Gradient and Vorticity Banding Phenomena in a Sheared Granular Fluid: Order Parameter Description

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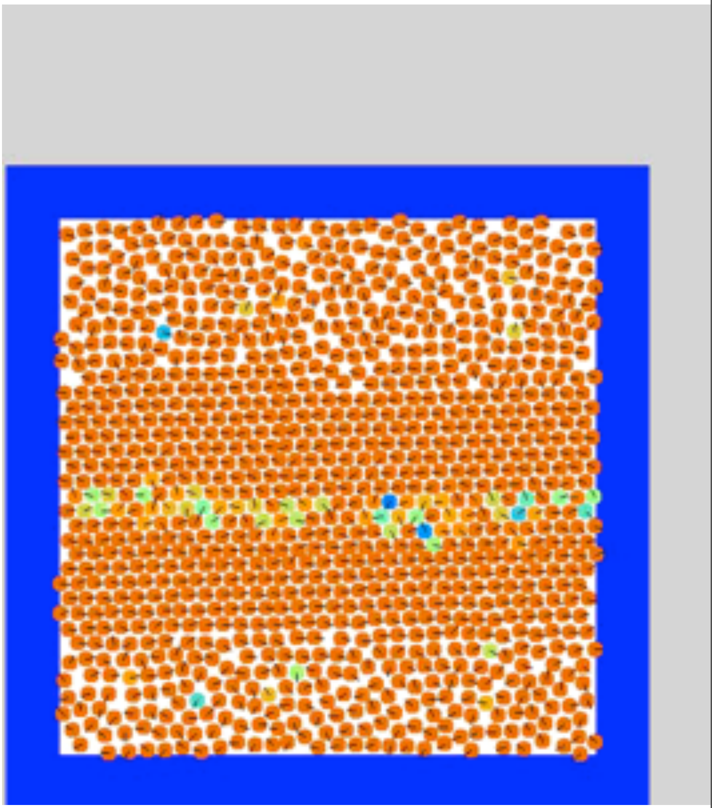
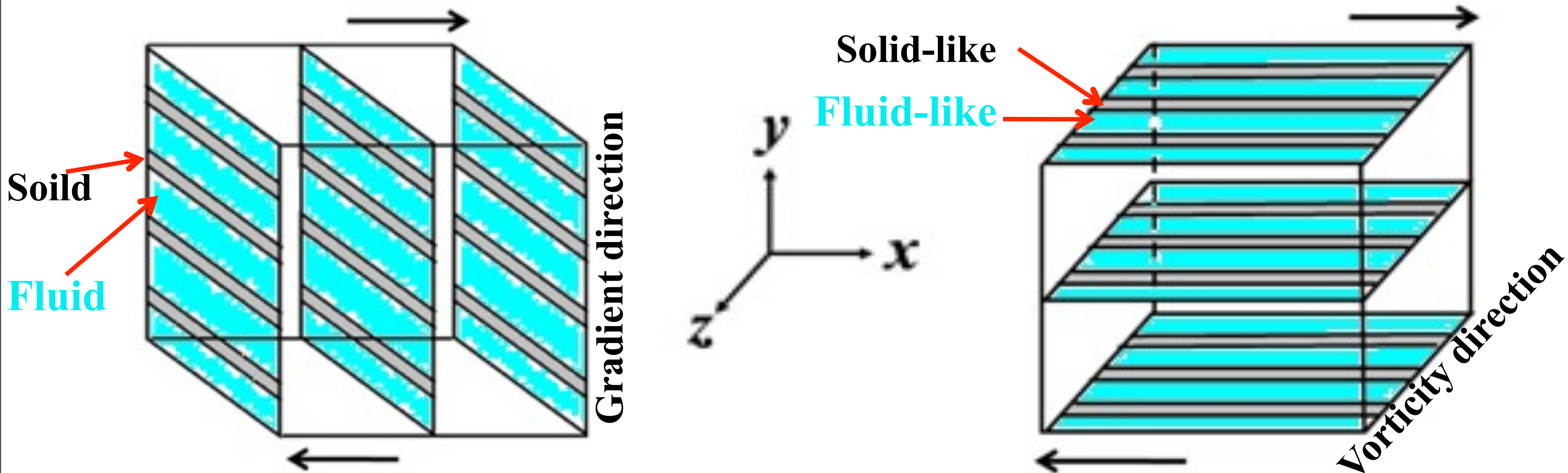
Outline of Talk

- **Shear-banding phenomena in granular and complex fluids**
- **Gradient Banding and Patterns in 2D granular PCF (Landau-Stuart Eqn.)**
- **Vorticity Banding in 3D gPCF**
- **Theory for Mode Interactions**
- **Spatially Modulated Patterns (Ginzburg-Landau Eqn.)**
- **Summary and Outlook**



Shear-banding ?

Sheared granular material (or any complex fluid) does not flow homogeneously like a simple fluid, but forms banded regions having inhomogeneous gradients in hydrodynamic fields.



Gradient Banding

Vorticity Banding

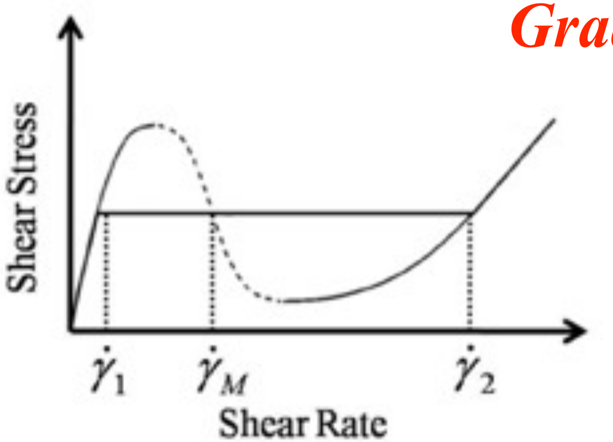
Origin of Shear-banding?

Multiple Branches in Constitutive Curve

⇩

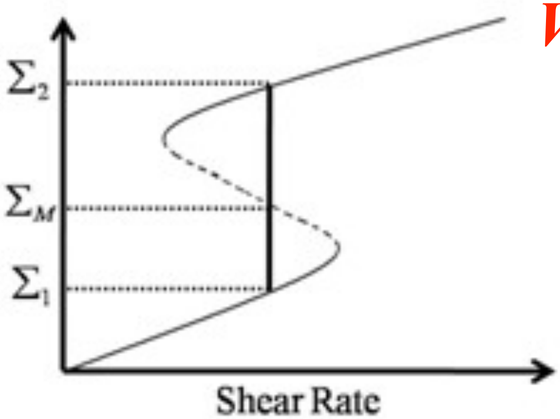
Non-monotonic Steady state Shear Stress vs. Shear Rate Curve

Gradient Banding



Shear Rate > 'Critical' shear rate
Flow breaks into bands of high and low shear rates with same shear stress along the gradient direction.

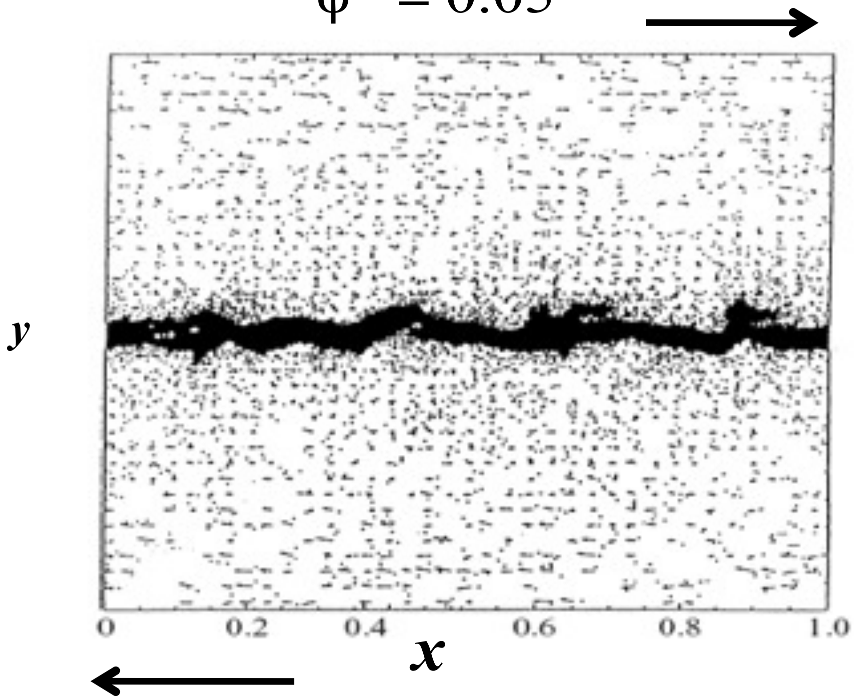
Vorticity Banding



Shear Stress > Critical Shear Stress
Flow breaks into bands of high and low shear stresses with same shear rates along the vorticity direction.

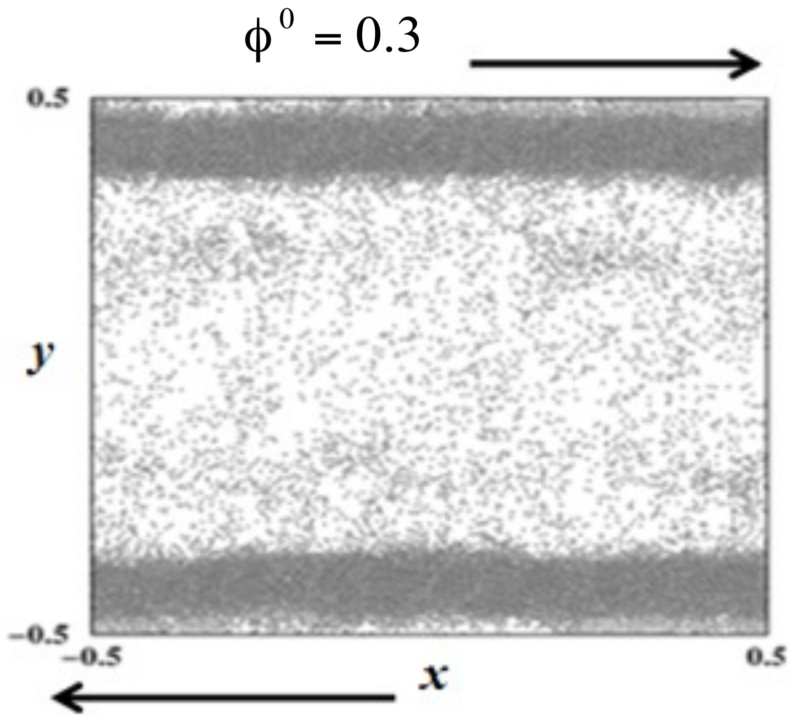
Gradient Banding in 2D-gPCF

$\phi^0 = 0.05$



Tan & Goldhirsch 1997

$\phi^0 = 0.3$



Alam 2003

$$\frac{\partial}{\partial x} = 0, \quad \frac{\partial}{\partial y} \neq 0,$$

Order-parameter description of shear-banding?

Shukla & Alam (2009, 2011a,b, 2013a,b)

Granular Hydrodynamic Equations
(Savage, Jenkins, Goldhirsch, ...)

Balance Equations

Mass $\frac{D\rho}{Dt} = -\rho\nabla.u$

Momentum $\rho\frac{Du}{Dt} = -\nabla.\sum$

Pseudo Thermal Energy

$$\frac{\dim}{2}\rho\frac{DT}{Dt} = -\nabla.q - \sum : \nabla u - D$$

ϕ : Volume fraction of particles

T : Granular temperature

u : Streamwise velocity

v : Normal velocity

$$\rho = \rho_p\phi$$

Navier-Stokes Order Constitutive Model

Stress $\sum = (p - \zeta(\nabla.u))I - 2\mu S$

$$S = \frac{1}{2}(\nabla u + \nabla u^T) - \frac{1}{\dim}(\nabla.u)I$$

Flux of pseudo-thermal energy

$$q = -\kappa\nabla T$$

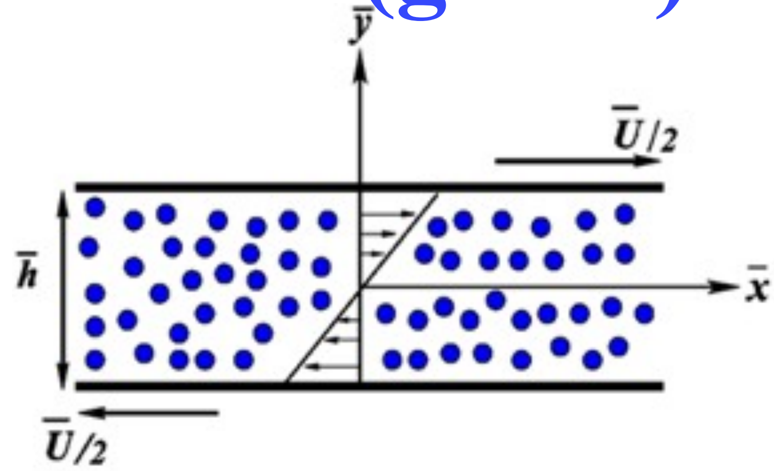
dim: Dimension of system

κ: Thermal Conductivity

μ: Shear Viscosity

D: Sink of granular energy

Plane Couette Flow (gPCF)



d : Particle diameter

Reference Length $\bar{h}$
Reference velocity $\bar{U}$
Reference Time $\bar{h}/U$

- **Base Flow :** Steady, Fully developed.
- **Boundary condition:** No Slip, Zero heat flux.

$$\frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) = 0$$

$$\frac{\partial p}{\partial y} = 0$$

$$\frac{\partial}{\partial y}\left(\kappa \frac{\partial T}{\partial y}\right) + \mu \left(\frac{\partial u}{\partial y}\right)^2 - D = 0$$

⇒

Uniform Shear Solution

$\phi^0 = const.$ $T^0 = const.$

$u^0(y) = y$

Control paramters

- $H = \bar{h}/d$ Couette Gap
- e Restitution Coeff.
- ϕ^0 Volume fraction or mean density

Linear Stability

$$\phi^0 = const. \quad T^0 = const. \qquad u^0(y) = y$$

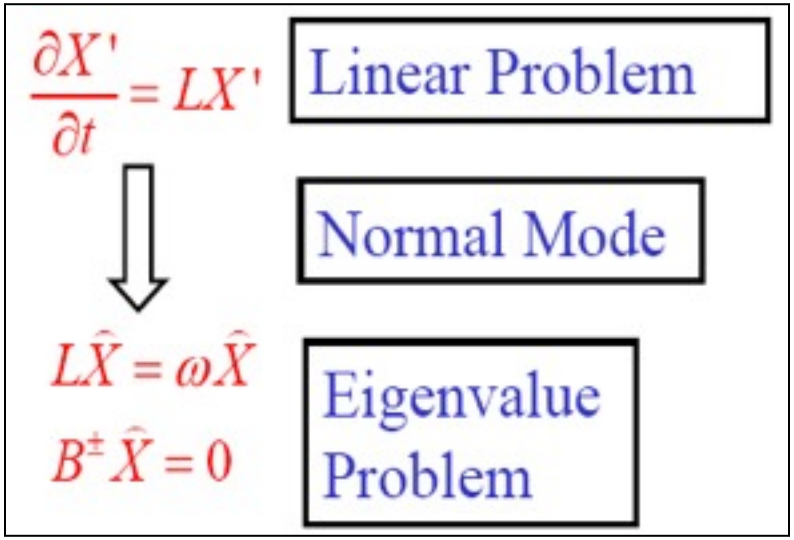
$+$

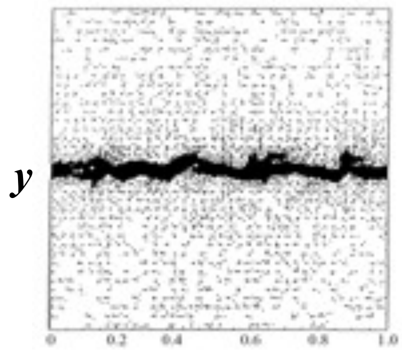
Perturbation (X')

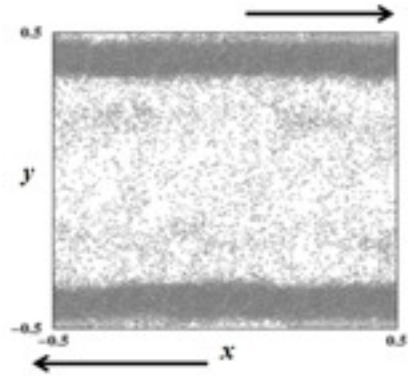
$= X_{total}$

If the disturbances are of infinitesimal magnitude, 'nonlinear terms' in disturbance eqns. can be neglected.

