar still linear in scalar (in up-down form):

$$R^\mu{}_\nu = \bar{R}^\mu{}_\nu - \phi k^\mu k^\lambda \bar{R}_{\lambda\nu} + \frac{1}{2}\left[\bar{\nabla}^\lambda \bar{\nabla}^\mu (\phi k_\lambda k_\nu) + \bar{\nabla}^\lambda \bar{\nabla}_\nu (\phi k^\mu k_\lambda) - \bar{\nabla}^2 (\phi k^\mu k_\nu)\right]$$

- Define single and zeroth copies in same way as earlier.
- Important technical point. Even after finding coords admitting KS, ambiguity in choices of scalar and vector. Invariant under:

$$k_\mu \to f k_\mu, \quad \phi \to \phi / f^2$$

- Doesn't affect gravity, of course. But does affect scalar and YM fields and the equations they satisfy

Reconciling the Ambiguity

- To understand better, rewrite Ricci tensor in form:

$$2(\bar{R}^{\mu}{}_{\nu} - R^{\mu}{}_{\nu}) = \left[\bar{\nabla}_\lambda F^{\lambda\mu} + \tfrac{(d-2)}{d(d-1)}\bar{R}A^{\mu}\right]k_\nu + X^{\mu}{}_{\nu} + Y^{\mu}{}_{\nu}$$

$$X^{\mu}{}_{\nu} \equiv -\,\bar{\nabla}_\nu\left[A^\mu\left(\bar{\nabla}_\lambda k^\lambda + \frac{k^\lambda\bar{\nabla}_\lambda\phi}{\phi}\right)\right]$$

$$Y^{\mu}{}_{\nu} \equiv F^{\rho\mu}\bar{\nabla}_\rho k_\nu - \bar{\nabla}_\rho\left(A^\rho\bar{\nabla}^\mu k_\nu - A^\mu\bar{\nabla}_\rho k_\nu\right)$$

- If full metric solves Einstein equation with a CC, LHS is

$$-16\pi G\left(T^{\mu}{}_{\nu} - \delta^{\mu}_{\nu}T/(d-2)\right)$$

- Now: contract w/ Killing vector of (either) metric. Yields EOM for single copy in d dimensions

$$\bar{\nabla}_\lambda F^{\lambda\mu} + \tfrac{(d-2)}{d(d-1)}\bar{R}A^{\mu} + \tfrac{V^\nu}{V^\lambda k_\lambda}\left(X^{\mu}{}_{\nu} + Y^{\mu}{}_{\nu}\right) = 8\pi G\,J^{\mu}$$

Dealing with Sources

- Have defined $$J^\mu \equiv -\frac{2V^\nu}{V^\rho k_\rho}\left(T^\mu{}_\nu - \delta^\mu_\nu \frac{T}{d-2}\right)$$
- Contract with Killing vector again - yields zeroth copy eqn:

$$\bar{\nabla}^2\phi = j - \frac{(d-2)}{d(d-1)}\bar{R}\phi - \frac{V_\nu}{(V^\mu k_\mu)^2}\left(V^\mu X^\nu{}_\mu + V^\mu Y^\nu{}_\mu + Z^\nu\right)$$

$$Z^\nu \equiv (V^\rho k_\rho)\bar{\nabla}_\mu\left(\phi\bar{\nabla}^{[\mu}k^{\nu]} - k^\mu\bar{\nabla}_\nu\phi\right)$$

- Have defined source as: $$j = \frac{V_\nu J^\nu}{V^\rho k_\rho}$$

- Will use timelike KV for stationary solns, and null KV for wave solns.
- KV allows us to find correct sources for the single and zeroth copies.
- EOMs not invariant under rescaling. Freedom allows us to choose KS vector and scalar so that copies satisfy *reasonable* EOMs.
- Localized source on gravitational side, yields localized source in the gauge and scalar theories.

