Induced Ellipticity for Inspiraling Binary Systems

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http://arxiv.org/abs/arXiv:1708.0856

http://arxiv.org/abs/arXiv:1802.057189

Introduction

- Successful detection of black hole mergers
- Rates predicted at tens/year
- What can we learn?
  - Black hole physics
  - But what else? Black hole environment?
- 3 stages: inspiral, merger, ringdown
- Inspiral “chirp” signal calculable
  - So should be gravitational perturbations to it
  - Should exist measurable, calculable differences due to tidal gravitational forces
- **Formation channels might lead to observables**
- Can **tidal effects** teach us about black hole neighborhoods?
  - Galaxy, globular cluster, isolated?

Introduction (cont'd)

- Interesting quantity is eccentricity
- GWs tend to circularize orbits
  - LIGO relies on circular templates
- However, eccentricity can be generated from surrounding matter, and survive even if source only temporary
  - Potentially distinguish GN and SMBH, GC, isolated (natal kick) generation
- So far, studied numerically (Antonini, Perets)
- Here present an analytical method for eccentricity distribution from galactic center black hole
- Account for both tidal forces and evaporation caused by environment

Utility?

- Gives insights into resulting distributions
- Makes it more efficient to probe the origin of the merger by studying distribution of e
- True measure of utility depends on what numbers turn out to be

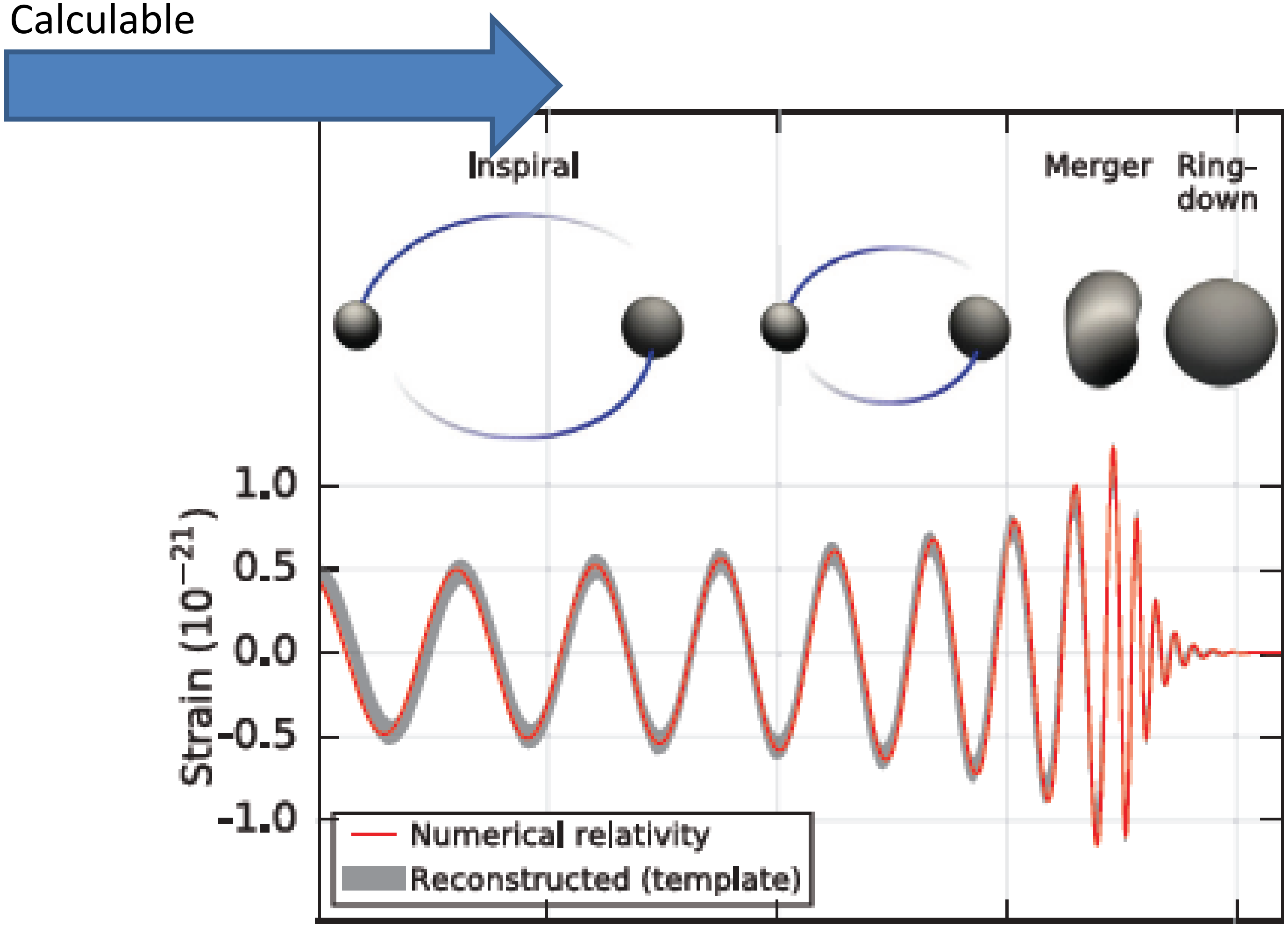
- Formation channels:
  - Isolated
    - Natal kick?
  - Dynamical: GC, SMBH
    - Hierarchical Triples

- Observables:
  - Mass, spin, eccentricity

- Integrate over initial distributions produces eccentricity distribution
  - Numerical
  - Analytical approaches

Merger History

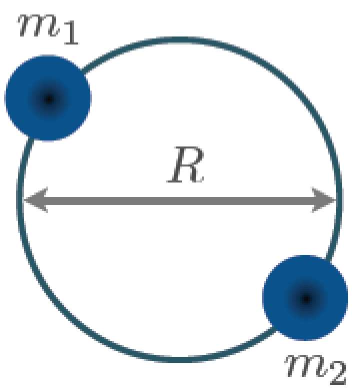
Analytically
Calculable



GW Emission from Inspiraling Binary

$$x = R\cos\omega t \quad y = R\sin\omega t \quad z = 0$$

- Assume circular, fixed orbit, point masses



$$\ddot{M}^{ab} = -2\mu\omega^2 R^2 \begin{pmatrix} \cos 2\omega t & \sin 2\omega t \\ \sin 2\omega t & -\cos 2\omega t \end{pmatrix}$$

$$h_{ij}(t, \mathbf{x}) = \frac{1}{r}\frac{2G}{c^4}\ddot{M}^{\mathrm{TT}}_{ij}(t - r/c)$$

$$\omega^2 = \frac{G(m_1 + m_2)}{R^3}.$$

$$A = \frac{1}{2^{1/3}}\left(\frac{R_c}{r}\right)\left(\frac{R_c}{\lambda}\right)^{2/3},$$

$$R_c = 2GM_c/c^2 \qquad \lambda = c/\omega_{GW}$$

- Chirp mass: $M_c \equiv (m_1 m_2)^{3/5}/(m_1 + m_2)^{1/5},$

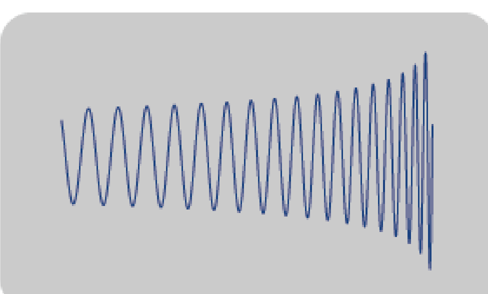
Inspiral from GW

- Radiation power: $P = \frac{32}{5}\frac{c^5}{G}\left(\frac{GM_c\omega_{GW}}{2c^3}\right)^{10/3},$
- Energy: $E = -\frac{Gm_1m_2}{2R} = -\left(\frac{G^2M_c^5\omega_{GW}^2}{32}\right)^{1/3}.$
- Solve $\dot{E} = -P$ for $f_{GW} = \omega_{GW}/(2\pi)$.

$$f_{GW}(\tau) = \frac{1}{\pi}\left(\frac{5}{256}\frac{1}{\tau}\right)^{3/8}\left(\frac{GM_c}{c^3}\right)^{-5/8} \simeq 151\mathrm{Hz}\left(\frac{M_\odot}{M_c}\right)^{5/8}\left(\frac{1\mathrm{s}}{\tau}\right)^{3/8},$$

$\tau = t_{\mathrm{coal}} - t$

$$\tau \simeq 3.00\mathrm{s}\left(\frac{M_\odot}{M_c}\right)^{5/3}\left(\frac{100\mathrm{Hz}}{f_{GW}}\right)^{8/3}.$$