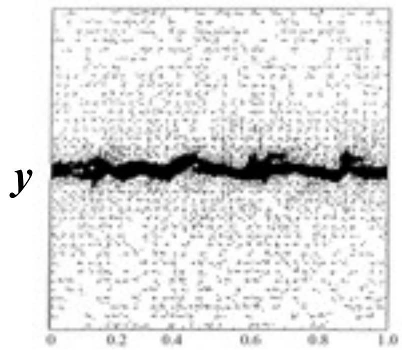
X'(x,y,z,t) ~ exp(\omega t)exp(ik_x x + i k_z z)

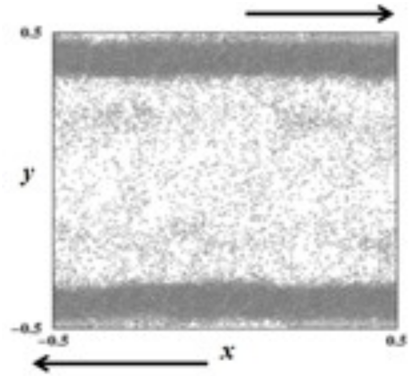






Can **'Linear Stability Analysis'** **able to predict 'Gradient-banding' in Granular Couette flow as observed in Particle Simulations?**



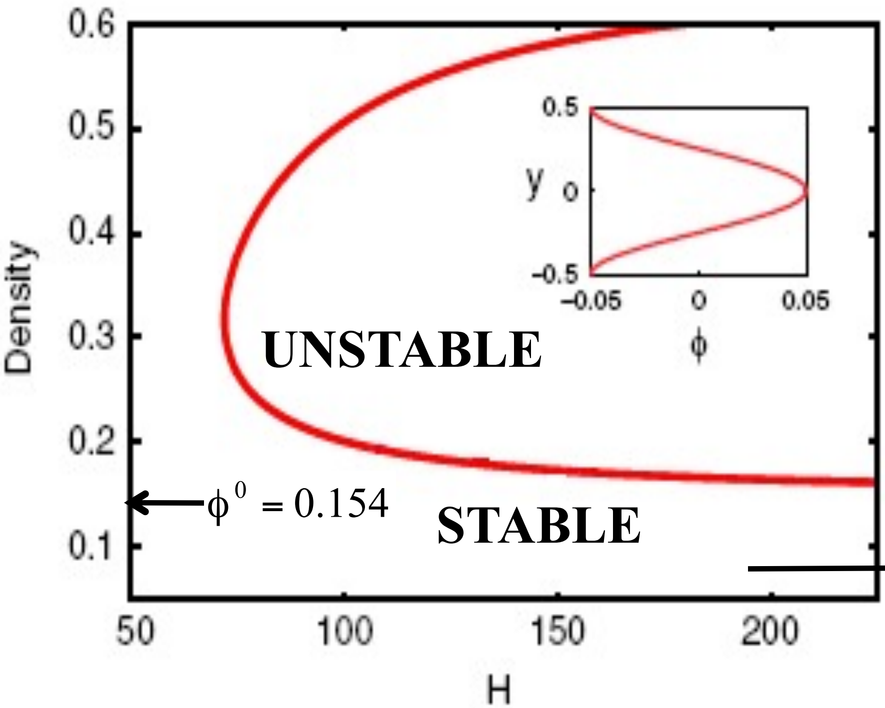


Can ‘Linear Stability Analysis’ able to predict ‘Gradient-banding’ in Granular Couette flow as observed in Particle Simulations?

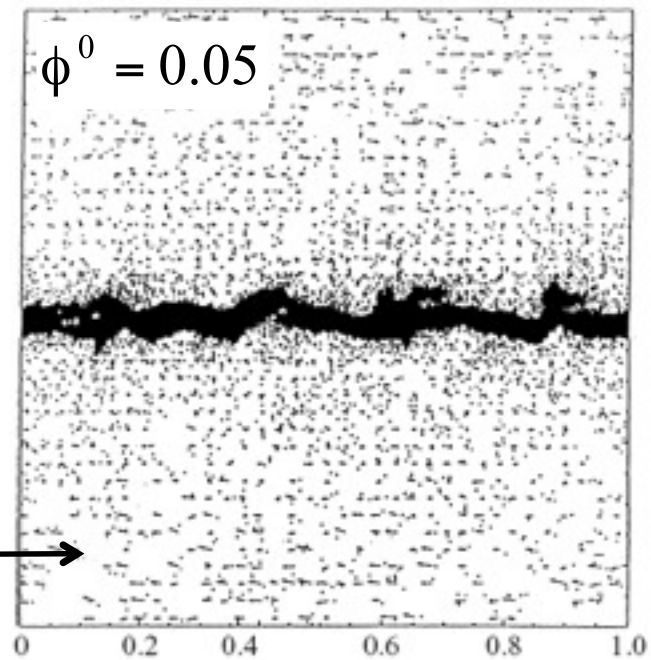
Not for all flow regime

Shukla & Alam 2009, PRL, 103, 068001

Linear Theory



Particle Simulation



Tan & Goldhirsch 1997 Phys. Fluids, 9

Flow remains 'uniform' in dilute limit

↓

Density segregated solutions are not possible in dilute limit

?

Flow is 'non-uniform' in dilute limit

↓

Density Segregated solutions are possible in dilute limit

One must look beyond Linear Stability

Nonlinear Stability Analysis: Center Manifold Reduction

(Carr 1981; Shukla & Alam, PRL 2009)

Dynamics close to critical situation is dominated by finitely many "critical" modes.

Disturbance, Critical Mode, Non-Critical Mode

$$X' = \phi + \psi$$

$$\phi = ZX^{[1;1]} + \tilde{Z}\tilde{X}^{[1;1]}$$

Amplitude, Linear Eigenvector

Z (t): complex amplitude of `finite-size' perturbation

$$\left(\frac{\partial}{\partial t} - L\right)\phi = N_2 + N_3 \qquad\qquad \left(\frac{\partial}{\partial t} - \omega\right) ZX_{11} = N_2 + N_3$$

$$\left(\frac{\partial}{\partial t} - L\right)\psi = N_2 + N_3 \quad \longrightarrow \quad \left(\frac{\partial}{\partial t} - L\right)\psi = N_2 + N_3$$

Taking the inner product of slow mode equation with adjoint eigenfunction of the linear problem and separating the like-power terms in amplitude, we get Landau-Stuart equation

$$c^{(0)} = a^{(0)} + ib^{(0)} = \omega$$

$$\left(\frac{\partial}{\partial t} - \omega\right) ZX_{11} = N_2 + N_3 \longrightarrow \frac{dZ}{dt} = c^{(0)}Z + c^{(2)}Z|Z|^2 + c^{(4)}Z|Z|^4 + ..$$

First Landau Coefficient

$$c^{(2)} = a^{(2)} + ib^{(2)}$$

Second Landau Coefficient

$$c^{(4)} = a^{(4)} + ib^{(4)}$$

Cont…

Adjoint

Distortion of mean flow

Second harmonic

$$c^{(2)} = \frac{\langle Y,\ N_2(X^{[0;2]}, X^{[1;1]}) + N_2(X^{[2;2]}, \tilde{X}^{[1;1]}) + N_3(X^{[1;1]}, X^{[1;1]}, \tilde{X}^{[1;1]})\rangle}{\langle Y, X^{[1;1]}\rangle}$$

$$\left(\frac{\partial}{\partial t} - L\right)\psi = \text{Nonlinear terms}$$

Enslaved Equation

Represent all non-critical modes

$$c^{(4)} = \frac{\langle Y,\ \theta(X^{[1;1]}, X^{[0;2]}, X^{[2;2]}, X^{[1;3]}, X^{[3;3]}, X^{[2;4]}, X^{[0;4]})\rangle}{\langle Y, X^{[1;1]}\rangle}$$

Other perturbation methods can be used:

e.g. Amplitude expansion method and multiple scale analysis

1st Landau Coefficient

Linear Problem $LX^{[1;1]} = c^{(0)} X^{[1;1]}$

Second Harmonic $L_{22} X^{[2;2]} = G_{22}$

Distortion to mean flow $L_{02} X^{[0;2]} = G_{02}$

Distortion to fundamental

$$L_{13} X^{[1;3]} = c^{(2)} X^{[1;1]} + G_{13}$$

Analytically solvable

Shukla & Alam (JFM 2011a)

Analytical expression of first Landau coefficient

$$c^{(2)} = \frac{\phi^a G_{13}^1 + u^a G_{13}^2 + v^a G_{13}^3 + T^a G_{13}^4}{\phi^a \phi^{[1;1]} + u^a u^{[1;1]} + v^a v^{[1;1]} + T^a T^{[1;1]}}$$

Analytical solution exists

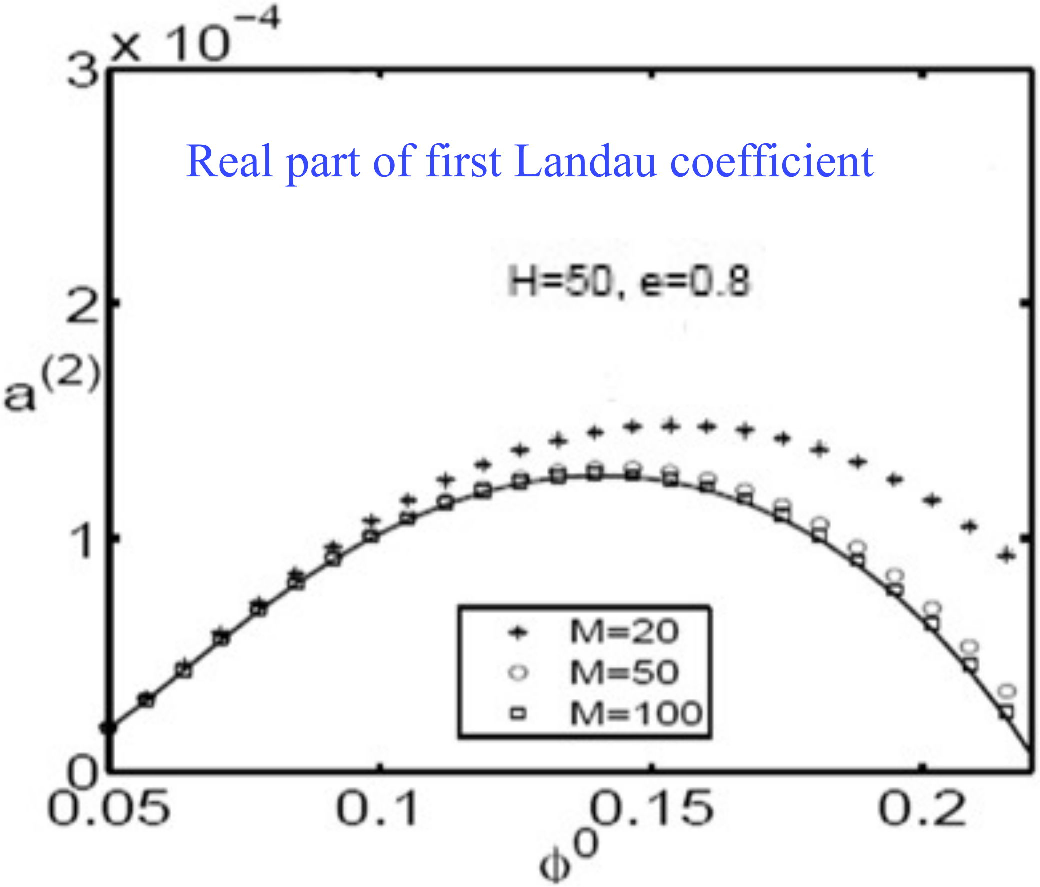
We have developed a spectral based numerical code to calculate Landau coefficients.

Numerical Method: comparison with analytical solution

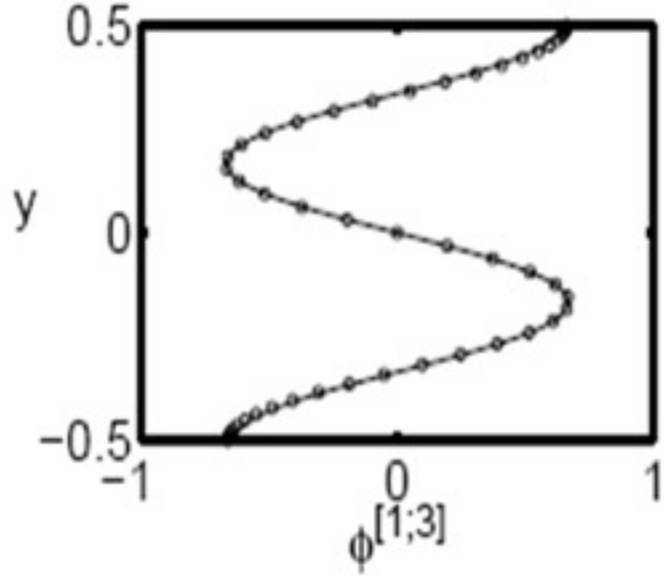
(i) Spectral collocation method,
(ii) SVD for inhomogeneous eqns.
(iii) Gauss-Chebyshev quadrature for integrals.

Shukla & Alam (JFM, 2011a)

Real part of first Landau coefficient



Distorted density eigenfunction



This validates spectral-based numerical code

Equilibrium Amplitude and Bifurcation

Cubic Landau Eqn: $Z = Ae^{i\theta}$

$$\frac{dZ}{dt} = c^{(0)}Z + c^{(2)}Z\,|\,Z\,|^2$$

$$\frac{dA}{dt} = a^{(0)}A + a^{(2)}A^3,$$

$$\frac{d\theta}{dt} = b^{(0)} + b^{(2)}A^2$$

Real amplitude eqn.

Phase eqn.

Cubic Solution

$$\frac{dA}{dt} = 0 \Rightarrow A = 0, \quad A = \pm\sqrt{-\frac{a^{(0)}}{a^{(2)}}}$$

Supercritical Bifurcation $a^{(0)} > 0, a^{(2)} < 0$

Subcritical Bifurcation $a^{(0)} < 0, a^{(2)} > 0$

$b^{(0)} = 0$ Pitchfork (stationary) bifurcation

$b^{(0)} \neq 0$ Hopf (oscillatory) bifurcation

Phase Diagram

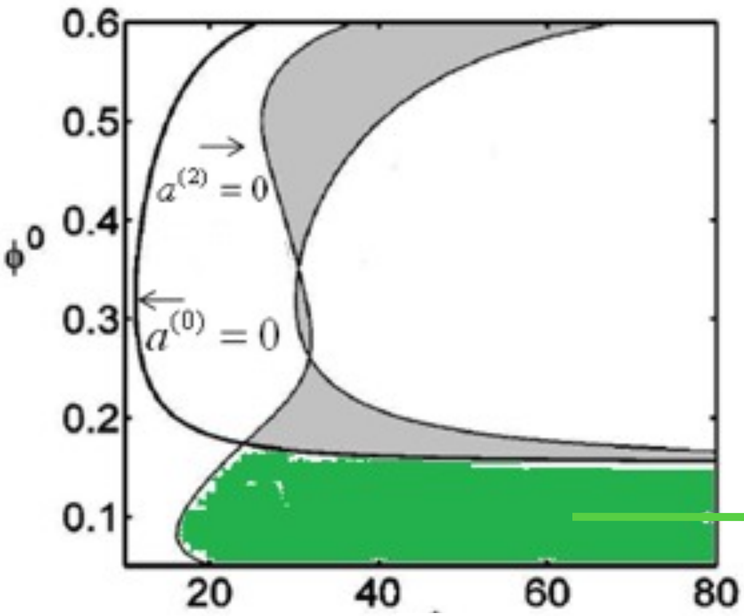
Constitutive equations are function of radial distribution function (RDF)

ϕ^0 : Mean Density

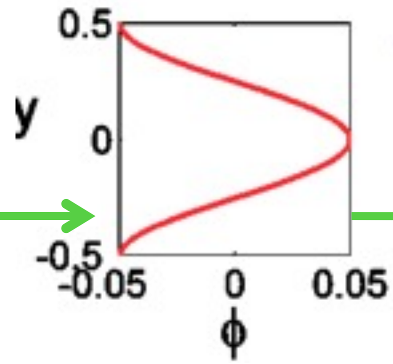
ϕ_m : Maximum Mean Density

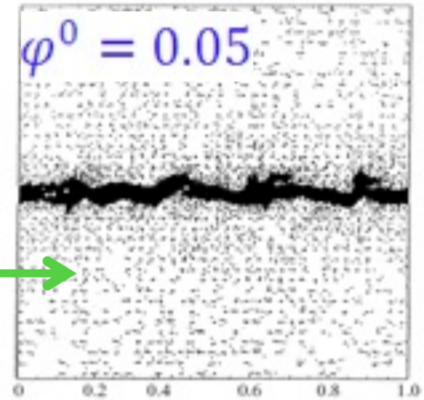
H : Couette Gap

$$\chi(\phi) = \frac{1}{1-(\phi/\phi_m)^{1/3}}$$



$$H^* = H\beta^{-1}\sqrt{(1-e^2)}$$





Gradient-banding in dilute flows

This agrees with MD simulations of
Tan & Goldhirsch 1997

△ *Nonlinear Stability theory and MD simulations both support gradient banding in 2D-GPCF* ***(PRL 2009)***