Example: (A)dS-Schwarzschild in d=4

- Admits KS form w/ (A)dS base in global coordinates …

$$\bar{g}_{\mu\nu}dx^\mu dx^\nu = -\left(1 - \frac{\Lambda r^2}{3}\right)dt^2 + \left(1 - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2 d\Omega^2$$

- .. and KS vector and scalar defined by

$$k_\mu dx^\mu = dt + \frac{dr}{1 - \Lambda r^2/3} \qquad \phi = \frac{2GM}{r}$$

- Full metric solves EE with a CC $\quad G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$
- Remove singularity at r=0 w/ localized source w/ stress-energy tensor

$$T^\mu{}_\nu = \frac{M}{2}\mathrm{diag}(0,0,1,1)\delta^{(3)}(\vec{r})$$

- Then, single copy satisfies Maxwell eqn on (A)dS with localized source $\quad \bar{\nabla}_\mu F^{\mu\nu} = g\, J^\nu$

$$J^\mu = M\,\delta^{(3)}(\vec{r})\,\delta^\mu_0$$

- Static point-particle with charge Q=M in (A)dS.
- Perfect analogy with the flat case.

- Zeroth copy satisfies

$$\left(\bar{\nabla}^2 - \frac{\bar{R}}{6}\right)\phi = j \quad \text{with localized source} \quad j = M\,\delta^{(3)}(\vec{r})$$

- So - unlike on flat background, zeroth copy satisfies the equation for conformally coupled scalar field.
- However, for d >4 non-minimal coupling exists but is not conformal.
- N.B.wrong choice of the Kerr-Schild vector yields unreasonable double copy - e.g. an extra non-localized term in current that changes total charge.

Time-dependent Solutions - Waves in d=4

- For -ve CC, 3 three types of wave solutions in vacuum in KS form.
  - Kundt waves (the only type for +ve CC, and same as pp then)
  - Generalized pp-waves
  - Siklos waves
- All Kundt spacetimes of Petrov-type N

- In these t-dep spacetimes, use null Killing vector to construct classical single and zeroth copies.

Kundt Waves in (A)dS

$$\bar{g}_{\mu\nu}dx^\mu dx^\nu = \frac{1}{P^2}\left[-4x^2 du\left(dv - v^2 du\right) + dx^2 + dy^2\right] \; , \qquad P = 1 + \frac{\Lambda}{12}(x^2+y^2)$$