Generalize: Eccentric Orbit

- Orbital frequency no longer constant

$$x = a(\cos u - e), \qquad y = b \sin u, \qquad z = 0,$$

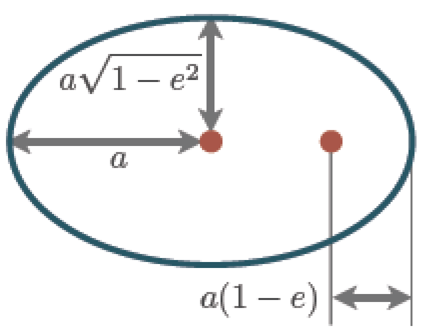
Eccentric anomaly

$$u - e \sin u = \omega_0 t \equiv \beta,$$

Polar coordinates

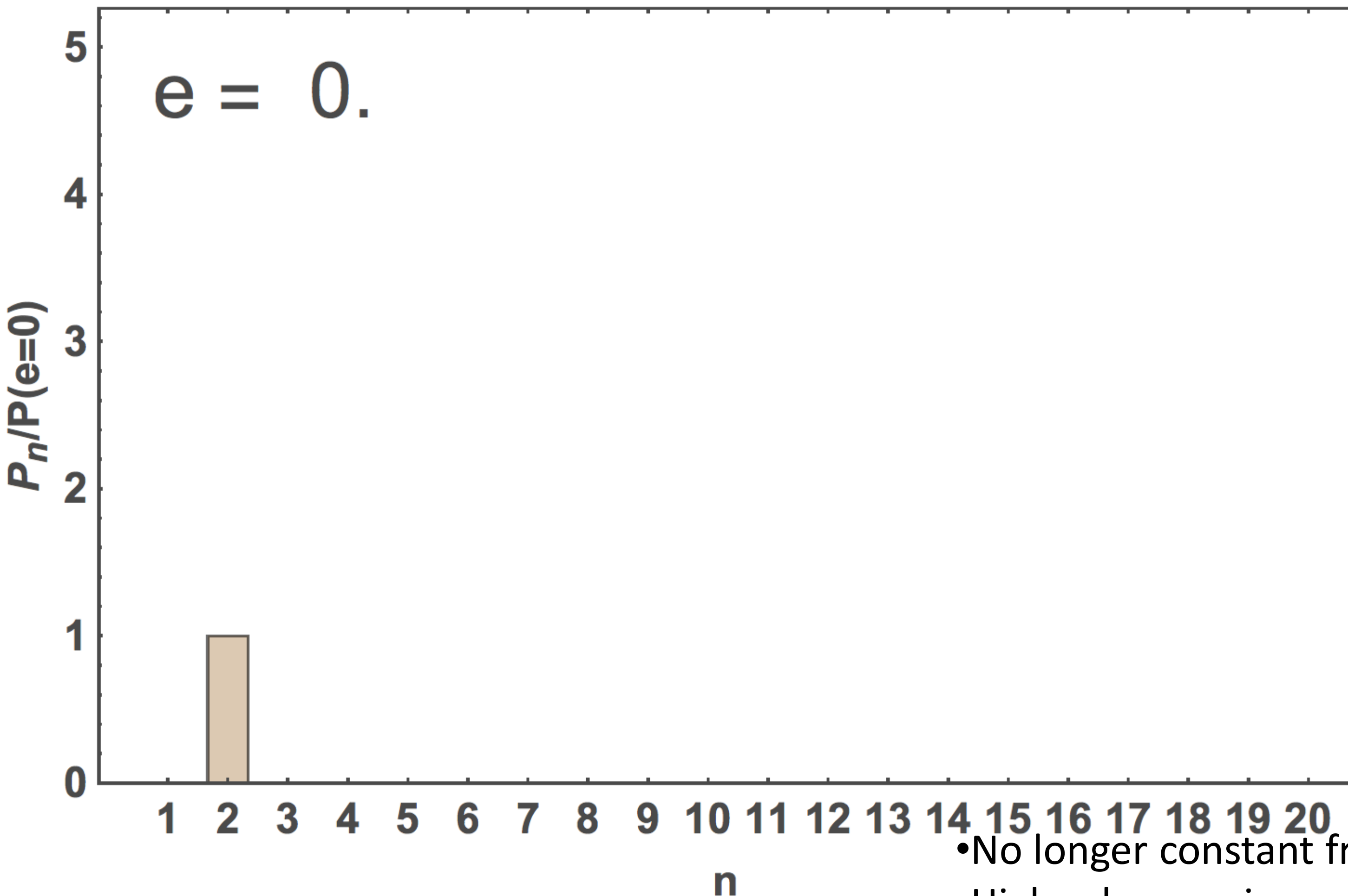
$$\frac{1}{r} = \frac{1 + e \cos \psi}{R}.$$

$$a = \frac{R}{1-e^2}, \qquad b = \frac{R}{\sqrt{1-e^2}}, \qquad \cos \psi = \frac{\cos u - e}{1 - e \cos u};$$



$$E = -\frac{G\mu m}{2a} \qquad J = \mu \sqrt{Gma(1-e^2)}$$

Sound and Shape of Eccentricity



- No longer constant frequency
- Higher harmonics
- Quadrupole dominates for small e
- Large e: $f_{\rm peak}(a,e) \simeq \dfrac{\sqrt{Gm}}{[a(1-e^2)]^{3/2}}$

Eccentricity loss during infall

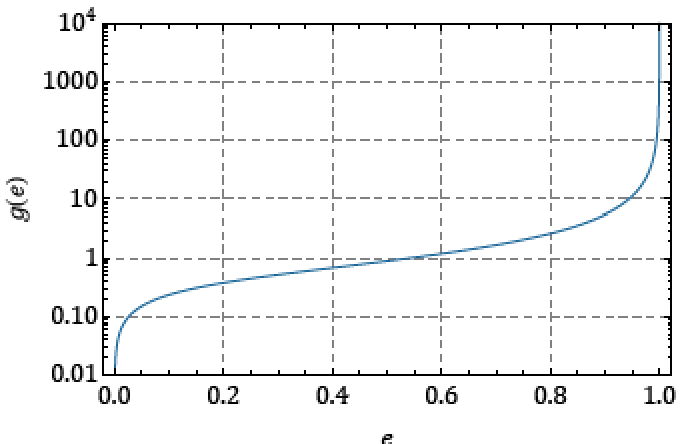
- Use dJ/dt, dE/dt from GW to derive
- da/dt, de/dt =>a(e)

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\frac{64}{5}\frac{G^3\mu m^2}{c^5 a^3}\frac{1}{(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right),$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = -\frac{304}{15}\frac{G^3\mu m^2}{c^5 a^4}\frac{e}{(1-e^2)^{5/2}}\left(1+\frac{121}{304}e^2\right).$$

$$\Rightarrow \quad a(e) = a_0\frac{g(e)}{g(e_0)},$$

$$g(e) = \frac{e^{12/19}}{1-e^2}\left(1+\frac{121}{304}e^2\right)^{870/2299} \simeq \begin{cases} e^{12/19} & e \ll 1 \\ \dfrac{1.1352}{1-e^2} & e \lesssim 1 \end{cases}$$



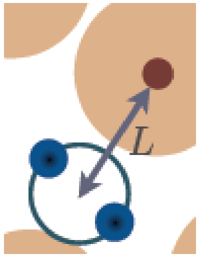
Note base frequency ~$1/a^{3/2}$

a depends on e so even base frequency dependence reflects eccentriity

Measurable?

- Large eccentricity: faster merger
  - Closer together
  - Higher harmonics
- Small eccentricity
  - Can measure at small eccentricity, even if merger began with large e
  - Detailed measurement of waveform
- Question become: can we drive eccentricity to larger values that survive into LIGO window?
- Assume e~o(.01) can be measured

Drive e with Point Source Tidal Force:Kozai Lidov



- Perturb: $$F_{\mathrm{tidal}} \simeq \frac{GMmR}{L^3}$$

- F_t/mv~ $$\omega_T \equiv \left.\frac{\mathrm{d}e}{\mathrm{d}t}\right|_{\mathrm{tidal}} \simeq \sqrt{\frac{GM^2R^3}{mL^6}}.$$

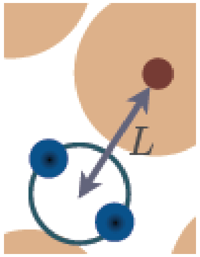
•Compare $$\frac{\omega_T}{\omega} = \frac{M}{m}\left(\frac{a}{L}\right)^3 = \left(\frac{\Omega}{\omega}\right)^2$$

•Rate of change smaller than both inner and outer orbital frequencies; perturbative

Tidal generation of eccentricity

- Competing effects
  - Gravitational wave emission is constant
  - Need coherent generation of eccentricity
  - Tidal force constant if nearby third body
- Need a hierarchical triple
  - otherwise unstable
- Can exist in cosmos
  - Galactic nuclei with SMBH
  - Dense globular clusters (binary-binary scattering)

Rate:Tidal modulation and GW modulation



$$\left.\frac{\mathrm{d}e}{\mathrm{d}t}\right|_{\mathrm{GW}} = -\frac{152}{15}\frac{G^3m^3}{c^5a^4}\frac{e}{(1-e^2)^{5/2}}\left(1+\frac{121}{304}e^2\right)$$

$$\simeq -\omega_C\frac{e}{(1-e^2)^{5/2}},$$

$$\left.\frac{\mathrm{d}e}{\mathrm{d}t}\right|_{\mathrm{tidal}} = \frac{15}{16}\sqrt{\frac{GM^2a^3}{2mL^6}}e(1-e^2)^{1/2}(1-\cos^2 I)\sin 2\gamma$$

$$\simeq \omega_T e(1-e^2)^{1/2}$$

$$\omega_C \equiv \frac{10G^3m^3}{c^5a^4} \qquad \omega_T \equiv \sqrt{\frac{GM^2a^3}{2mL^6}}$$