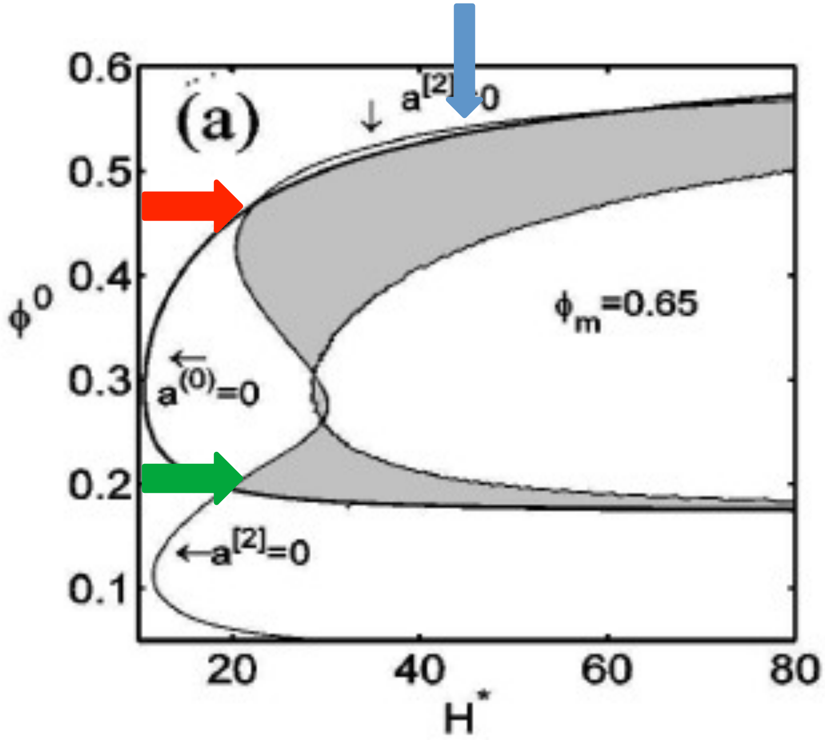
Cont…

(JFM 2011a)

Carnahan-Starling RDF

$$\chi(\phi) = \frac{1-\phi/2}{(1-\phi/\phi_m)^3}$$

Change of constitutive relations leads to three degenerate points



Stable Solutions

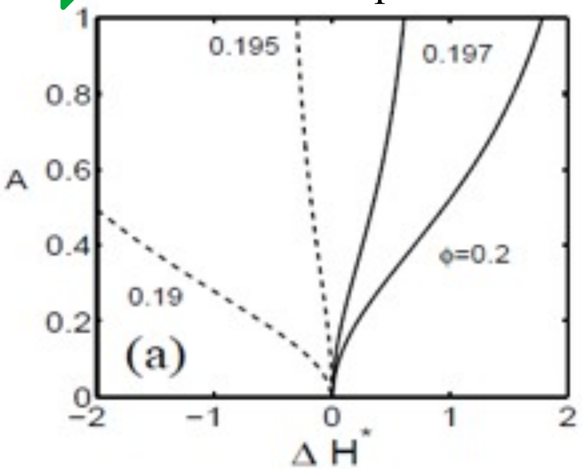
Unstable Solutions

0.559

Subcritical -> supercritical

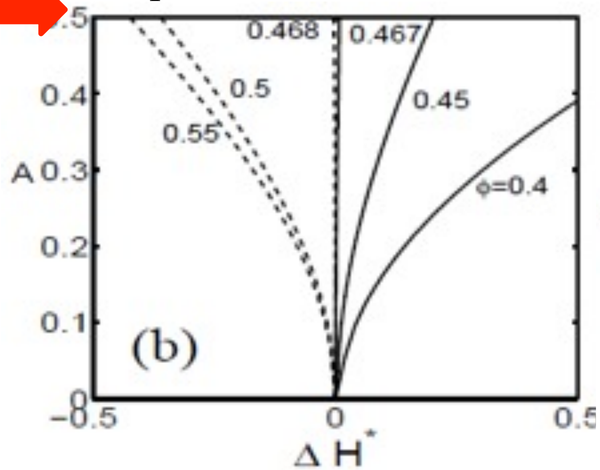
0.196

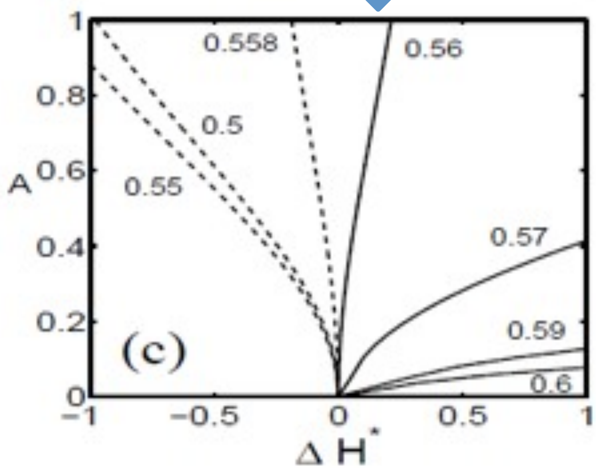
Subcritical -> supercritical



0.467

Supercritical-> subcritical





Paradigm of Pitchfork Bifurcations

Supercritical

$$\phi^0 > \phi_c^{s2}$$

Subcritical

$$\phi_c^{s1} < \phi^0 < \phi_c^{s2} \approx 0.559$$

Supercritical

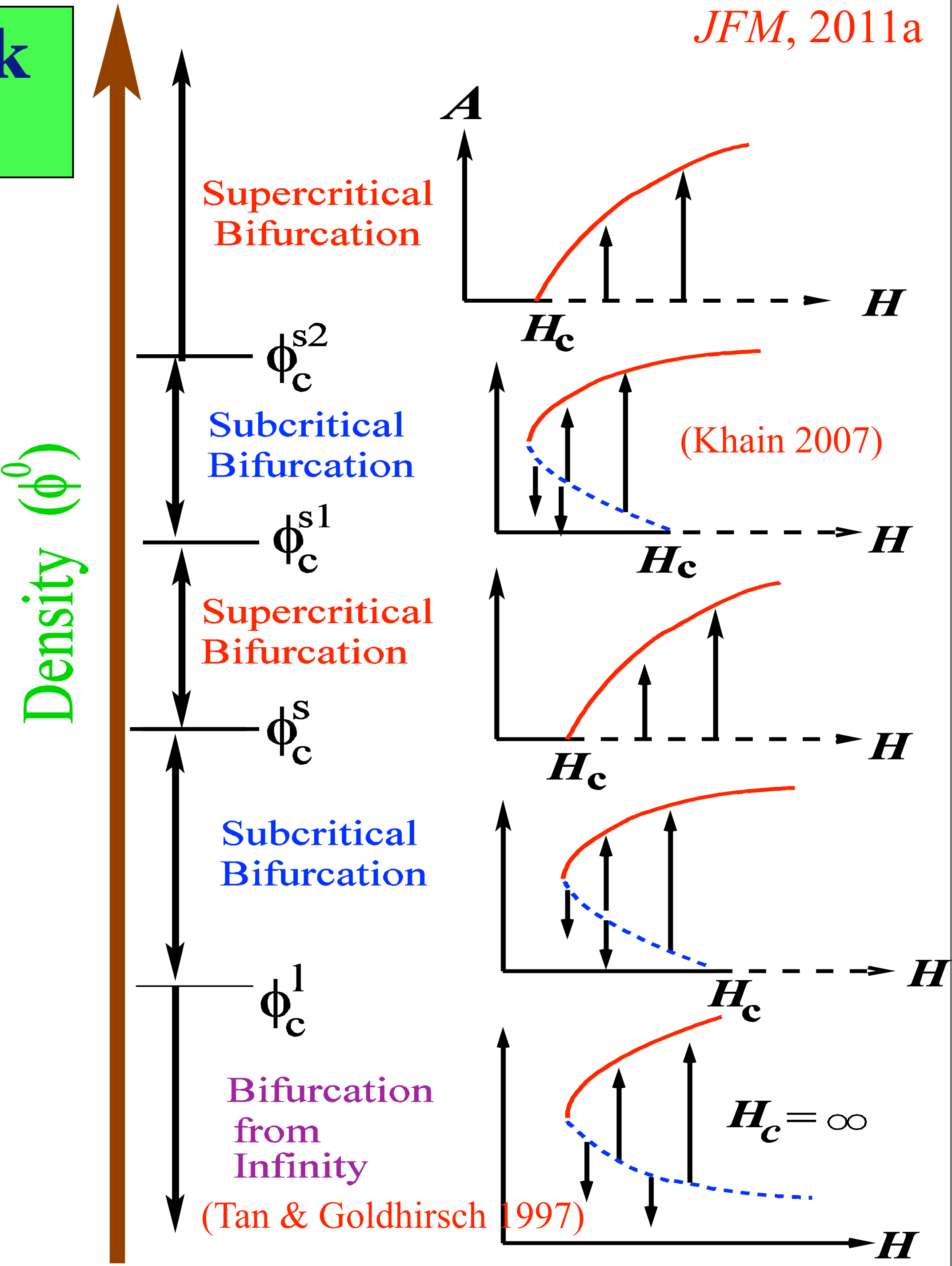
$$\phi_c^{s} < \phi^0 < \phi_c^{s1} \approx 0.467$$

Subcritical

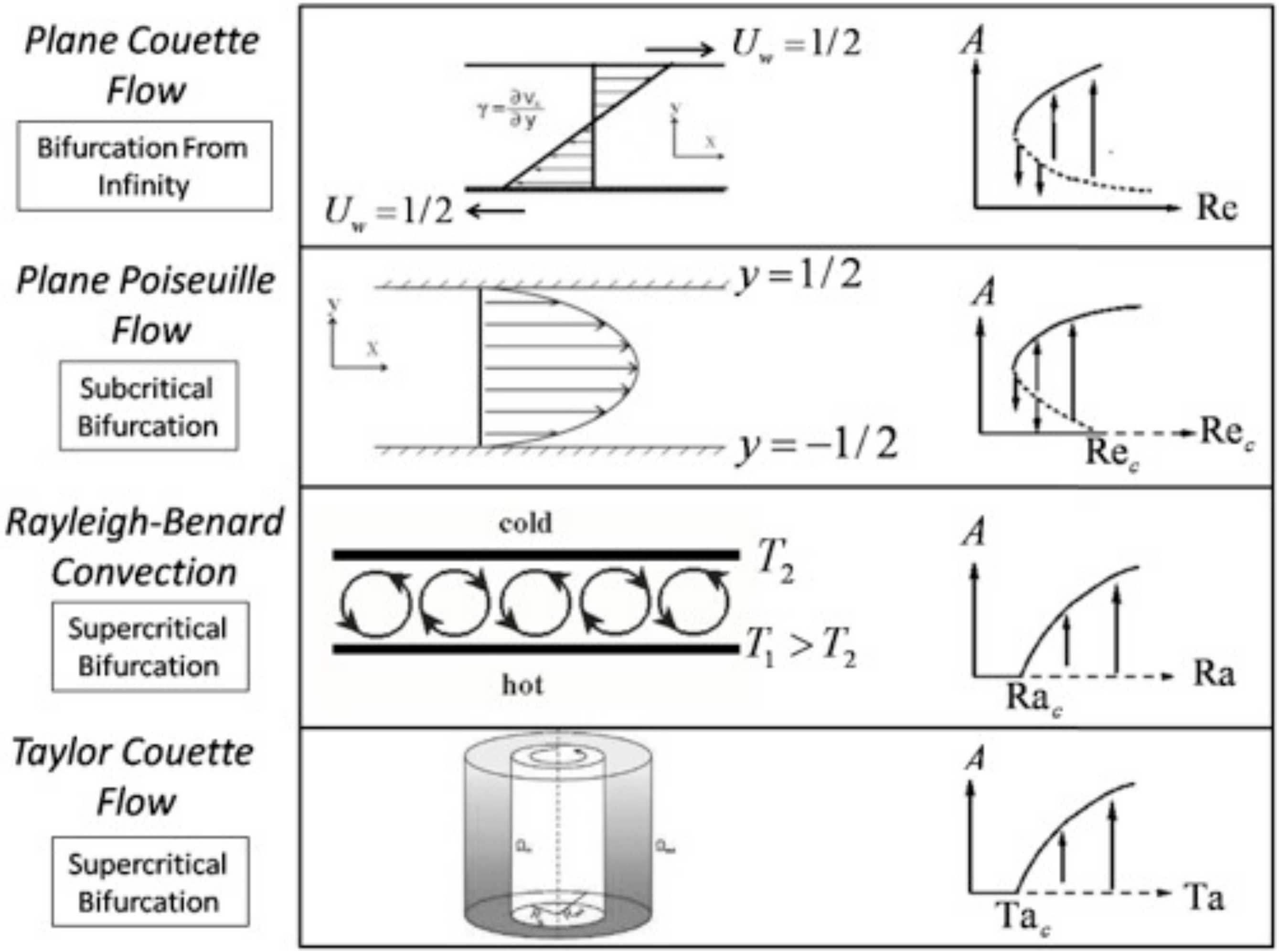
$$\phi_c^{l} < \phi^0 < \phi_c^{s} \approx 0.196$$

Bifurcation from infinity

$$\phi^0 < \phi_c^{l} \approx 0.174$$



Incompressible Newtonian Fluids



All in one!

Granular Plane Couette flow

admits all types of Pitchfork bifurcations

Conclusions

➢Problem is analytically solvable.

➢ **Landau-Stuart equation** describes gradient-banding transition in a sheared granular fluid.

➢Landau coefficients suggest that there is a "sub-critical" (bifurcation from infinity) finite amplitude instability for "dilute" flows even though the dilute flow is stable according to linear theory.

➢This result agrees with previous MD-simulation of gPCF.

➢gPCF serves as a **paradigm** of pitchfork bifurcations.

➢Analytical solutions have been obtained.

➢An spectral based numerical code has been validated.

*References: Shukla & Alam (2011a), J. Fluid Mech., vol **666**, 204-253*
*Shukla & Alam (2009) Phys. Rev. Lett., vol **103** , 068001.*

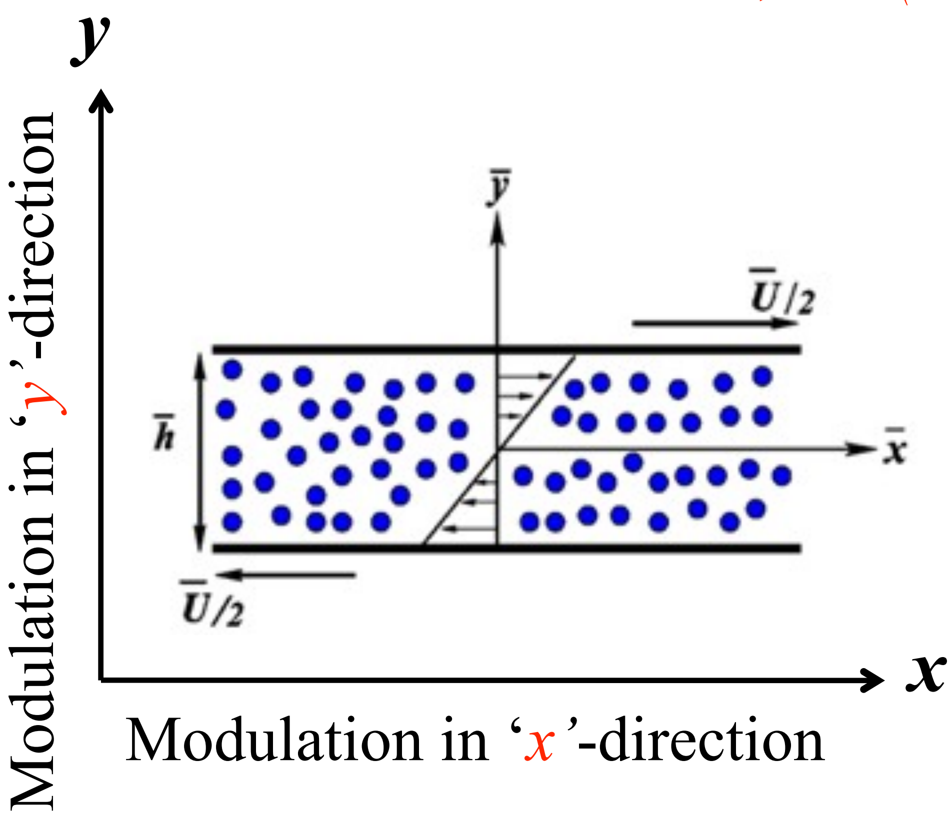
``Gradient-banding" and Saturn's Ring?