$$k_\mu = \frac{x}{P}\,\delta^u_\mu \; , \qquad \phi = \frac{P}{x}H(u,x,y)$$

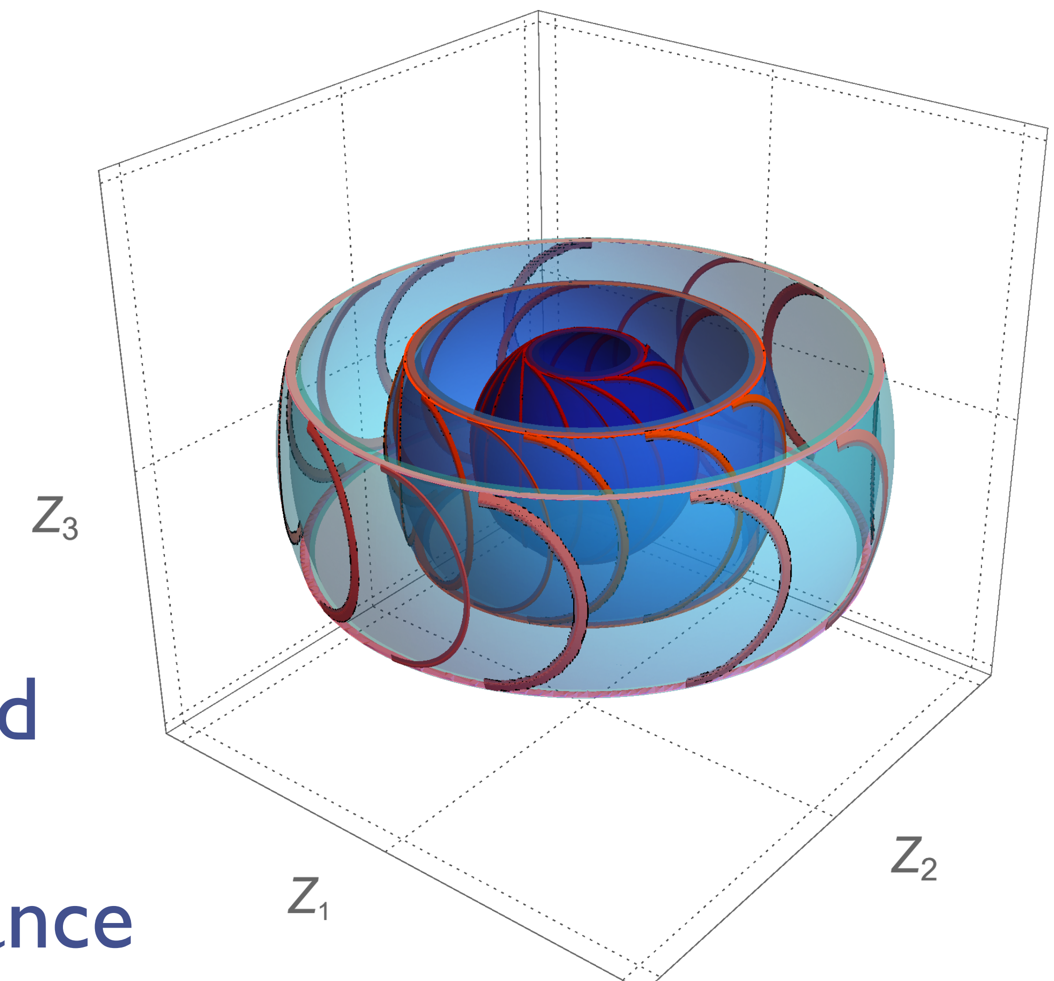
Light cone coordinates

- Full metric solves EE with a CC in vacuum if H(u,x,y) satisfies:

$$\left[\partial_x^2 + \partial_y^2 + \frac{2\Lambda}{3P^2}\right] H(u,x,y) = 0$$

- Singularity of metric at x=0 corresponds to expanding torus in dS and to expanding hyperboloid in AdS.
- In dS, wavefronts are tangent to the expanding torus.
- Copy EOMs are:

$$\bar{\nabla}_\mu F^{\mu\nu} + \frac{\bar{R}}{6} A^\nu = 0 \qquad \bar{\nabla}^2 \phi = 0$$



- Copies correspond to waves in gauge and scalar theory w/ same wavefronts.
- N.B. single copy has broken gauge invariance due to the mass term proportional to the Ricci scalar.

Unusual Example - BTZ Black Hole (AdS-3)

- Metric in KS form w/ Minkowski base metric

$$\bar{g}_{\mu\nu}dx^\mu dx^\nu = -dt^2 + \frac{r^2}{r^2+a^2}dr^2 + (r^2+a^2)d\theta^2$$

$$k_\mu = \left(1, \frac{r^2}{r^2+a^2}, -a\right) \ , \quad \phi = 1 + 8GM + \Lambda r^2$$

- Corresponding single copy satisfies Abelian YM EOMs with source a constant charge density filling all space (using CC density)

$$J^\mu = 4\rho\delta^\mu_0$$

- Non-zero components of the field strength tensor show non-rotating case gives electric field; rotating case yields electric and magnetic fields
- This holds even though there are no gravitons in d=3.
- Seems possible to consider the copy of *geometry*.
- Perhaps unrelated to scattering amplitudes double copy, since no gravitons scattering.

Open Questions and Directions

- Have shown how, in examples, to fix ambiguity in defining copies.
- Extract Maxwell's equations from contraction of Ricci tensor and KV using Einstein equations.
- Require single and zeroth copies obtained to be "reasonable".
- Remains to identify exact property required by null vector and scalar.
- t-indep. and t-dep. copies satisfy different equations
- Reasons for these differences remain to be resolved.

- Would like to extend idea to metrics in a non-Kerr-Schild form.
- Perhaps first to "extended Kerr-Schild form".
- Possible application of our work in curved spacetimes in AdS/CFT.
- Holographic duals to the gravitational AdS solutions may be able to be extended to the CFT side of the duality.
- How exactly does this connect to amplitudes? Is it useful?

- Work on all this underway.

Conclusions

- Have constructed examples of classical double copy in maximally symmetric backgrounds.
- Some black hole copies straightforward extensions of double copy in flat space. Other solutions have more involved interpretations.
- (A)dS-Schwarzschild (and (A)dS-Kerr) single copy corresponds to field sourced by static (rotating) electric charge in (A)dS respectively.
- Black strings and black branes copy to charged lines and charged planes in (A)dS.
- Black hole in AdS-3. Rotating BTZ black hole gives rise to a single copy which produces a magnetic field, even though there are no gravitons in d=3. A copy of the geometry?
- For waves, single copy satisfies Maxwell's equation w/ extra term proportional to Ricci scalar. Zeroth copy EOM is a free scalar field.
- Since not all gravitational features have a copy, remains to be seen whether instabilities of black holes, black strings, or black branes yield instabilities in the gauge and scalar theories.

Thank You!