Tidal Sphere of Influence

- Comparing rates of GW-circularization and tidal effect

$$\frac{\omega_T}{\omega} = \frac{M}{m}\left(\frac{a}{L}\right)^3 = \left(\frac{\Omega}{\omega}\right)^2$$ <1

$$\frac{\omega_T}{\omega_C} \simeq \frac{M}{m}\left(\frac{a}{R_m}\right)^{5/2}\left(\frac{a}{L}\right)^3$$ >1

- Sufficiently large *a* : tidal modulation fast enough. Find critical separation—after GW only

$$\dot{e}\big|_{\rm tidal} = \dot{e}\big|_{\rm GW}$$

$$L_i = a\left(\frac{M}{m}\right)^{1/3}\left(\frac{a}{R_m}\right)^{5/6}(1-e^2)$$

$$a_i = \left[L^6 R_m^5 \left(\frac{m}{M}\right)^2 \frac{1}{(1-e_i^2)^6}\right]^{1/11}$$

Kozai-Lidov resonance : coherent generation

Interchange between inclination and eccentricity

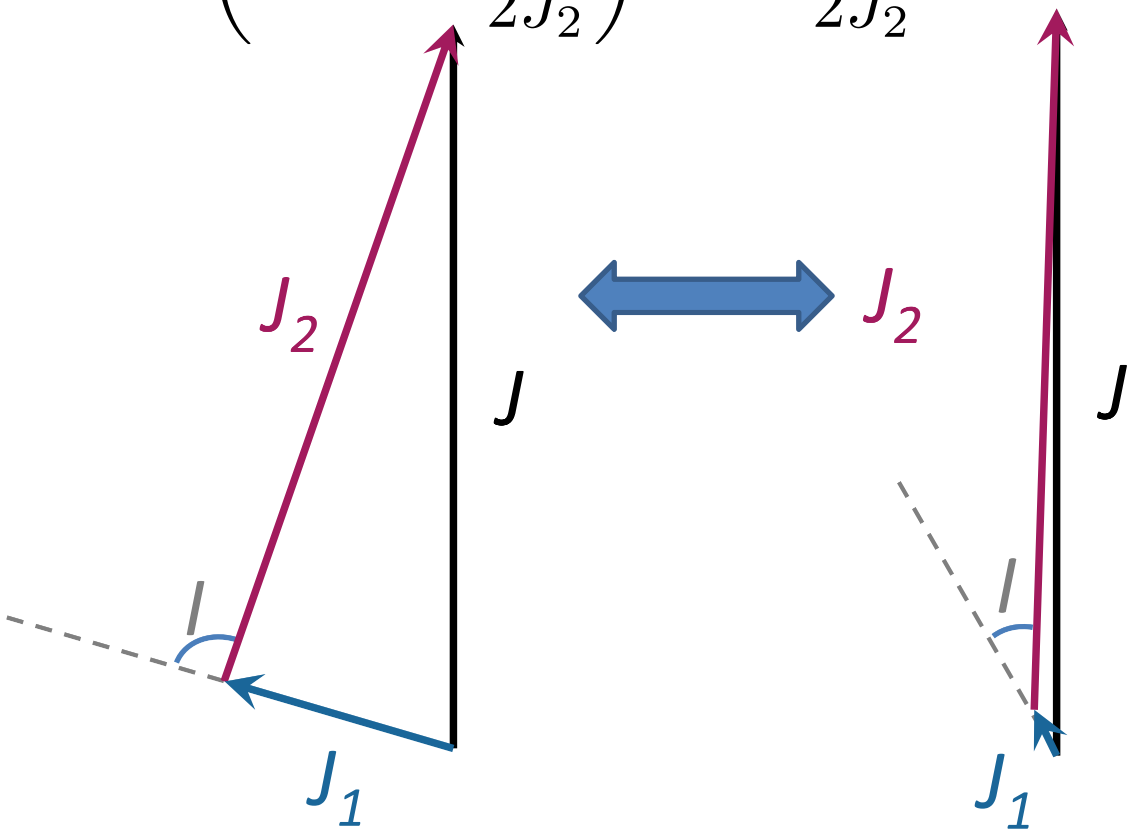
$$J_1\left(\cos I + \frac{J_1}{2J_2}\right) = \frac{J^2 - J_2^2}{2J_2} = \mathrm{const.}$$

$$J_1 = \mathrm{const.} \times \sqrt{1 - e_1^2}$$

|J|=const.

$|J_2|$=const.

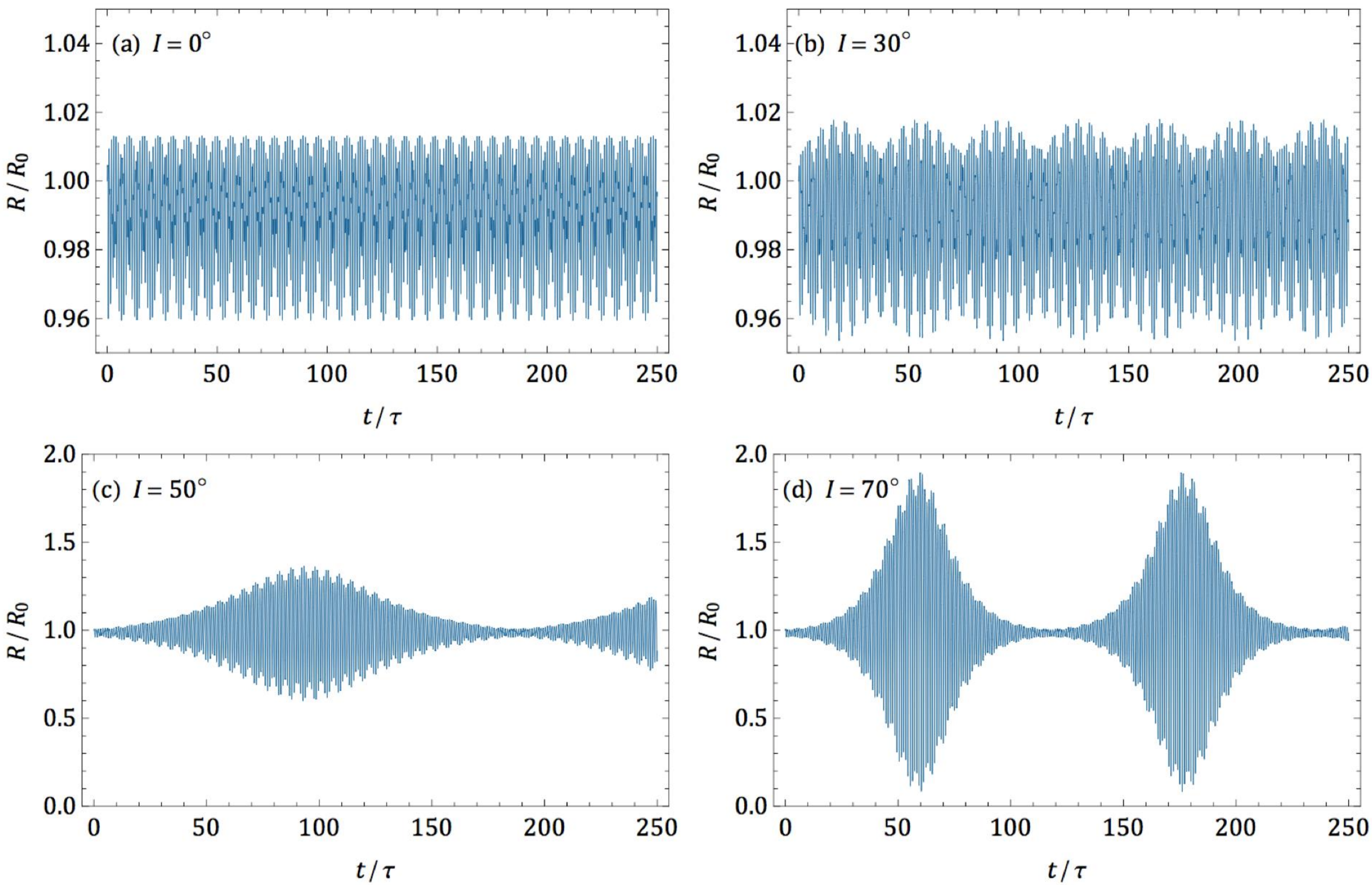
$|J_1| \propto (1-e^2)^{1/2}$



highly inclined

highly eccentric

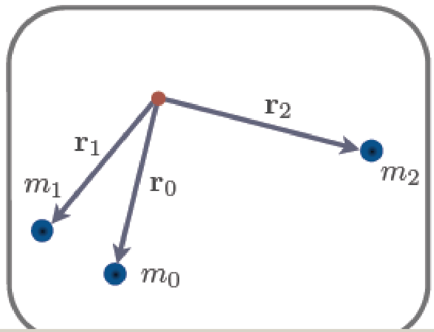
Critical Angle for Eccentricity to Develop Need High Inclinatoin