➤Self gravity?... other effects needed...

References: Schmitt & Tscharnuter (1995, 1999) Icarus
Salo, Schmidt & Spahn (2001) Icarus,
Schmidt & Salo (2003) Phys. Rev. Lett.

Patterns in 2D-gPCF

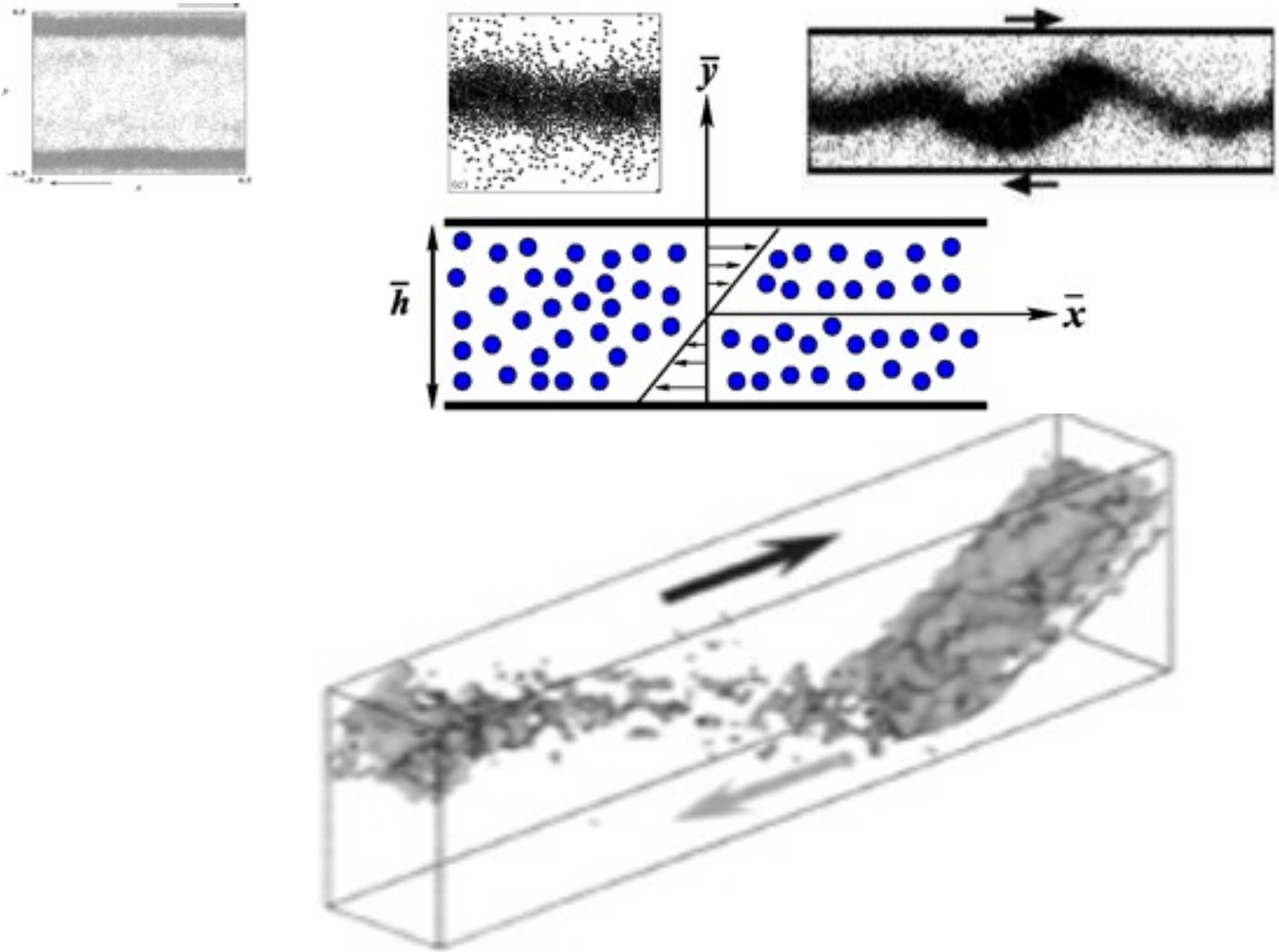
Shukla & Alam , JFM (2011b) vol. 672, 147-195



Flow is linearly unstable due to stationary and traveling waves, leading to particle clustering along the flow and gradient directions

Particle Simulations of Granular PCF

(Conway and Glasser 2006)



$$\frac{dA}{dt} = f(A,t)$$

$$\frac{\partial A}{\partial t} + a_1 \frac{\partial A}{\partial x} + a_2 \nabla^2 A = g(A,t,x,\cdots)$$

Stability of 2D-gPCF when subject to **"finite amplitude perturbation"**

Seeking an **order parameter theory** for stationary and traveling wave instabilities…

Linear Theory

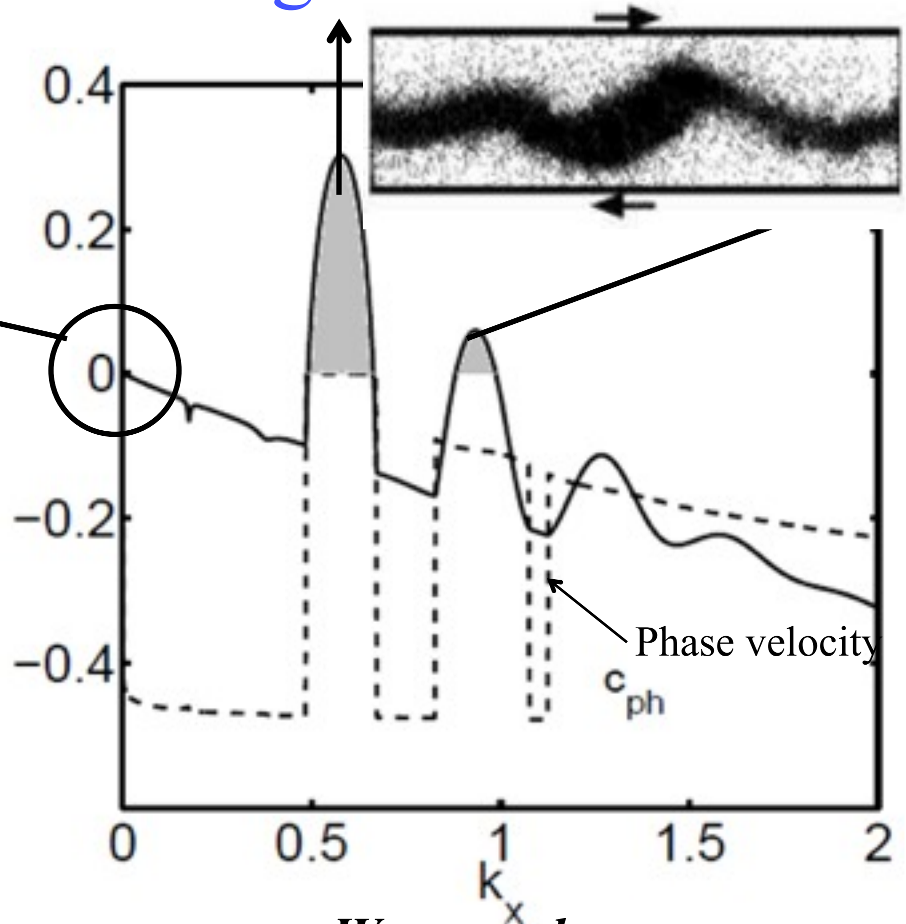
$\phi = 0.2, H = 100, e = 0.8$

1st peak $k_x \approx O(1)$

Standing wave instability

Long-wave instability

$k_x \approx 0$



Wavenumber ⟶

$k_x \approx O(1)$

2^{nd} peak

Traveling wave instability

$a^{(0)}$ Growth rate

c_{ph} Phase velocity

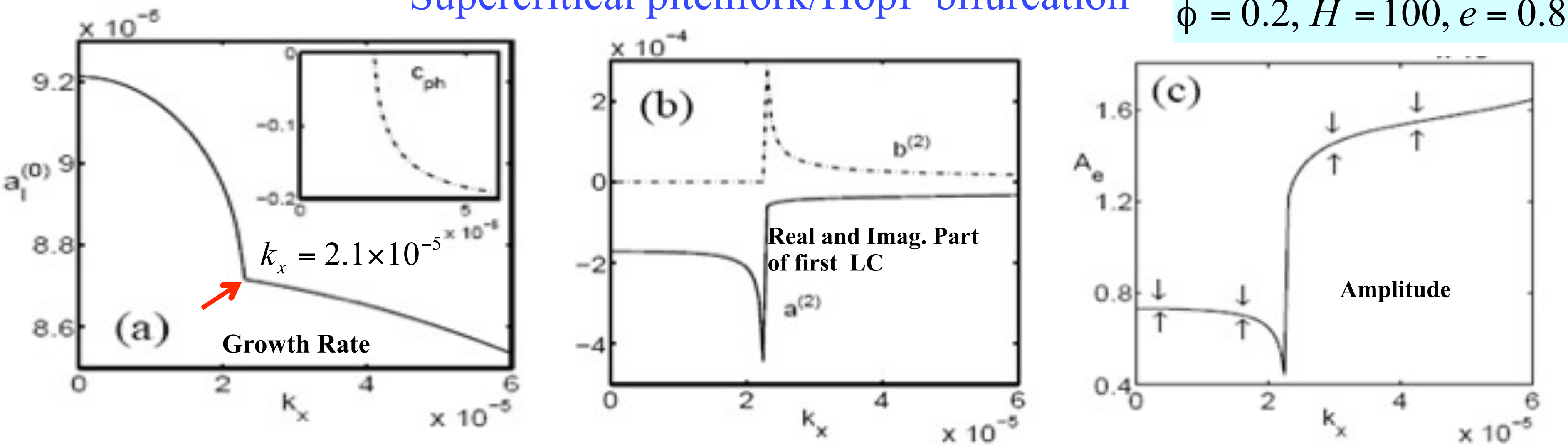
$$c_{ph} = -\frac{b^{(0)}}{k_x}$$

Long-Wave Instabilities

$k_x \approx 0$

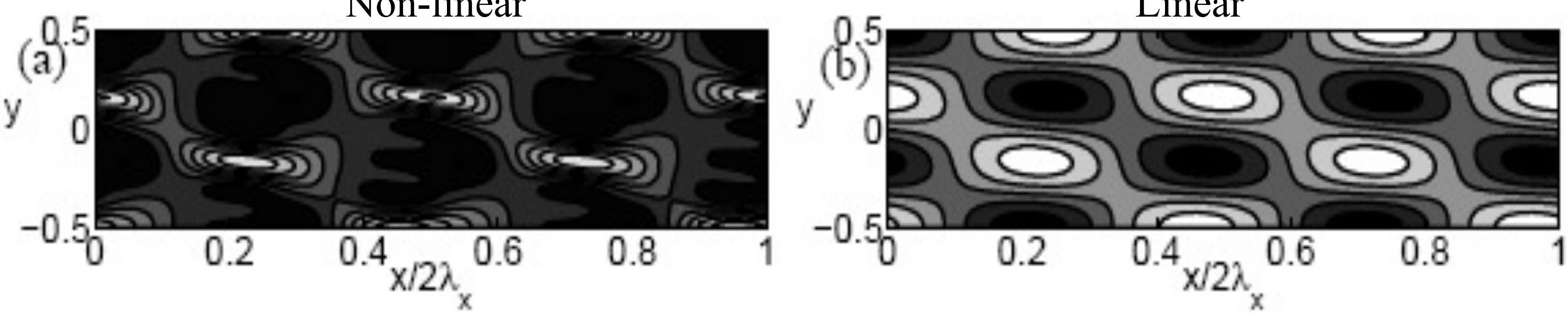
Supercritical pitchfork/Hopf bifurcation

$\phi = 0.2, H = 100, e = 0.8$



Non-linear

Linear

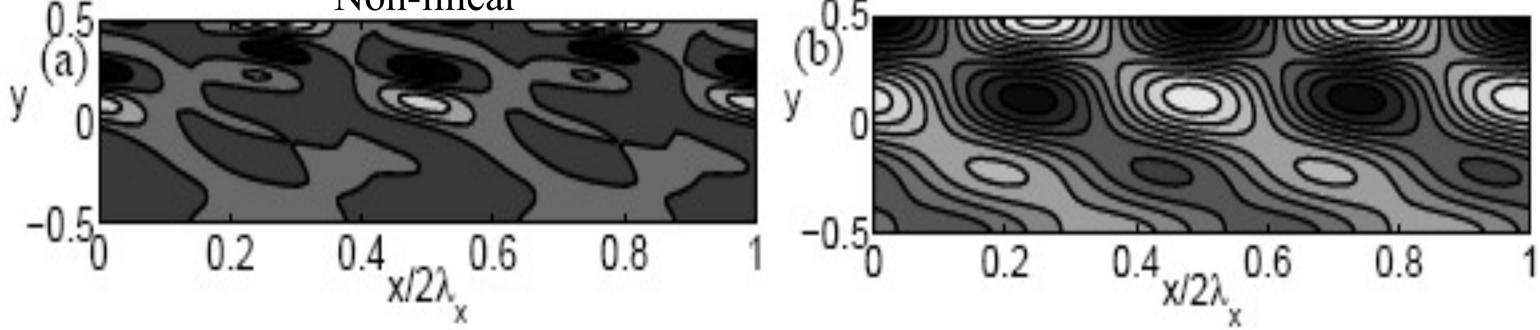


$k_x = 10^{-5}$

SW Density Patterns

Non-linear

Linear



$k_x = 4\times10^{-5}$

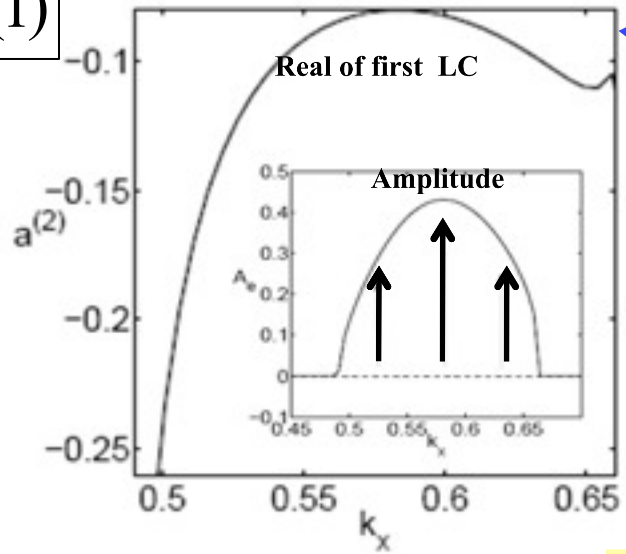
TW Density Patterns

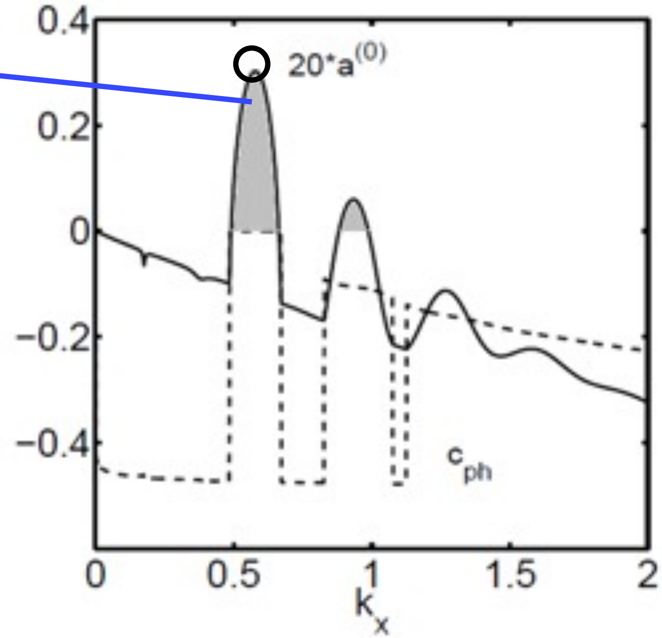
Stationary Instability

$\phi = 0.2, H = 100, e = 0.8$

Supercritical pitchfork bifurcation

$k_x \approx O(1)$

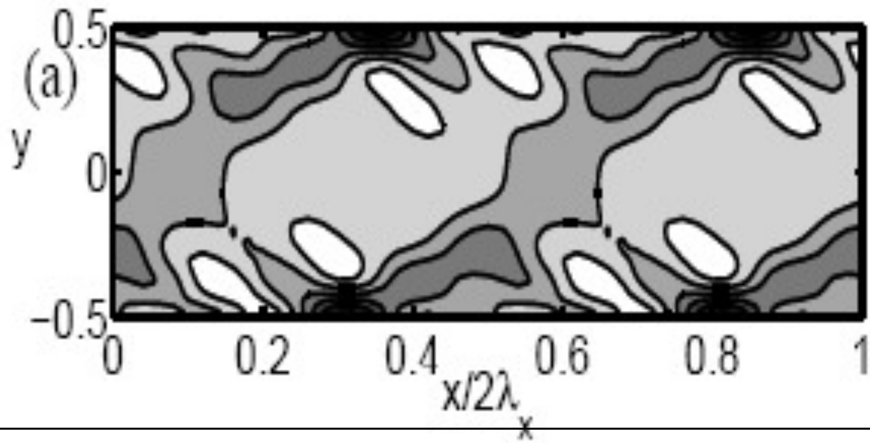




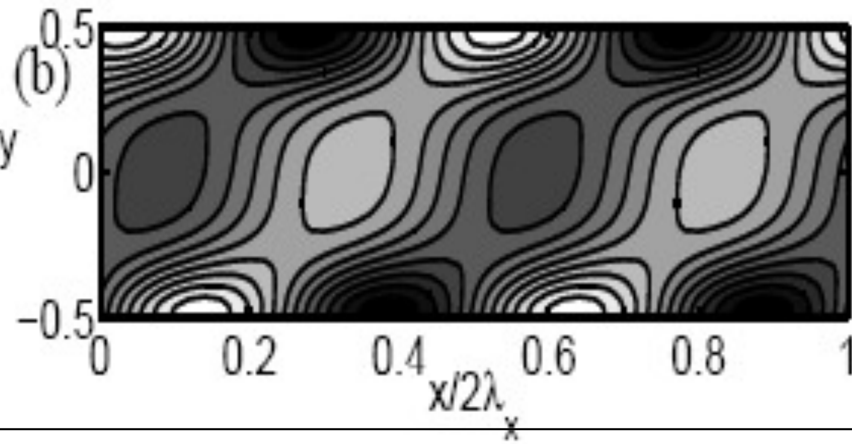
SW density patterns

$k_x = 0.58$

Non-linear



Linear



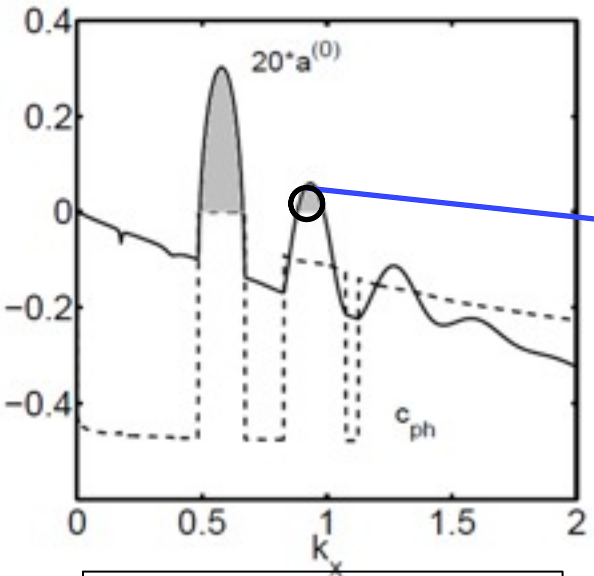
Structural features are different from long-wave stationary instability

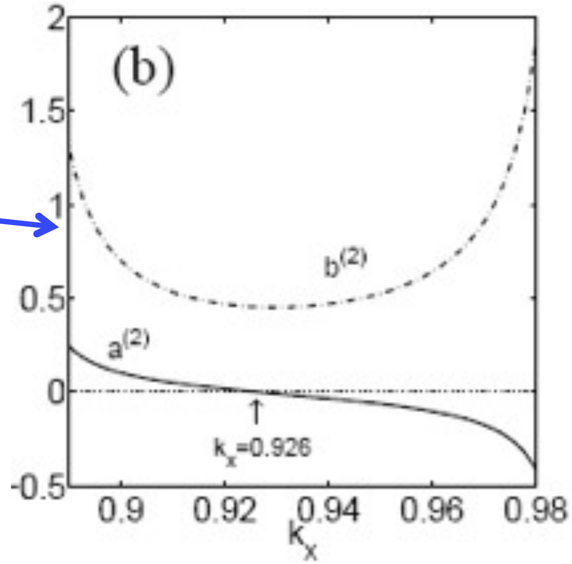
Travelling Instabilities

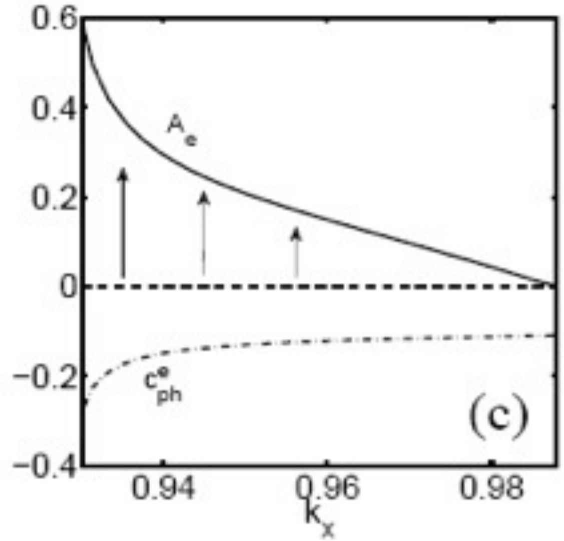
$\phi = 0.2, H = 100, e = 0.8$

Supercritical Hopf bifurcation

$k_x \approx O(1)$





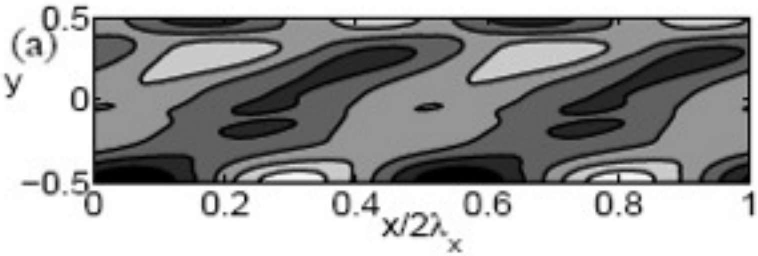


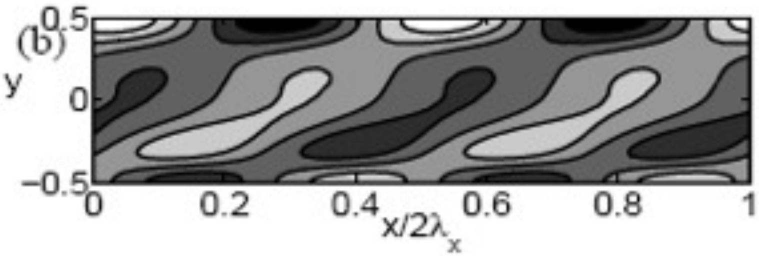
$$c_{ph}^{e} = -\frac{\omega}{k_x} = c_{ph} - \frac{b^{(2)} A^2}{k_x}$$

Non-linear

Linear

$k_x = 0.935$





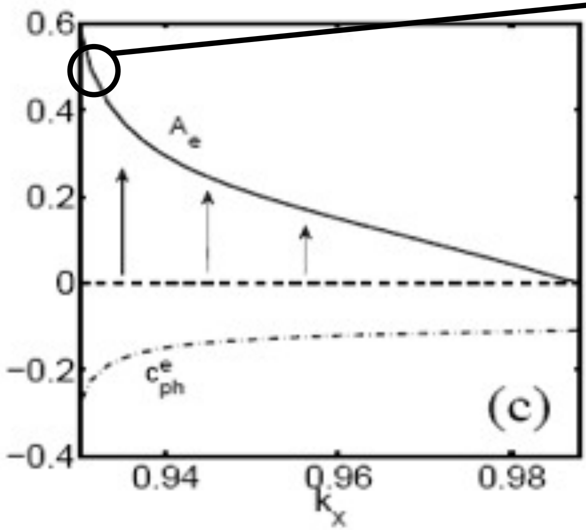
Supercritical Hopf Bifurcation/ Limit Cycle Solutions

$$\frac{dZ}{dt} = c^{(0)}Z + c^{(2)}Z\,|\,Z\,|^2$$

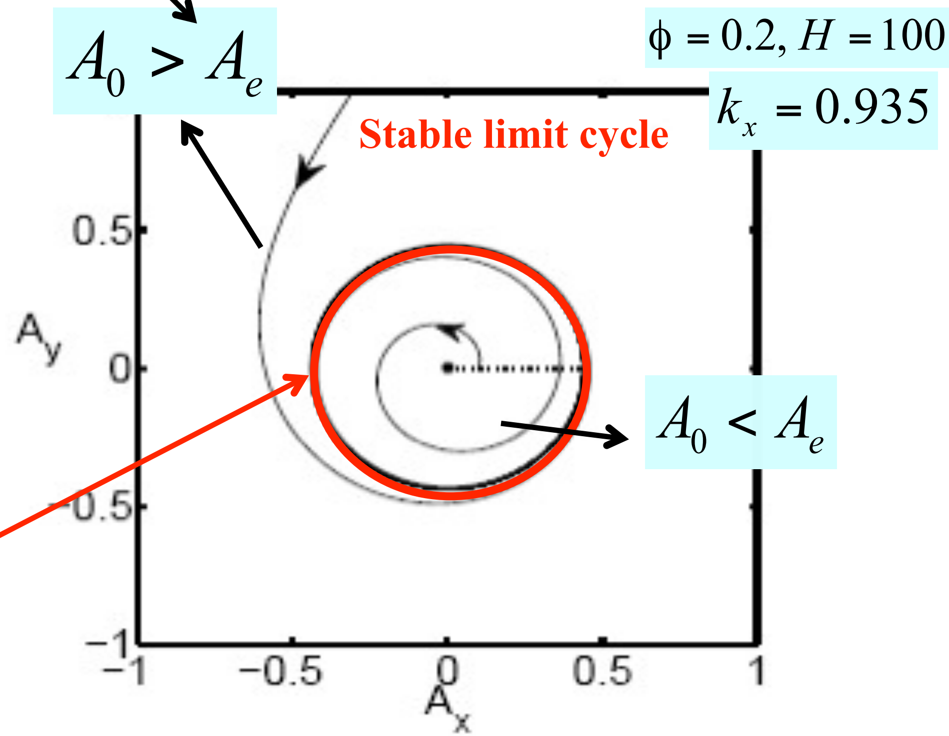
$$\frac{dA}{dt} = a^{(0)}A + a^{(2)}A^3, \quad \longrightarrow \quad A^2(t) = \frac{a^{(0)}A_0^2}{[a^{(0)} + a^{(2)}A_0^2]\exp(-2a^{(0)}t) - a^{(2)}A_0^2},$$

$$\frac{d\theta}{dt} = b^{(0)} + b^{(2)}A^2 \qquad \theta(t) = \theta_0 + b^{(0)}t - \frac{b^{(2)}a^{(0)}}{a^{(2)}}\ln\left[\frac{a^{(0)} + a^{(2)}A_0^2(1-\exp(-2a^{(0)}t))}{a^{(0)}}\right]$$

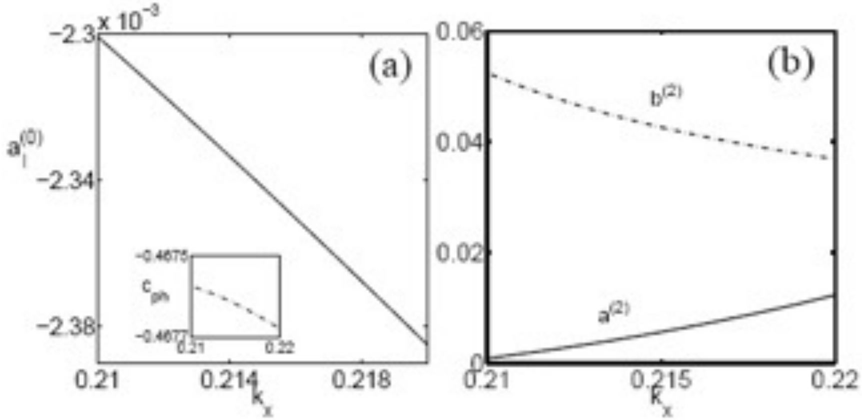
$$A_0 = A(t=0), \theta_0 = \theta(t=0)$$