Can we find an analytical solution

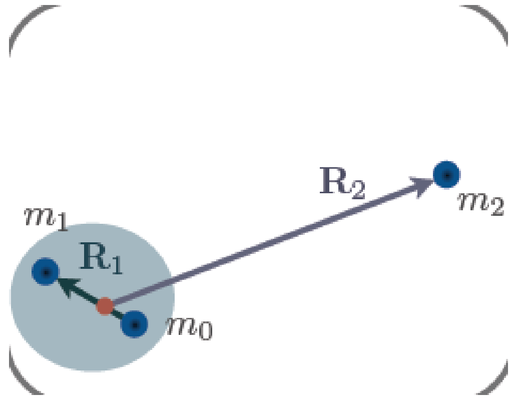
- Analytical solution at least in principle lets us relate measurable quantity (e) directly to parameters of environment in which BBH formed
- Distribution of e depends on initial parameters
- With solution, don’t need to numerically scan over all parameters
- Can directly relate to density distribution

Three-Body Systems
We are interested in hierarchical triples



$$H = \frac{1}{2m_0}|\mathbf{p}_0|^2 + \frac{1}{2m_1}|\mathbf{p}_1|^2 + \frac{1}{2m_2}|\mathbf{p}_2|^2 - \frac{Gm_0m_1}{|\mathbf{r}_0 - \mathbf{r}_1|} - \frac{Gm_0m_2}{|\mathbf{r}_0 - \mathbf{r}_2|} - \frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

Jacobi Coordinates: Hierarchical

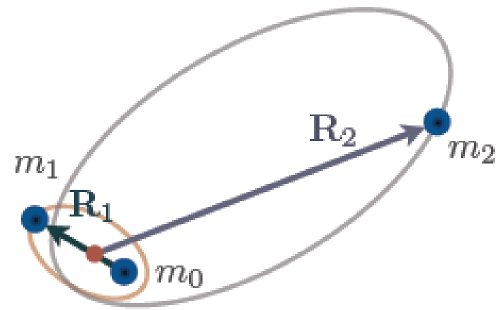


$$m = m_0 + m_1$$
$$M = m + m_2$$
$$\mu_1 = \frac{m_0 m_1}{m}$$
$$\mu_2 = \frac{m m_2}{M}$$

$$\mathrm{R} = \frac{1}{M}(m_0 \mathbf{r}_0 + m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2)$$
$$\mathrm{R}_1 = \mathbf{r}_1 - \mathbf{r}_0$$
$$\mathrm{R}_2 = \mathbf{r}_2 - \frac{1}{m}(m_0 \mathbf{r}_0 + m_1 \mathbf{r}_1)$$

$$\mathbf{P} = M\dot{\mathbf{R}}$$
$$\Pi_1 = \mu_1 \dot{\mathbf{R}}_1$$
$$\Pi_2 = \mu_2 \dot{\mathbf{R}}_2$$

Exploit Hierarchy: Orbit-Orbit Coupling and Multipole Expansion



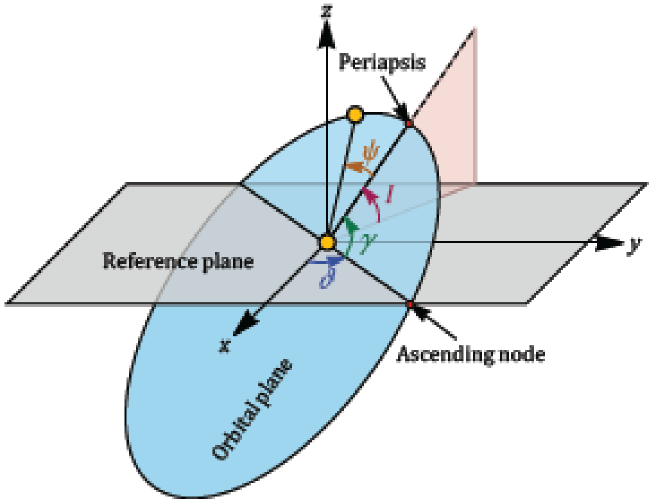
$$H = \frac{1}{2M}|\mathbf{P}|^2 + \left(\frac{1}{2\mu_1}|\mathbf{\Pi}_1|^2 - \frac{Gm\mu_1}{|\mathbf{R}_1|}\right) + \left(\frac{1}{2\mu_2}|\mathbf{\Pi}_2|^2 - \frac{GM\mu_2}{|\mathbf{R}_2|}\right) + H'$$

$$H' = \frac{Gmm_2}{|\mathbf{R}_2|} - \frac{Gm_0m_2}{|\mathbf{R}_2 + \frac{m_1}{m}\mathbf{R}_1|} - \frac{Gm_1m_2}{|\mathbf{R}_2 - \frac{m_0}{m}\mathbf{R}_1|}$$

$$H' = \frac{Gmm_2}{|\mathbf{R}_2|} - \frac{Gm_0m_2}{|\mathbf{R}_2 + \frac{m_1}{m}\mathbf{R}_1|} - \frac{Gm_1m_2}{|\mathbf{R}_2 - \frac{m_0}{m}\mathbf{R}_1|} = \sum_{\ell=2}^{\infty} H^{(\ell)}$$

$$H^{(2)} = -\frac{Gm_0m_1m_2}{2m}\frac{R_1^2}{R_2^3}(3\cos^2\varphi - 1)$$

Quadrupole: Integrable System



Angles to characterize both orbits
Angles to characterize relative orbital planes
Average over orbits

$$H^{(2)} = -\frac{Gm_0m_1m_2}{2m}\frac{R_1^2}{R_2^3}(3\cos^2\varphi - 1)$$

$$\overline{H}^{(2)} = -K(W + \frac{5}{3})$$

$$K \equiv \frac{3Gm_0m_1m_2}{8m}\frac{a_1^2}{a_2^3(1-e_2^2)^{3/2}}$$

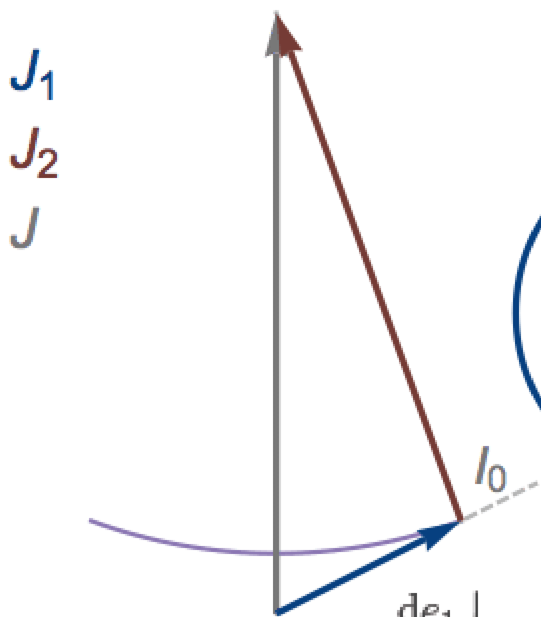
$$W \equiv (-2+\cos^2 I)(1-e_1^2) + 5e_1^2(\cos^2 I - 1)\sin^2\gamma$$

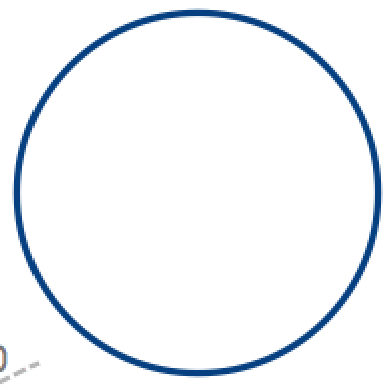
(Lidov 1961, 1962; Kozai 1962; Lidov, Ziglin 1976)

Interchange

Conserved:

$$E, \mathrm{J}, E_1, E_2, J_2 \rightarrow a_1, a_2, e_2$$



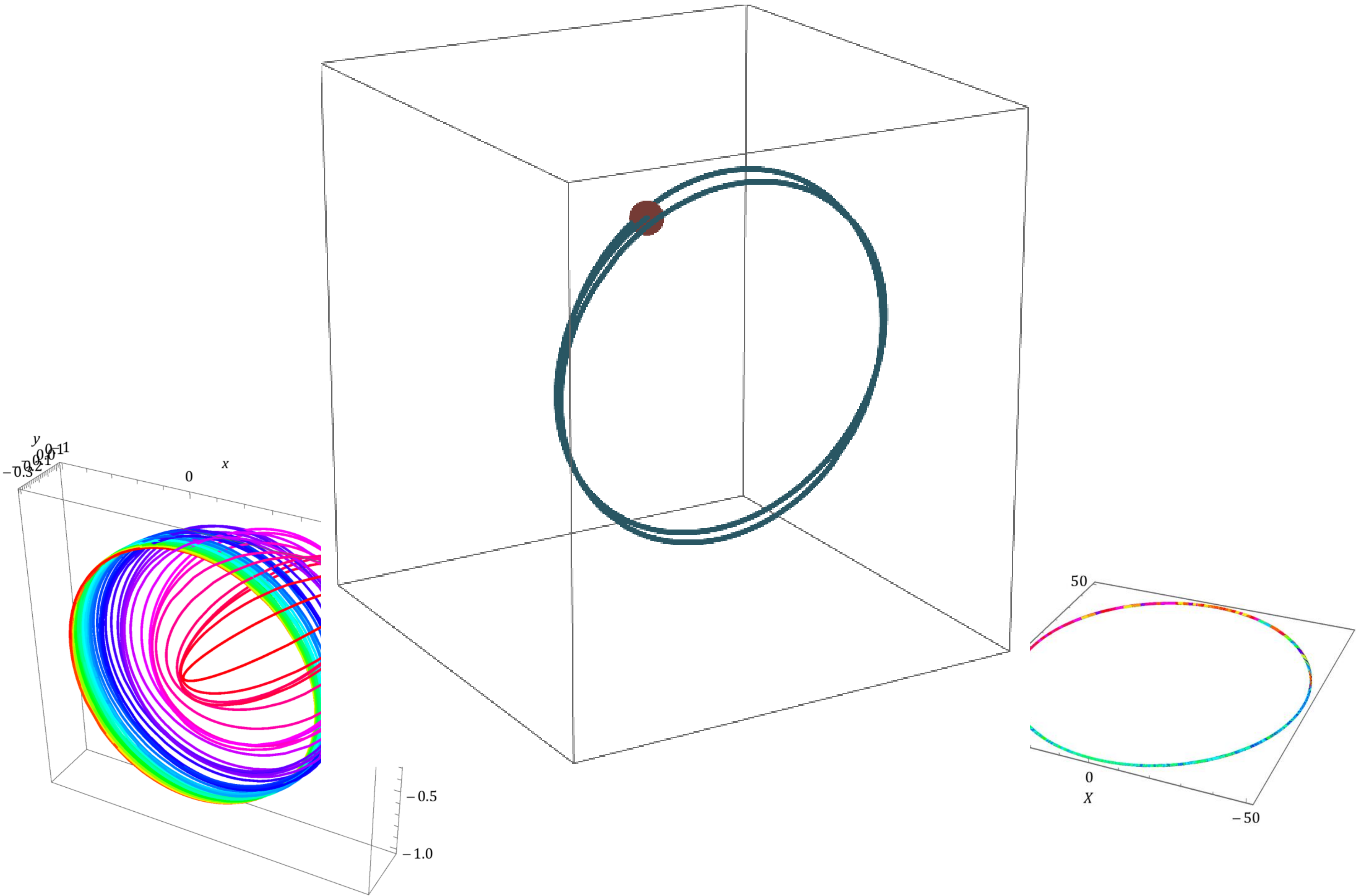


Dynamical:conjugate

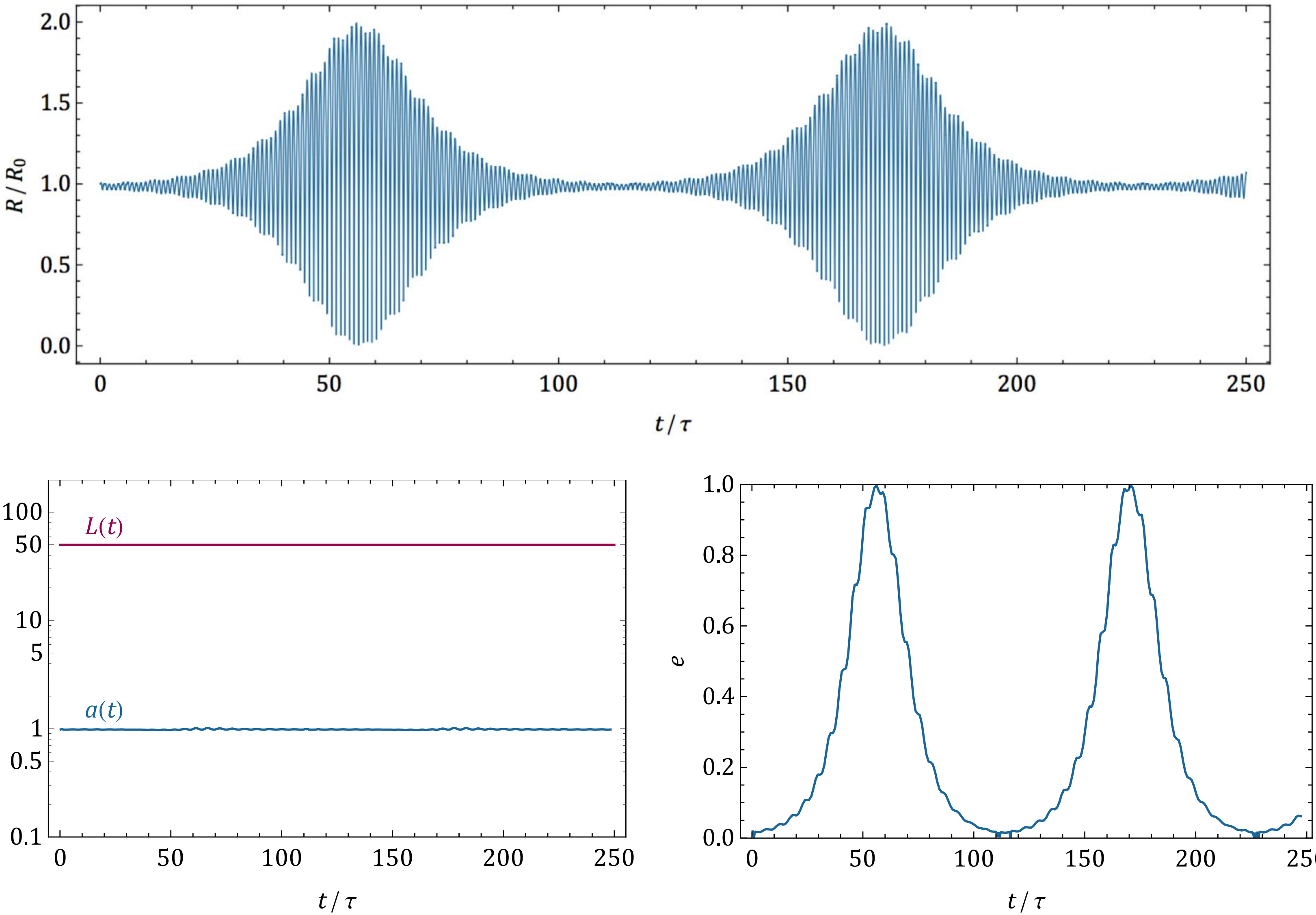
$$\left.\frac{\mathrm{d}e_1}{\mathrm{d}t}\right|_{\mathrm{KL}} = \frac{5K}{J_{\gamma 1}} e_1(1-e_1^2)(1-\cos^2 I)\sin 2\gamma_1$$

$$\left.\frac{\mathrm{d}\gamma_1}{\mathrm{d}t}\right|_{\mathrm{KL}} = 2K\left[\frac{1}{J_{\gamma 1}}\left(2(1-e_1^2) - 5(1-e_1^2-\cos^2 I)\sin^2\gamma_1\right) + \frac{1}{J_{\gamma 2}}\left(1-e_1^2+5e_1^2\sin^2\gamma_1\right)\cos I\right]$$

Argument of periapsis



y
x
0
−0.5
−1.0
50
0
X
−50



R / R0
t / τ
L(t)
a(t)
a / R0, L / R0
t / τ
e
t / τ

Does eccentricity survive to LIGO?

- Tidal modulation increases or decreases e
- Rate slower than orbital frequencies
  - Many orbits while e develops
- But GW always decreases it
- Need tidal effect to work fast enough that GW won't erase it
- Want tidal modulation frequency greater than circularization from GW rate

Tidal and GW

- Don’t expect KL indefinitely
- GW becomes important
- PN effect destroys resonance and allows GW to take over
  - No longer in tidal sphere of influence
- Want to know how much eccentricity remains

So how much e remains?

- Enters LIGO window

$$a_{\rm LIGO} = \left(\frac{Gm}{f_{\rm LIGO}^2}\right)^{1/3} \frac{1}{1-e^2}.$$

  Compare to binary orbit size when tidal force no longer dominates
- Follow inspiral to LIGO a due to GW analytically
- Need “initial” e distribution: note independent of background density profile so just one function
- Then can find how much e lost as it inspirals

In fact can do better

- Include PN and GW explicitly

$$\left.\frac{d\gamma_1}{dt}\right|_{PN} = \frac{3}{c^2 a_1(1-e_1^2)}\left(\frac{Gm}{a_1}\right)^{3/2}$$

$$H_{PN} = -\frac{K\Theta_{PN}}{\sqrt{1-e_1^2}}$$

$$\Theta_{PN} = \frac{8Gm^2 a_2^3(1-e_2^2)^{3/2}}{c^2 m_2 a_1^4}$$

Useful to have conservative Hamiltonian description

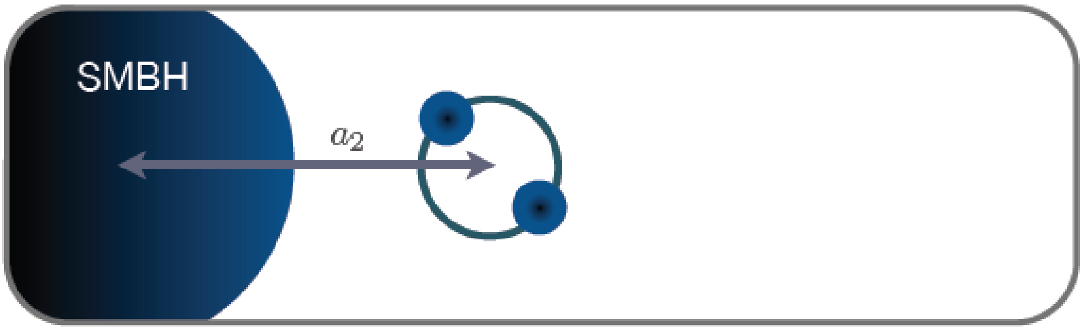
GW (Peters Equation) as before: E, J no longer conserved