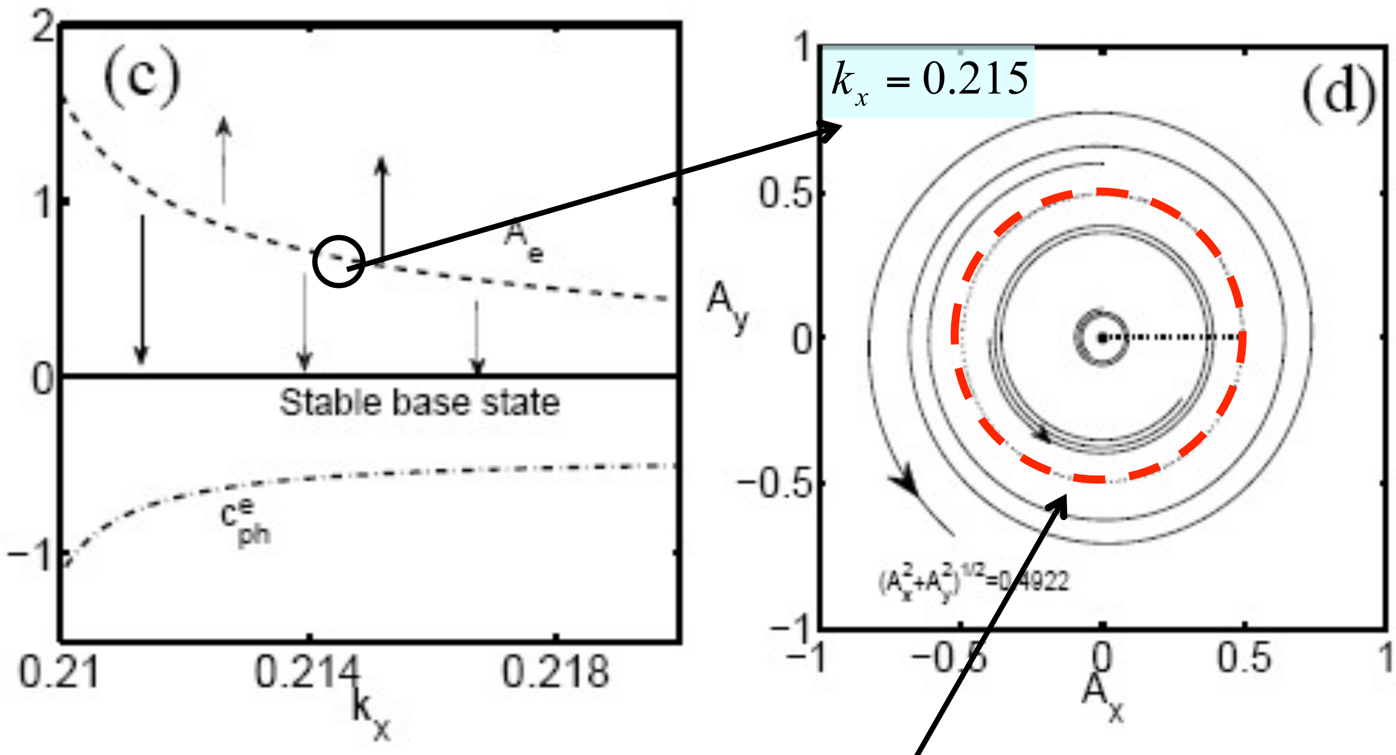
Stable limit cycle



Subcritical Hopf Bifurcation/ Limit Cycle Solutions

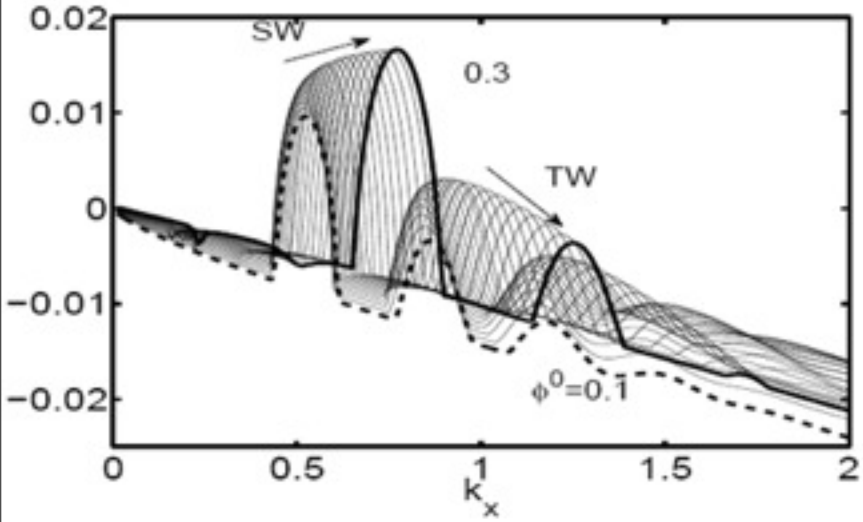


moderate values of k_x

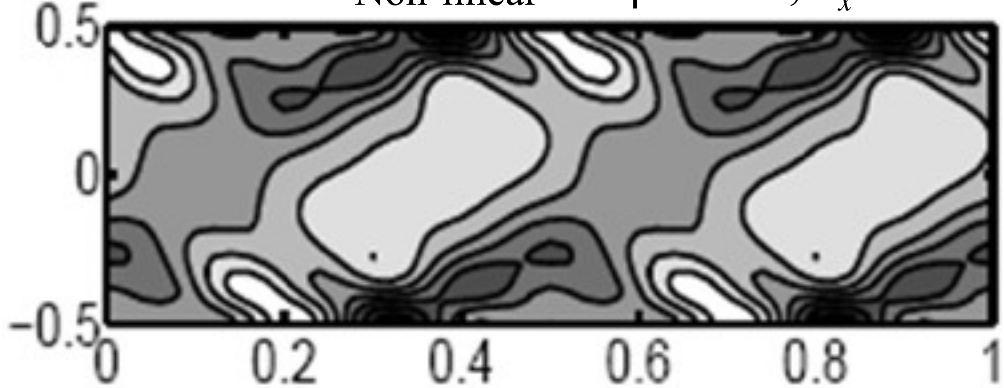


Both orbits spiral away from the unstable limit cycle

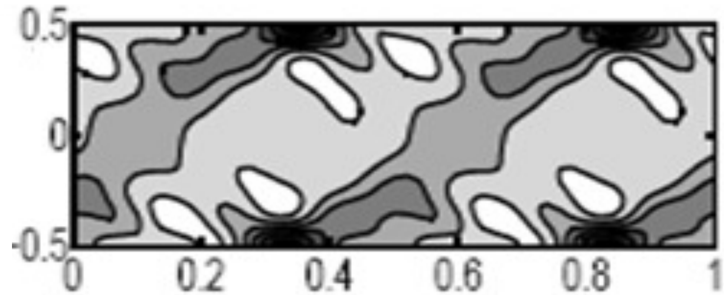
Dominant Stationary Instabilities

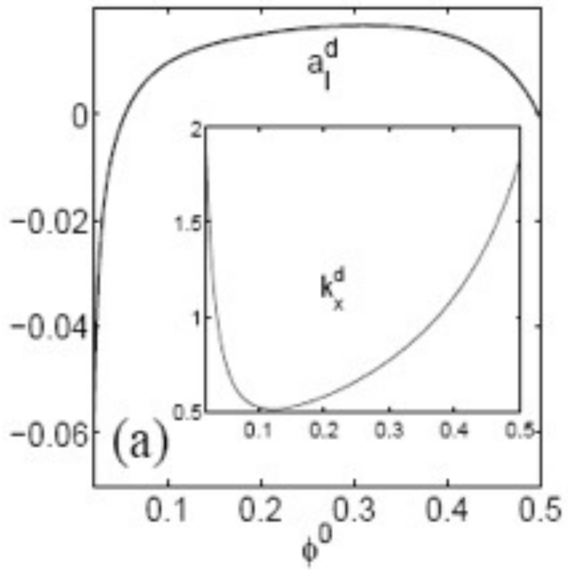


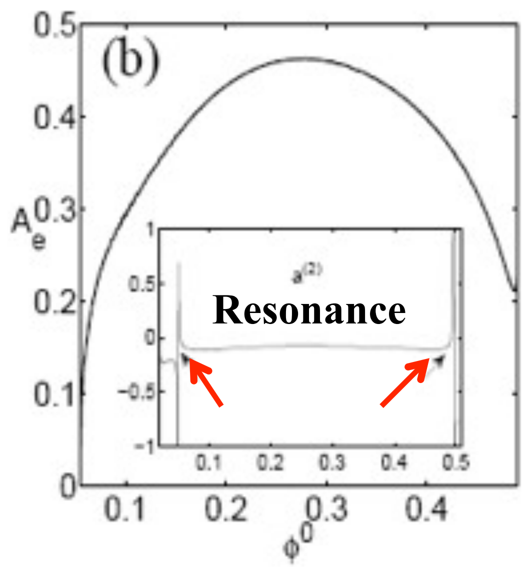
Non-linear $\phi^0 = 0.1, k_x = 0.52$



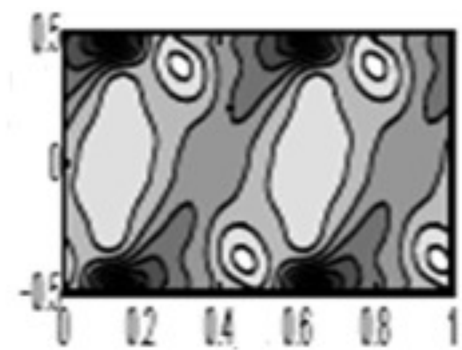
Non-linear $\phi^0 = 0.3, k_x = 0.77$







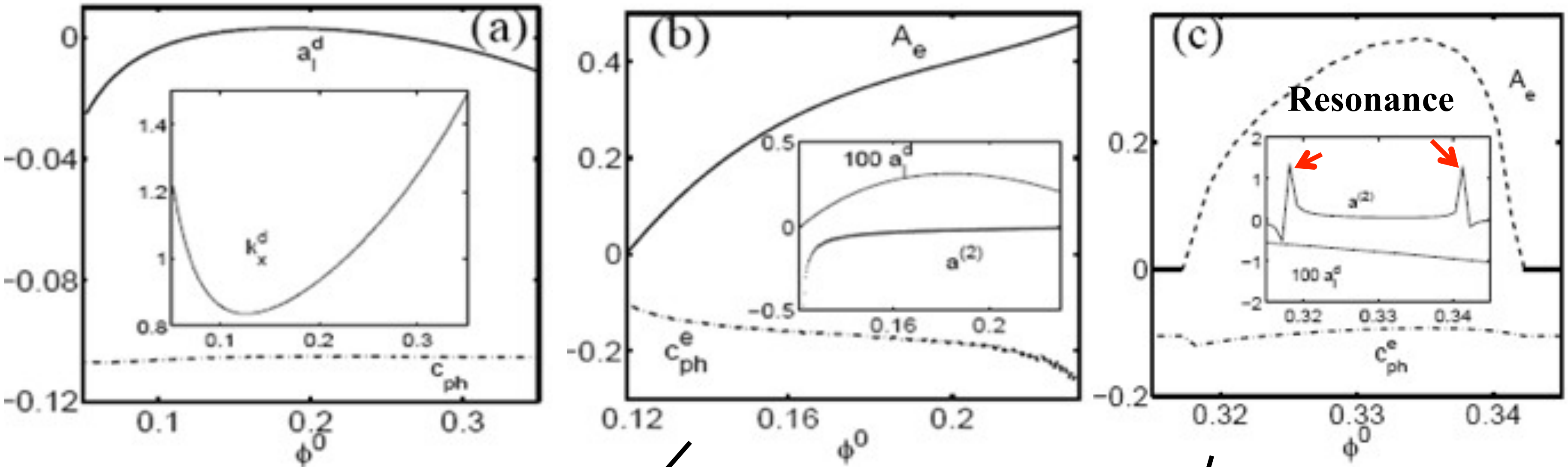
Non-linear $\phi^0 = 0.45, k_x = 1.39$



Density patterns are structurally similar at all densities

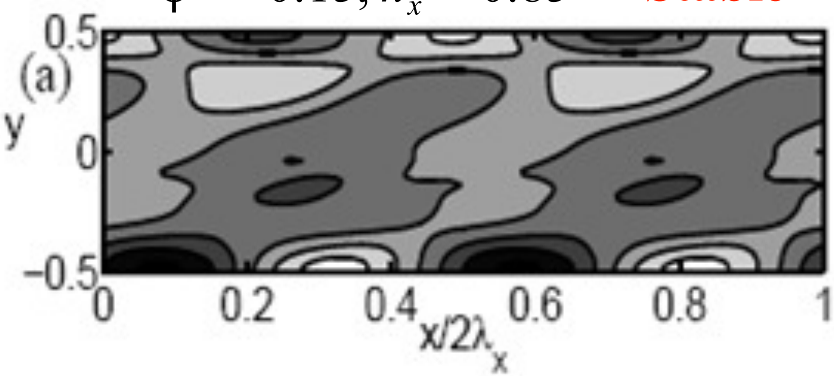
Dominant Traveling Instabilities

Supercritical Hopf Bifurcation | Subcritical Hopf Bifurcation



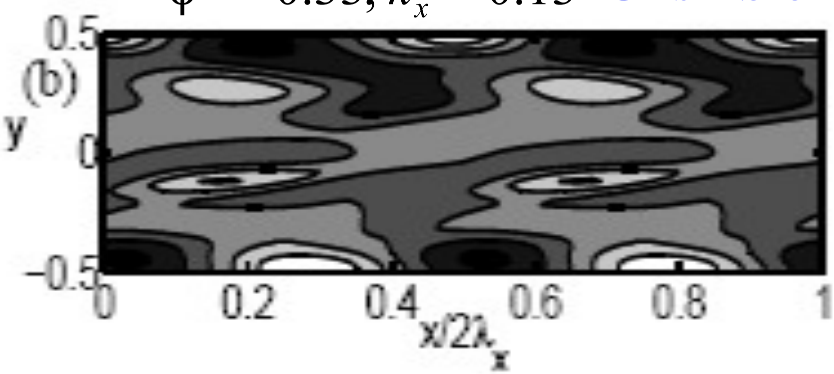
Non-linear

$\phi^0 = 0.15, k_x = 0.85$ **Stable**



Non-linear

$\phi^0 = 0.33, k_x = 0.13$ **Unstable**



Conclusions

- The origin of nonlinear states at long-wave lengths is tied to the corresponding subcritical / supercritical nonlinear gradient-banding solutions (discussed in 1st Part of talk).

- For the dominant stationary instability nonlinear solutions appear via supercritical bifurcation.

- Structure of patterns of supercritical stationary solutions look similar at any value of density and Couette gap.

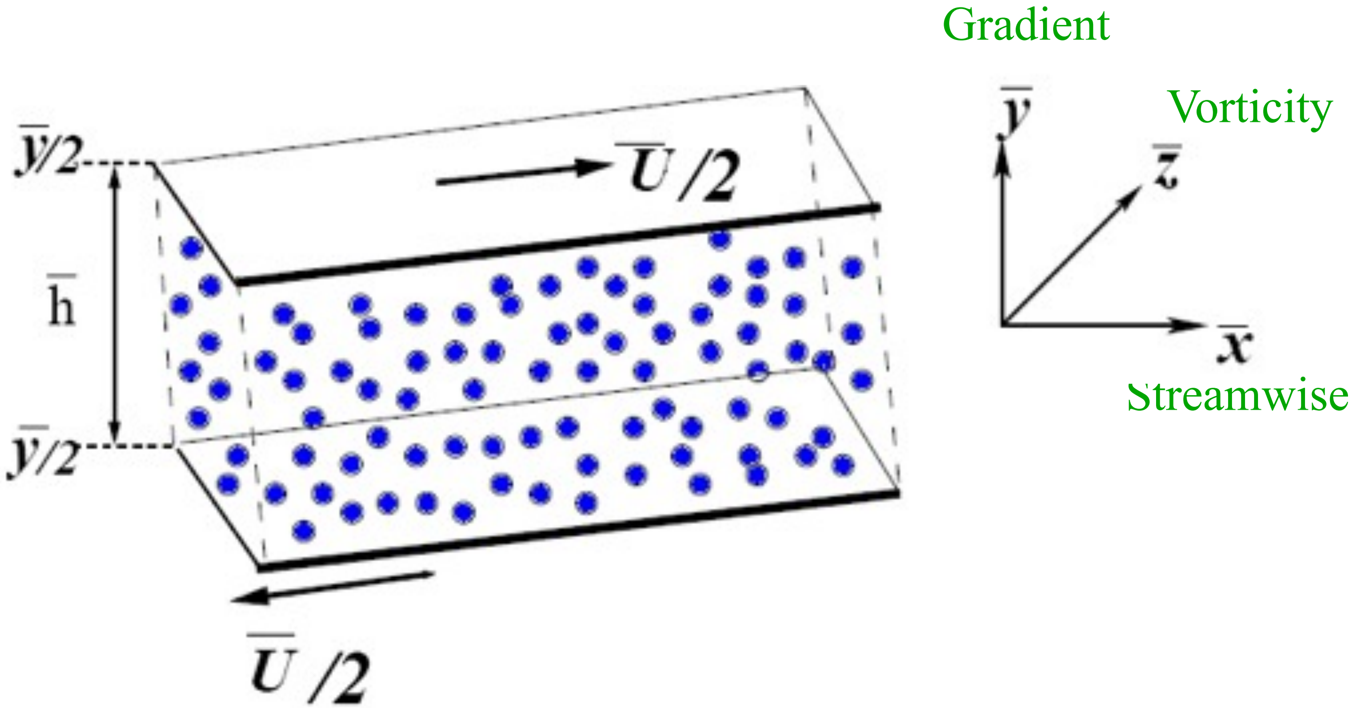
- For the dominant traveling instability, there are supercritical and subcritical **Hopf** bifurcations at small and large densities.
- Uncovered mean flow resonance at quadratic order.

References: *Shukla & Alam (2011b), J. Fluid Mech., vol. 672, p. 147-195.*
Shukla & Alam (2011a), J. Fluid Mech., vol ***666****, p. 204-253.*
Shukla & Alam (2009) Phys. Rev. Lett., vol ***103*** *, 068001.*

Vorticity Banding in 3D-gPCF

Pure Spanwise Perturbations

$$\frac{\partial}{\partial x} = 0, \ \frac{\partial}{\partial y} = 0, \ \frac{\partial}{\partial z} \neq 0$$



Shukla & Alam (2013b, JFM)

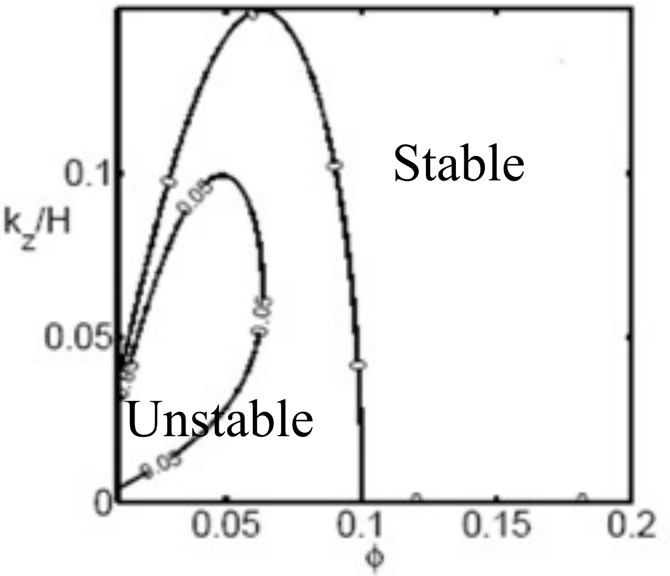
Linear Vorticity Banding

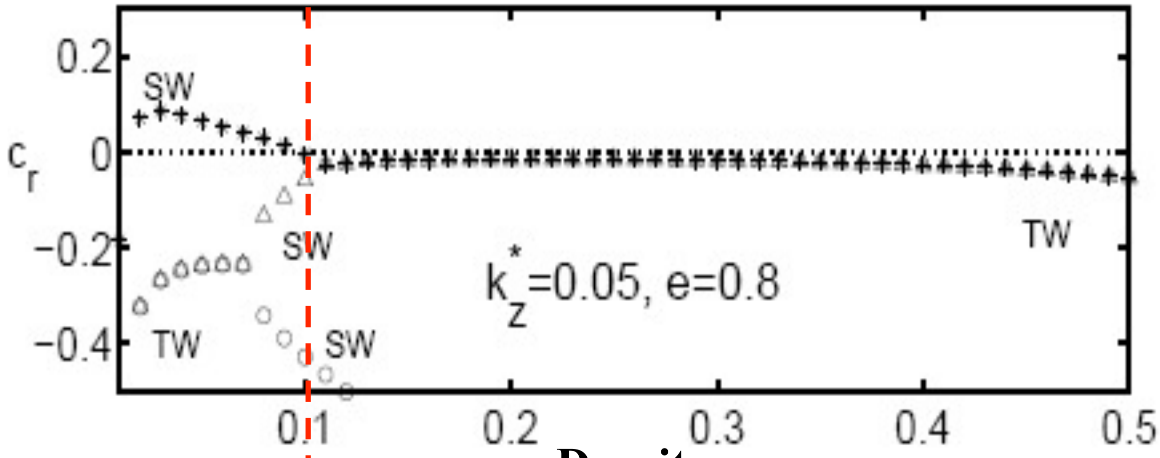
Dispersion relation

$$\left(\omega + \frac{\mu^0 k_z^2}{\phi^0 H^2}\right)^2 (\omega^3 + a_2\omega^2 + a_1\omega + a_0) = 0$$