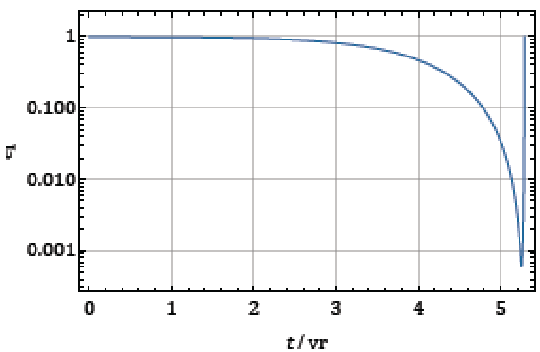
$$\left.\frac{da_1}{dt}\right|_{GW} = -\frac{64}{5}\frac{G^3\mu_1 m^2}{c^5 a_1^3}\frac{1}{(1-e_1^2)^{7/2}}\left(1+\frac{73}{24}e_1^2+\frac{37}{96}e_1^4\right)$$

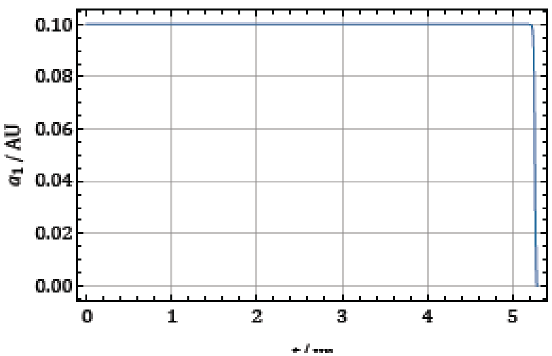
$$\left.\frac{de_1}{dt}\right|_{GW} = -\frac{304}{15}\frac{G^3\mu_1 m^2}{c^5 a_1^4}\frac{e_1}{(1-e_1^2)^{5/2}}\left(1+\frac{121}{304}e_1^2\right)$$

Critical to calculation that change in orbital radius dominated by large eccentricity region

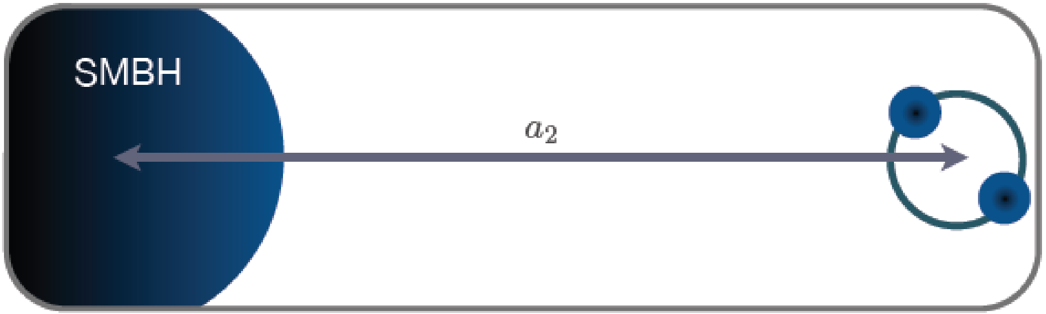
Case we don't calculate: fast merger

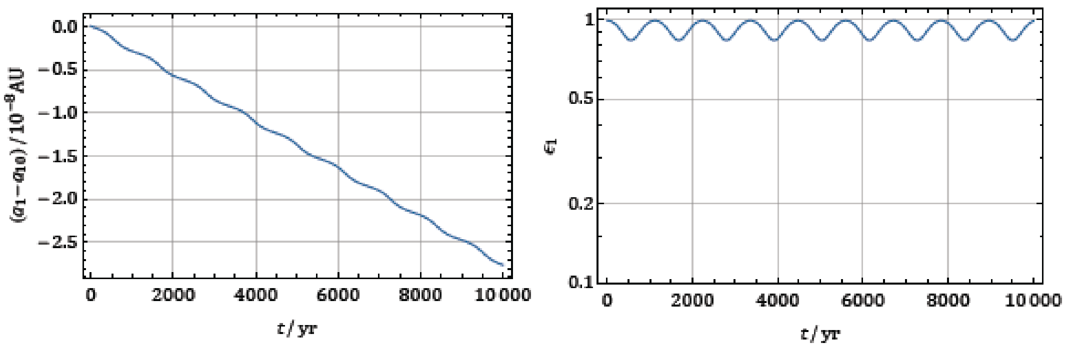




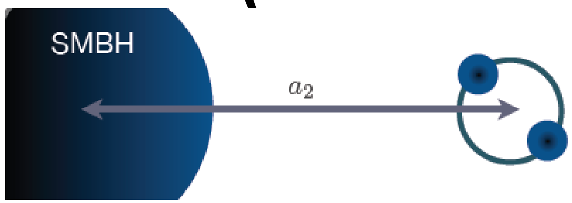


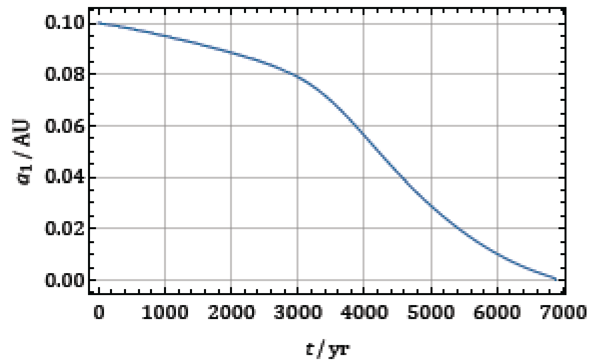
Case we don't consider here: isolation limit

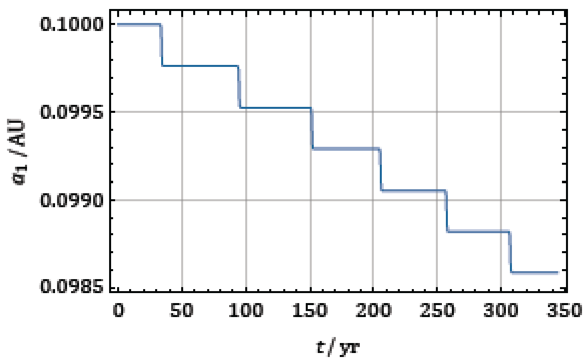




We calculate:
KL-boosted (but several cycles)







Find lifetime of fictitious binary with the max e

Correct for amount of time spent with that e $\sqrt{1-e_{1\max}^2}$ of time near $e_{1\max}$

Merger Time

$$\tau_{\rm slow} \simeq \frac{\tau_{\rm iso}}{\sqrt{1-e_{1\max}^2}} \simeq \frac{5}{256}\frac{c^5 a_{10}^4}{G^3 m_0 m_1 m}(1-e_{1\max}^2)^3$$

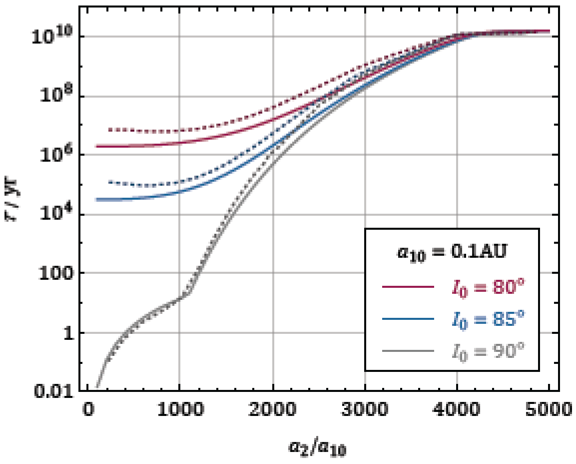
$e_{1\max}$ can be calculated from the conservation of E & J

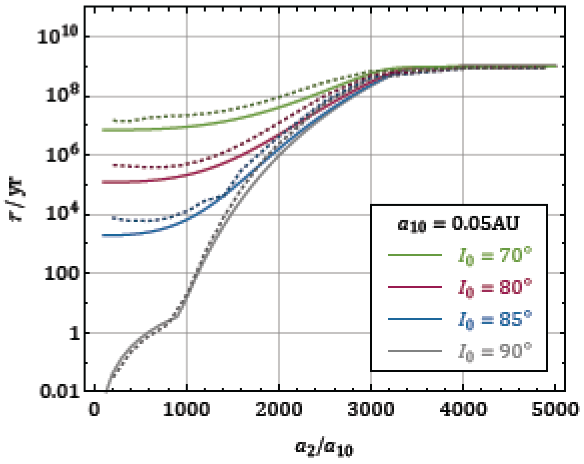
$$H(e_{10},\gamma_{10}) = H(e_{1\max},\gamma_1=\pi/2)$$

Use PN Hamiltonian formulation here…

Comparison with numerical results

(Antonini & Perets 2012)