Analytically solvable

Pure spanwise GPCF





Density →

Pitchfork bifurcation | Supercritical Hopf bifurcation

Gradient-banding modes stationary **modes at all density.**

Vorticity-banding modes stationary **at dilute limit &**
traveling **in moderate-to-dense limit.**

Nonlinear Stability

Shukla & Alam (2013b, JFM)

Linear Problem $LX^{[1;1]} = c^{(0)} X^{[1;1]}$

Second Harmonic $L_{22} X^{[2;2]} = G_{22}$

Distortion to mean flow $L_{02} X^{[0;2]} = G_{02}$

Distortion to fundamental

$$L_{13} X^{[1;3]} = c^{(2)} X^{[1;1]} + G_{13}$$

} Analytically solvable

Analytical expression for first Landau coefficient

$$c^{(2)} = \frac{\phi^a G_{13}^1 + w^a G_{13}^4 + T^a G_{13}^5}{\phi^a \phi^{[1;1]} + w^a w^{[1;1]} + T^a T^{[1;1]}}$$

Adjoint Eigenfunction

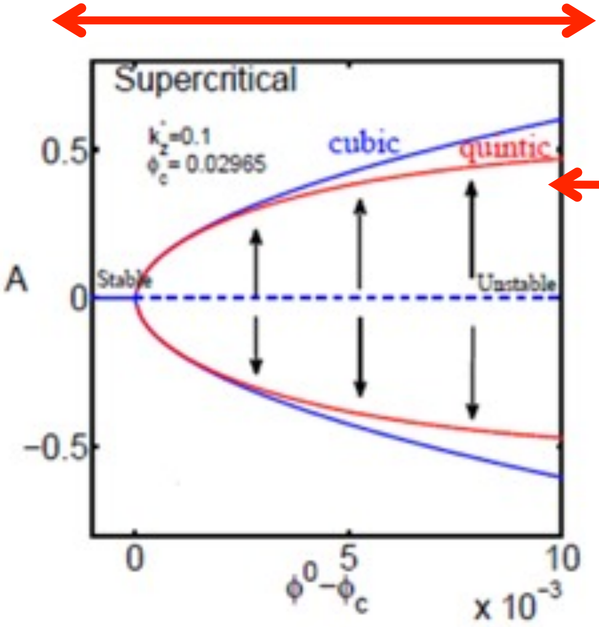
(ϕ^a, w^a, T^a)

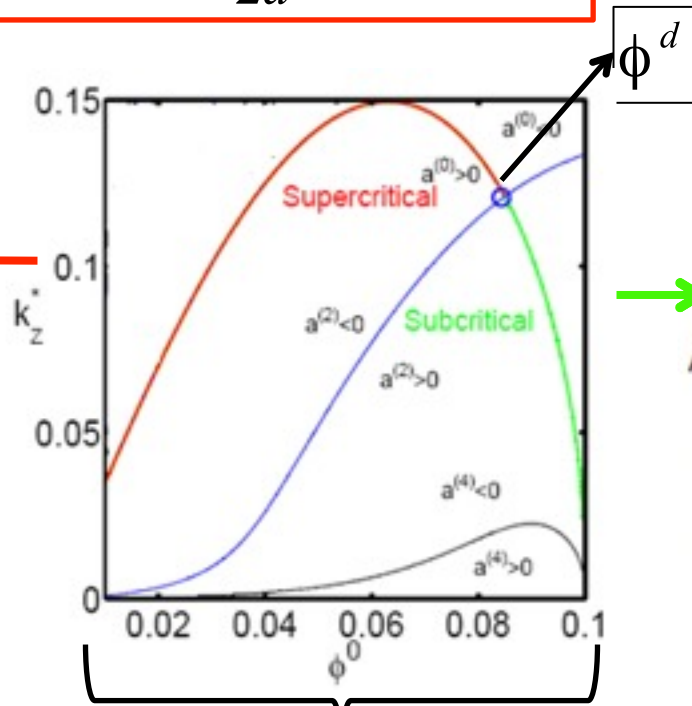
Analytical solution exists at any order in amplitude

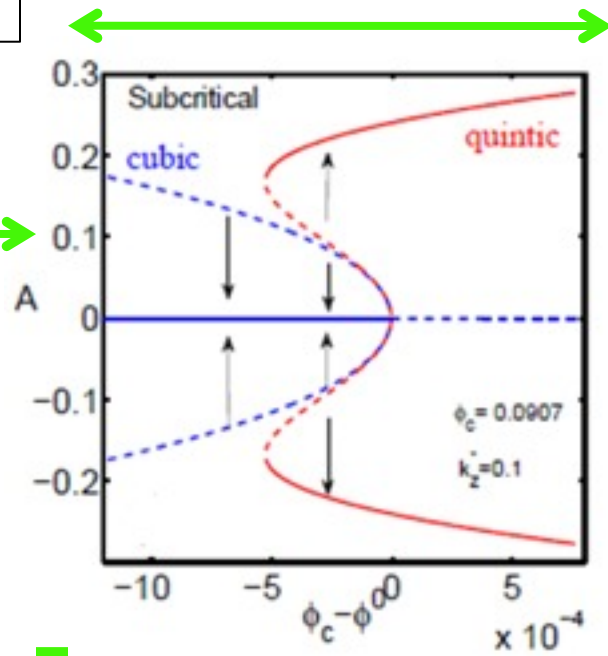
Nonlinear Vorticity Banding

$$A_e = \pm\sqrt{-\frac{a^{(0)}}{a^{(2)}}}$$

$$A_e = \pm\sqrt{\frac{-a^{(2)} \pm \sqrt{(a^{(2)})^2 - 4a^{(0)}a^{(4)}}}{2a^{(4)}}}$$







$\phi^0 < 0.08$

$\phi^d = 0.08$

$0.08 < \phi^0 \le 0.1$

Supercritical Pithfork Bifurcation

Subcritical Pitchfork Bifurcation

Density

Significance of higher order nonlinear correction

Nonlinear disturbance $$\left(\frac{\partial}{\partial t} - L\right)X' = \sum_{i=2}^{\infty} N_i$$

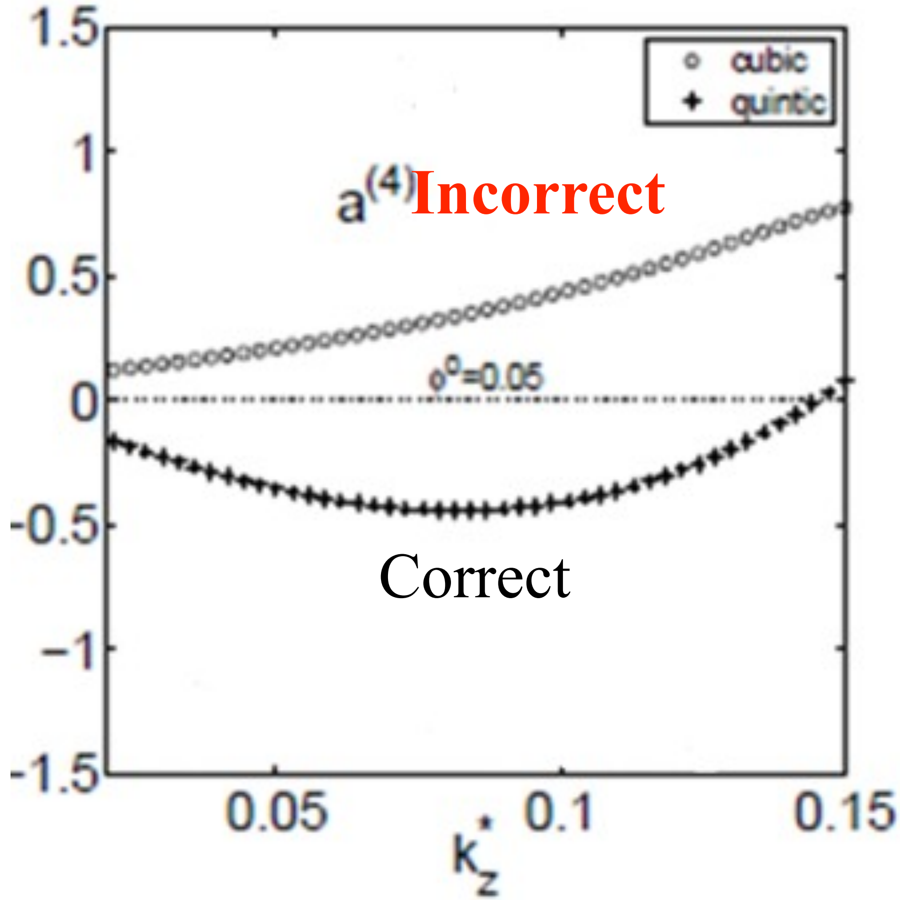
$$c^{(n)} = c^{(n)}(N_2, N_3, N_4, N_5, \ldots, N_{n+1})$$

Cubic correction

$$c^{(4)} = c^{(4)}(N_2, N_3, \cancel{N_4}, \cancel{N_5})$$

Quintic correction

$$c^{(4)} = c^{(4)}(N_2, N_3, N_4, N_5)$$

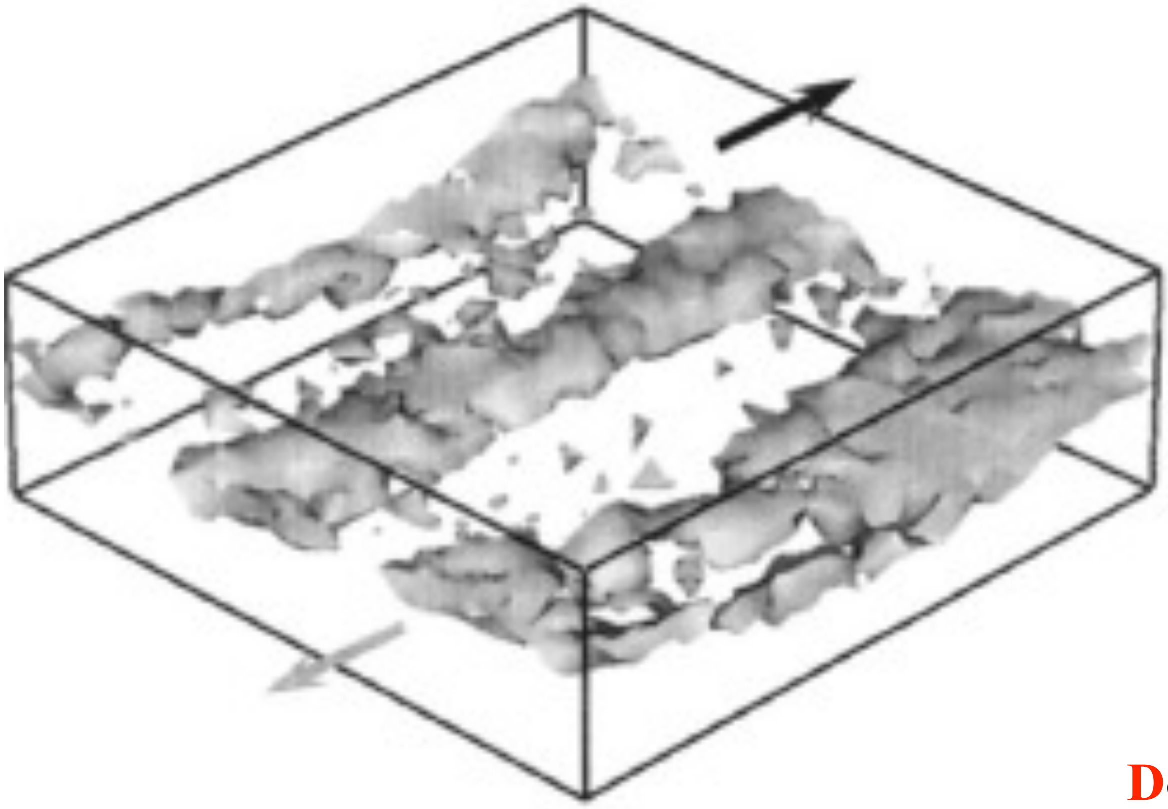


n-th order Landau coeff. need (n+1)-th order nonlinear term for correct results.

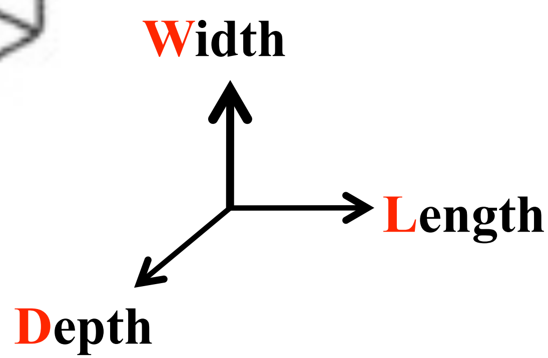
Vorticity Banding in Dilute 3D Granular Flow

(Conway and Glasser. Phys. Fluids, 2006)

Particle density iso-surfaces for $\phi = 0.05, e = 0.6$

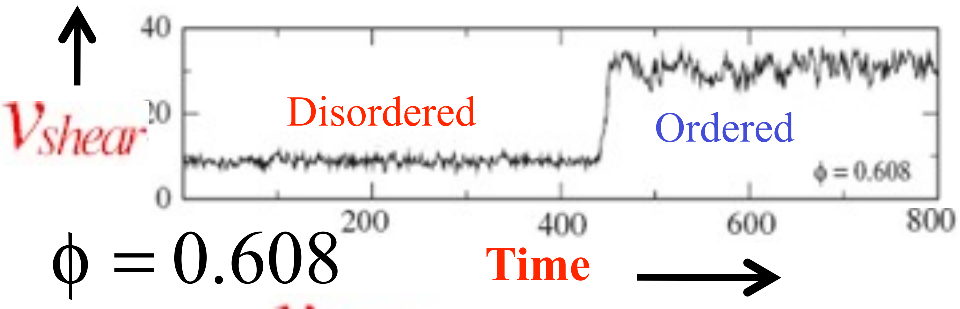


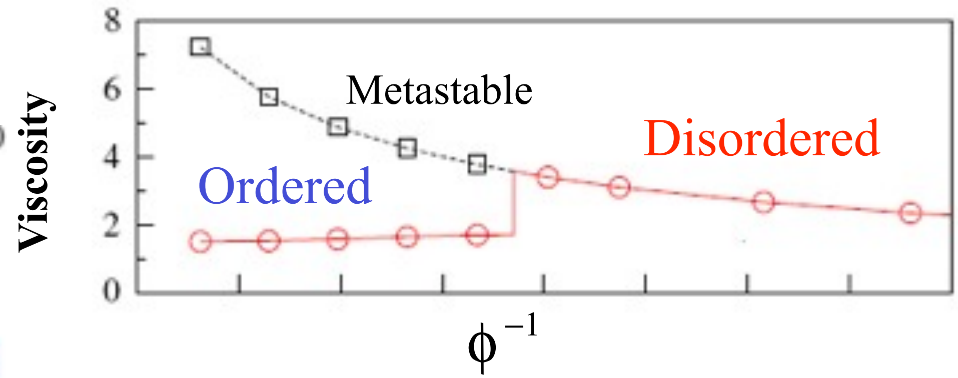
$$\frac{L}{W} = \frac{D}{W} = 3$$

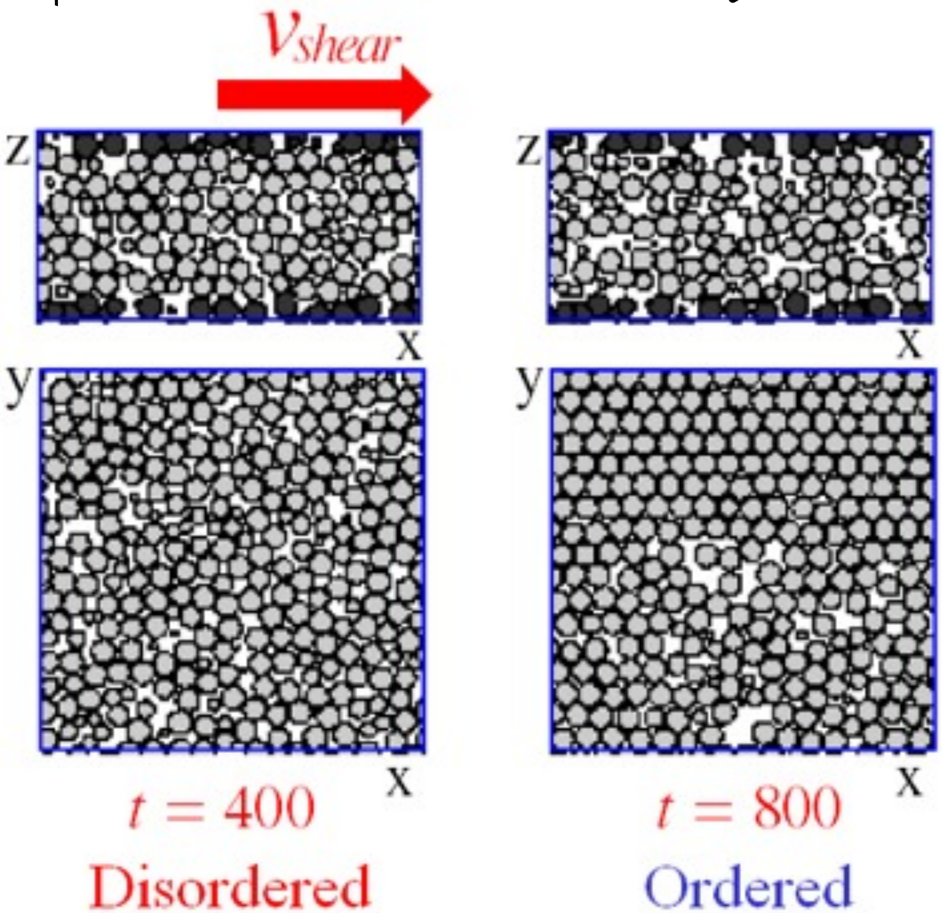


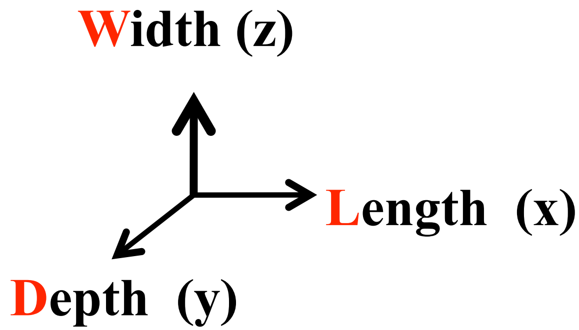
Vorticity Banding in Dense 3D Granular Flow