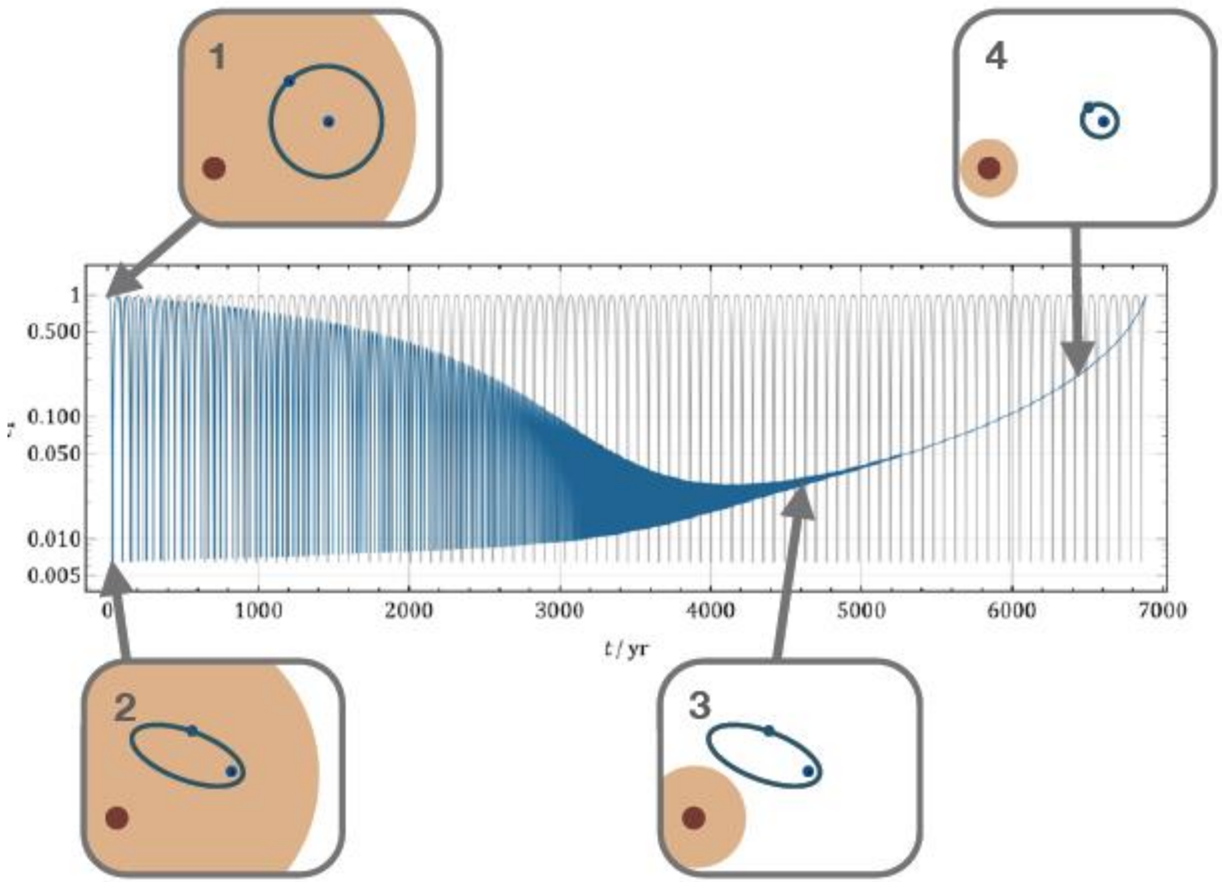




Works well!!

What about eccentricity?

- Now that we know merger time can postulate an isolated binary with that merger time, mass, and initial semi-major axis
- Eccentricity distribution follows that of the isolated one in the end-- where KL turned off



Explicitly…

$$\tau_{\rm slow} \simeq \frac{\tau_{\rm iso}}{\sqrt{1-e_{1\max}^2}} \simeq \frac{5}{256}\frac{c^5 a_{10}^4}{G^3 m_0 m_1 m}(1-e_{1\max}^2)^3$$

$$\frac{5}{256}\frac{c^5 a_{10}^4}{G^3 m_0 m_1 m}\widehat{\epsilon}_1^{\,7/2} = \tau = \frac{5}{256}\frac{c^5 a_{10}^4}{G^3 m_0 m_1 m}\epsilon_{1\min}^3$$

$$\rightarrow \quad \boxed{\widehat{\epsilon}_1 = \epsilon_{1\min}^{6/7}} \qquad (\epsilon \equiv 1-e^2)$$

- Eccentricity after KL damped

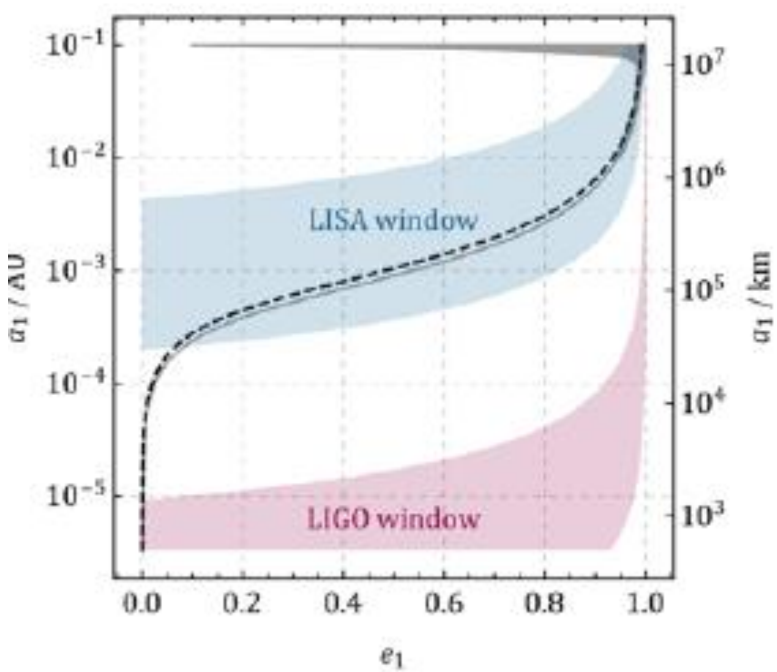
$$e_1 = g^{-1}\left[\frac{a_1}{a_{10}} g\left(\sqrt{1-\epsilon_{1\min}^{6/7}}\right)\right]$$

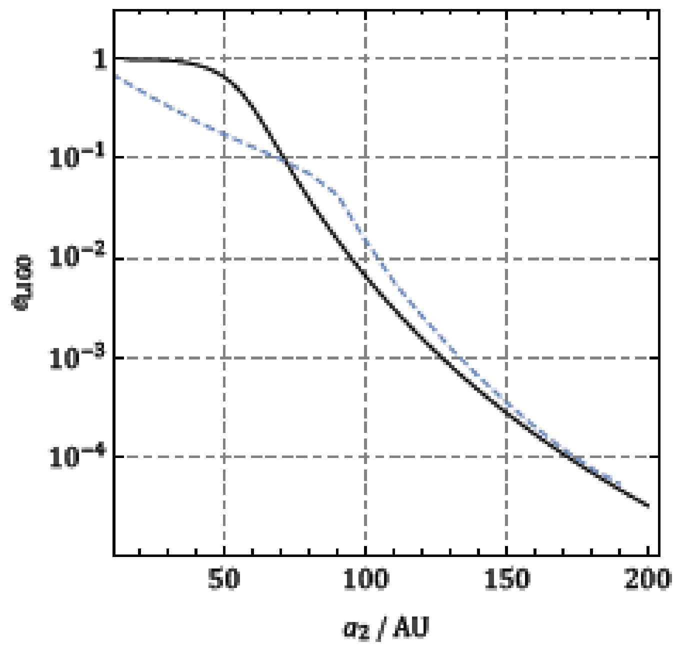
- At the LIGO threshold

$$e_{\rm LIGO} = g^{-1}\left[\frac{a_{\rm LIGO}}{a_{10}} g\left(\sqrt{1-\epsilon_{1\min}^{6/7}}\right)\right]$$

$$a_{\rm LIGO} \simeq 513{\rm km} \times (m/M_\odot)^{1/3}$$

Comparison to numerical results

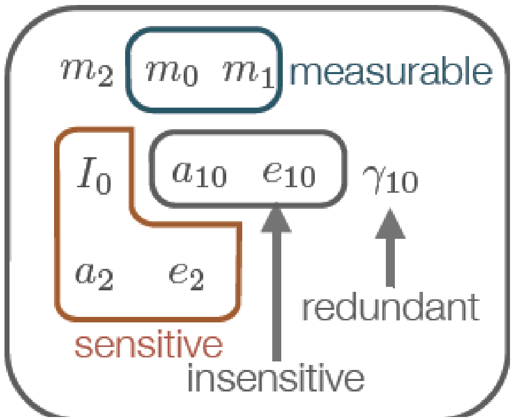




Works well away from large e

What to do with this result?

Lots or parameters
Only a few relevant



$a_2^{2-\beta} \mathrm{d}a_2$ with $\beta = 2$ for the cusp model and $\beta = 0.5$ for the core model.

Make some assumptions: hopefully test in the end

$f(I_0)\mathrm{d}I_0 \propto \mathrm{d}\cos I_0$ $\quad\quad$ $\propto \mathrm{d}e_2^2,$ $\quad\quad$ thermal

Distribution in a2 tells us about density distribution of black holes--origin

Core vs cusp: $a_2^{2-\beta} \mathrm{d}a_2$ with $\beta = 2$ for the cusp model and $\beta = 0.5$ for the core model.

Additional constraints: Evaporation and Tidal Disruption

- This was all for an isolated binary in presence of BH
- In reality, binary inside galaxy
- Evaporation can occur: depends on L
- To date, competition done with simulation
- In first analysis we used a cutoff L beyond which evaporation dominates
- Now with analytical result, we can compare to analytical result for evaporation

- We also require no tidal disruption from SMBH

Evaporation and disruption

- Evaporation of binaries by scattering with ambient matter: require merge, not evaporate

$$\sigma_{\rm evap} = \frac{40\pi G}{3}\frac{m_\star^2}{m}\frac{a}{v_\star^2}$$ (Hut, 1983)

$$\tau_{\rm evap}^{-1} = n_\star \langle \sigma_{\rm evap} v_\star \rangle = \frac{40\pi G a}{3}\frac{\rho_\star m_\star}{m}\langle v_\star^{-1}\rangle$$

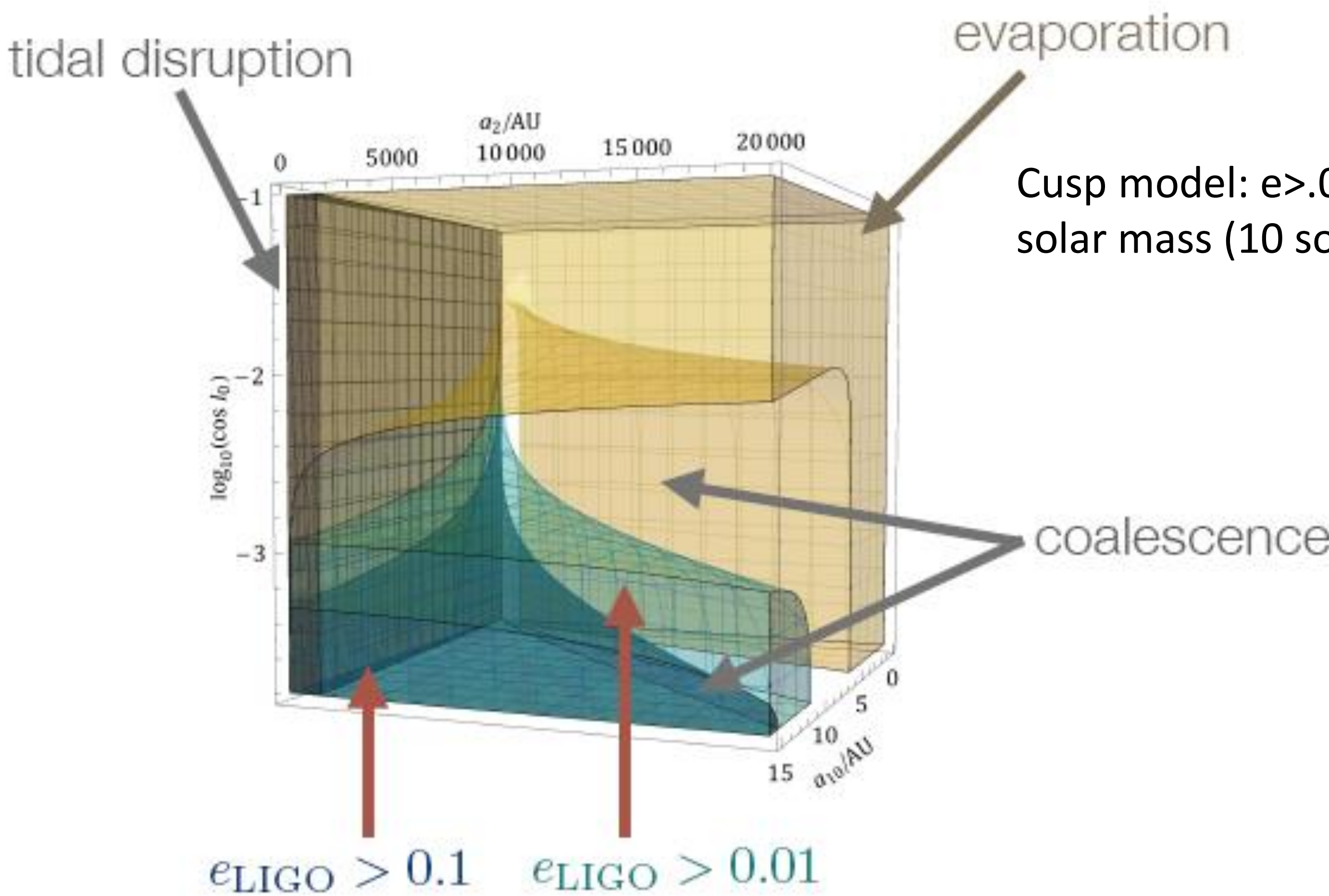
$$\tau_{\rm evap} = \frac{3m\bar{v}}{40\sqrt{2\pi}G\rho_\star m_\star a_{10}} \qquad \bar{v}^2 \sim \frac{GM_{\rm SMBH}}{a_2}$$

$$\tau < \tau_{\rm evap}$$

Tidal disruption constraint:

$$a_2(1-e_2) > \left(\frac{4m}{m_2}\right)^{1/3} a_{10}$$

Sample Result with all Constraints



Cusp model: e>.01: 5% (25%) for solar mass (10 solar mass) objects

Should occur at measurable rate

Can in principle use to distinguish different density distributions

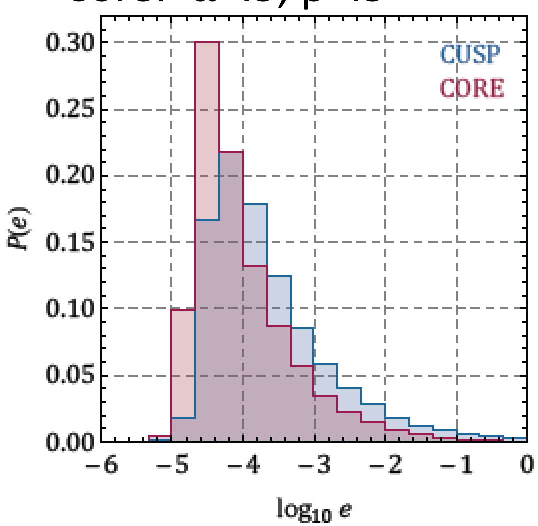
- Eg Core vs Cusp, Different masses

Background and bh distributions: bh number density, background matter density

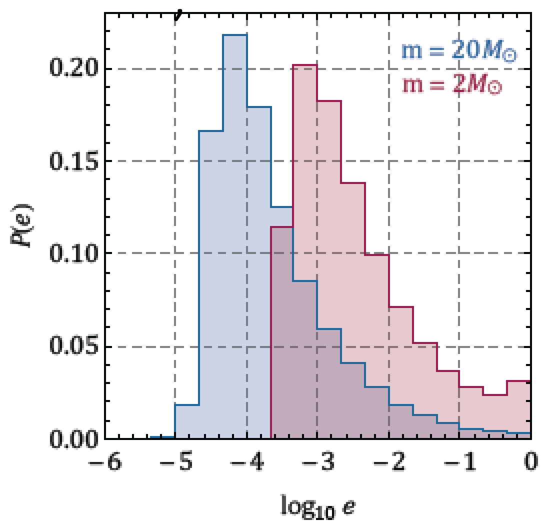
$a_2^{2-\beta}\mathrm{d}a_2$

$$\rho_\star(a_2) = \rho_0 \left(\frac{a_2}{a_{20}}\right)^{-\alpha}$$

Cusp: α=7/4, β=2;
Core: α=.5, β=.5



α=7/4, β=2, α=7/4, β=7/4



Also some analytical understanding of dependencies

$$e_{\rm LIGO} = g^{-1}\left[\frac{a_{\rm LIGO}}{a_{10}} g\left(\sqrt{1-\epsilon_{\rm 1min}^{6/7}}\right)\right]$$

$$e_{\rm LIGO} \simeq 1.22\left(\frac{a_{\rm LIGO}}{a_{10}}\right)^{19/12} \epsilon_{\rm 1min}^{-19/14}$$

Big initial e, small final e

Very large I

$$e_{\rm LIGO} \simeq 0.01\left(\frac{20M_\odot}{m}\right)^{1235/252}\left(\frac{m_2}{4\times 10^6 M_\odot}\right)^{19/7}\left(\frac{a_{10}}{0.1{\rm AU}}\right)^{779/84}\left(\frac{100{\rm AU}}{b_2}\right)^{57/7}$$

Vs smaller I and suppressed PN

$$e_{\rm LIGO} \simeq 0.01\left(\frac{m}{20M_\odot}\right)^{19/36}\left(\frac{0.1{\rm AU}}{a_{10}}\right)^{19/12}\left(\frac{\cos 88.8^\circ}{\cos I_0}\right)^{19/7}$$

Interesting that m, a dependence reversed
In end, first case dominates: stronger dependence and more of parameter space

Early stages but promising

- Analytical result means we don't have to calculate e distribution numerically
- Only numerics is integrating over initial parameters
  - No Monte Carlo
- Will however require lots of statistics in end
- Also sometimes near SMBH, sometimes isolated (natal kicks), sometimes GN
- We want to find ways to distinguish options
- Or disentangle components
- Clearly information is there
  - Want to know where black holes come from
  - Distributions of matter surrounding them
  - Ultimately is it standard or nonstandard
- Goal to retrieve the information
- Early stages so hopeful!

- Thank you