(Grebenkov, Ciamarra, Nicodemi, Coniglio, PRL 2008, vol 100)







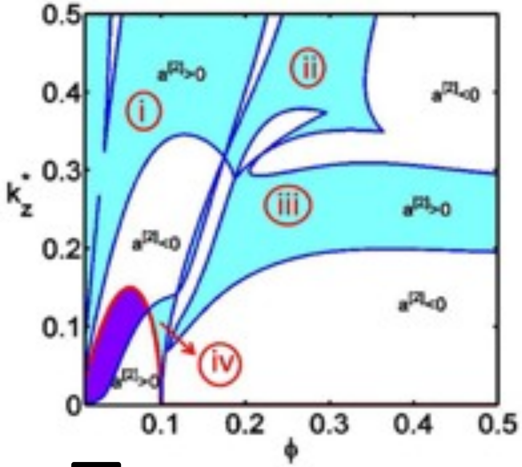


Ordered state ⇒ lower viscosity

Disordered state ⇒ higher viscosity

Conclusions

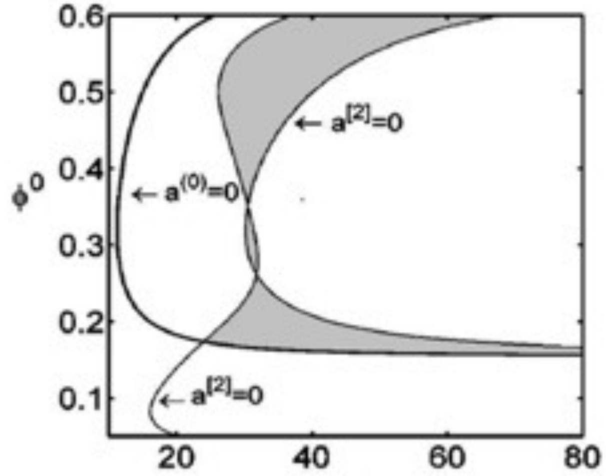
Vorticity Banding



Supercritical Region

Subcritical Region

Gradient Banding



$$H^* = H\beta^{-1}\sqrt{(1-e^2)}$$

$$\phi^0 = 0.1$$

Pitchfork Bifurcation — Subcritical and supercritical

Hopf Bifurcation — Subcritical

Density

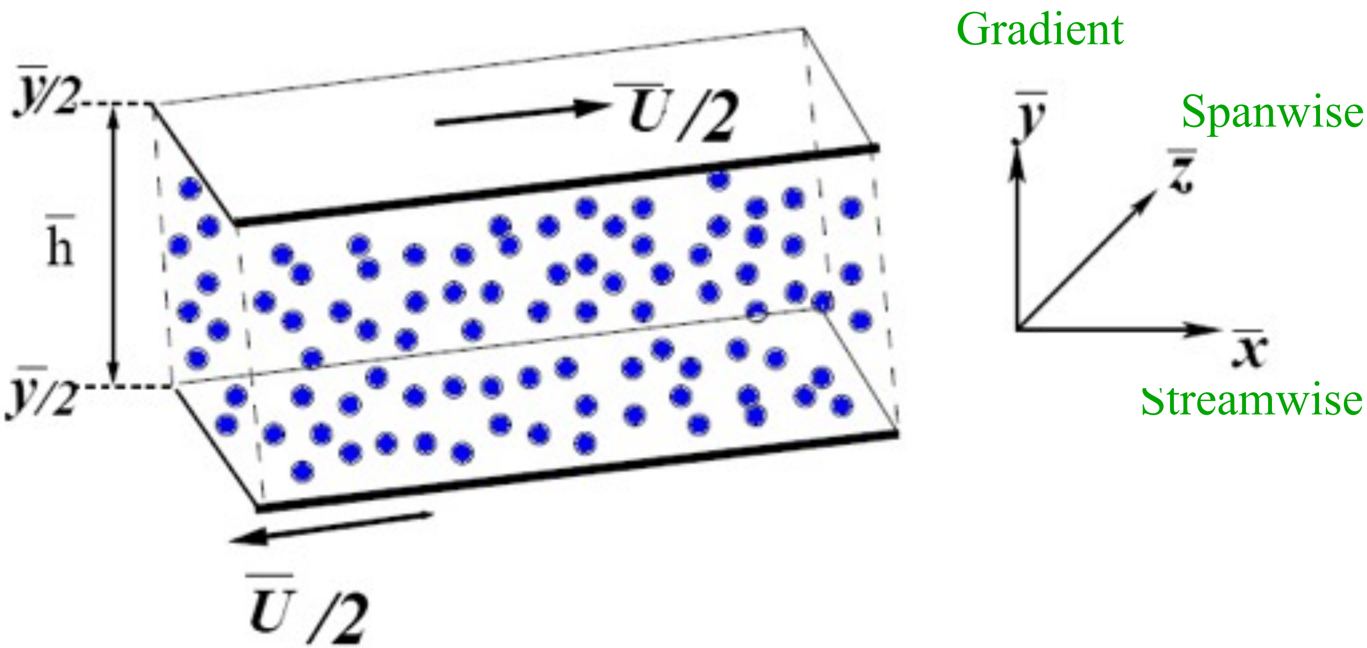
Pitchfork Bifurcation

Density

Analytical solution exists at any order.

Higher order nonlinear terms are important to get correct bifurcation scenario.

Patterns in three-dimensional gPCF



Shukla & Alam (2013c, preprint)

Nonlinear Stability

Shukla & Alam (2013) (Preprint for PoF)

Linear Problem $LX^{[1;1]} = c^{(0)} X^{[1;1]}$

Second Harmonic $L_{22} X^{[2;2]} = G_{22}$

Distortion to mean flow $L_{02} X^{[0;2]} = G_{02}$

Distortion to fundamental $L_{13} X^{[1;3]} = c^{(2)} X^{[1;1]} + G_{13}$

} Analytically solved

Analytical Expression of first Landau Coefficient

$$c^{(2)} = \frac{\phi^a G_{13}^1 + u^a G_{13}^2 + v^a G_{13}^3 + w^a G_{13}^4 + T^a G_{13}^5}{\phi^a \phi^{[1;1]} + u^a u^{[1;1]} + v^a v^{[1;1]} + w^a w^{[1;1]} + T^a T^{[1;1]}}$$

Adjoint Eigenfunction

$(\phi^a, u^a, v^a, w^a, T^a)$

Analytical solution exists

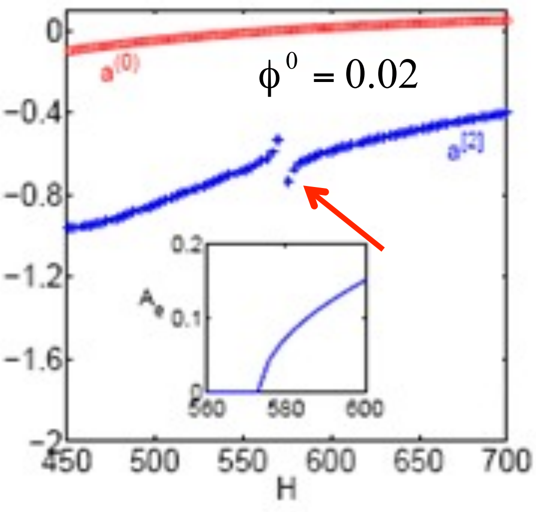
Dilute Flows

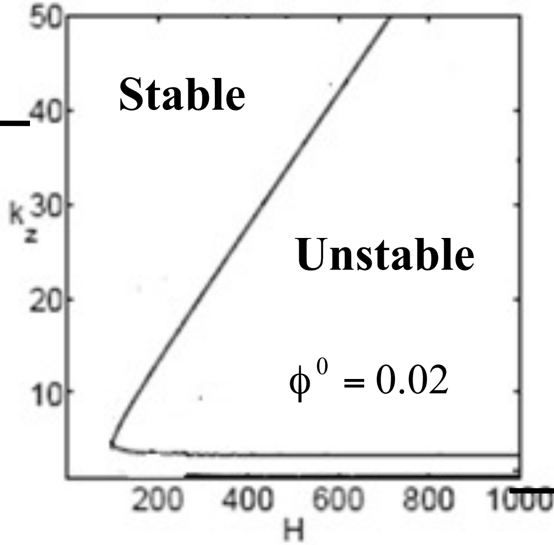
Supercritical bifurcation

Subcritical bifurcation

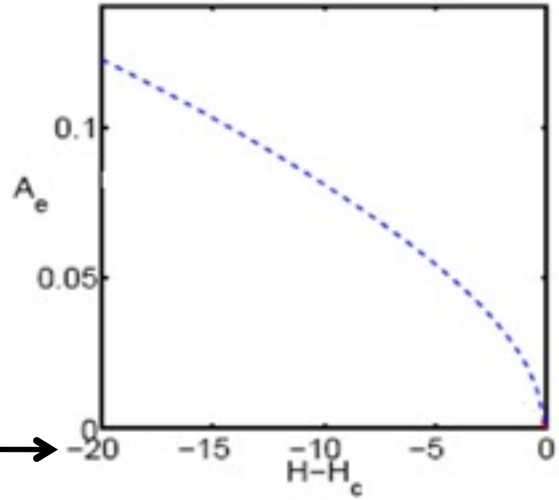
Linear Stability Curve

$k_z = 40$





$k_z = 2$



Large wavenumbers

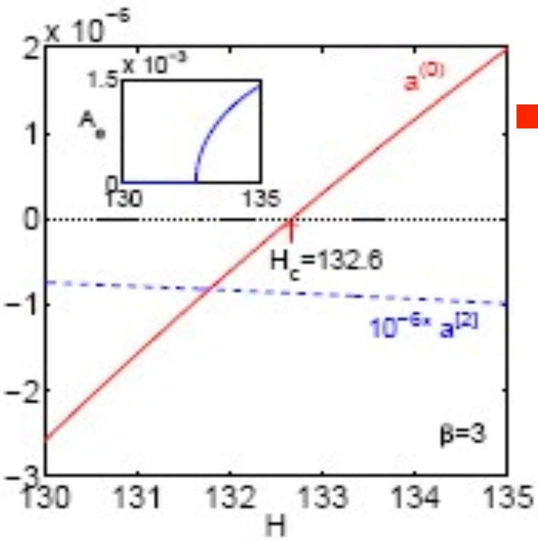
Small wavenumbers

Moderate-to-Dense Flows

Linear Stability Curve

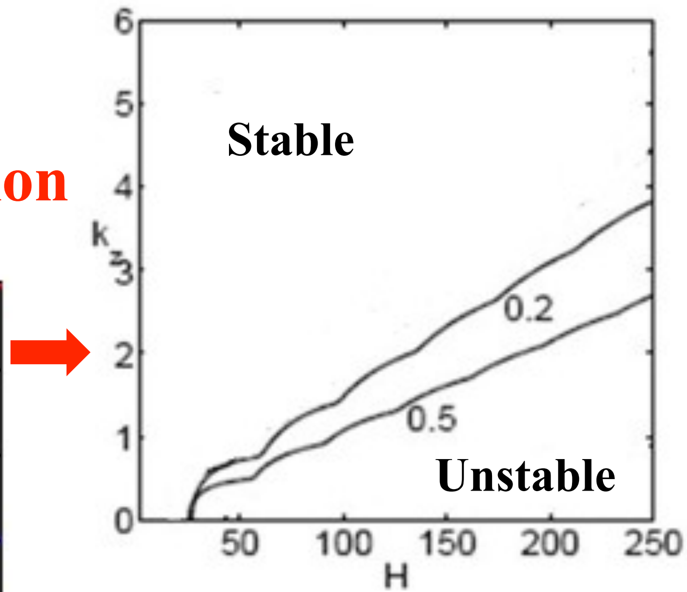
Moderate Flows

Supercritical bifurcation



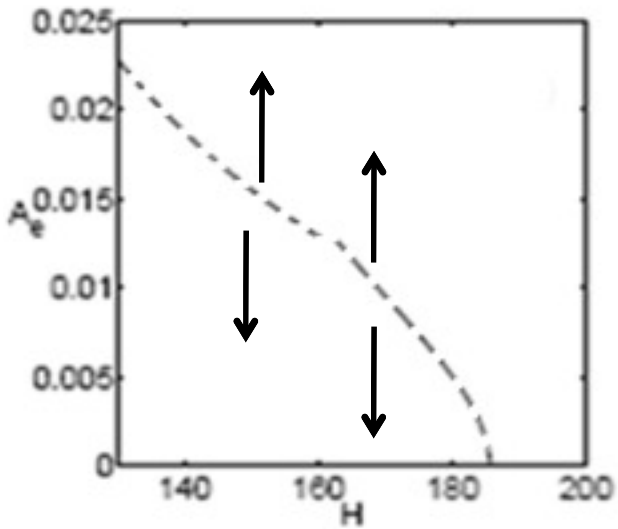
$\phi^0 = 0.2$

$k_z = 2, e = 0.8$



Dense Flows

Subcritical bifurcation



$\phi^0 = 0.5$

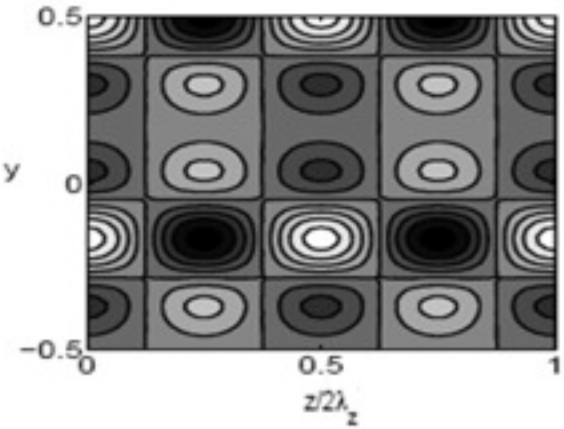
$k_z = 2, e = 0.8$

Linear and Nonlinear Density Patterns

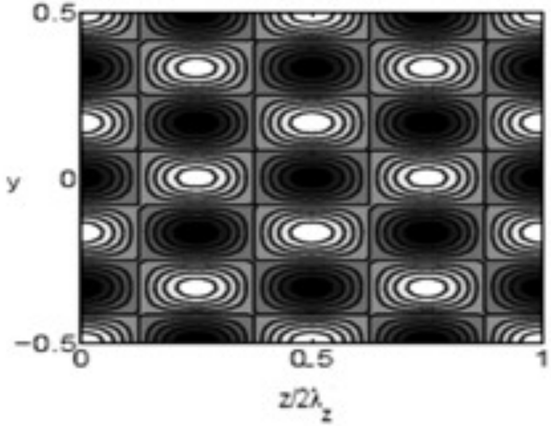
Stable, supercritical patterns

$H = 135, \phi^0 = 0.2,$
$e = 0.8, k_z = 2$

Nonlinear



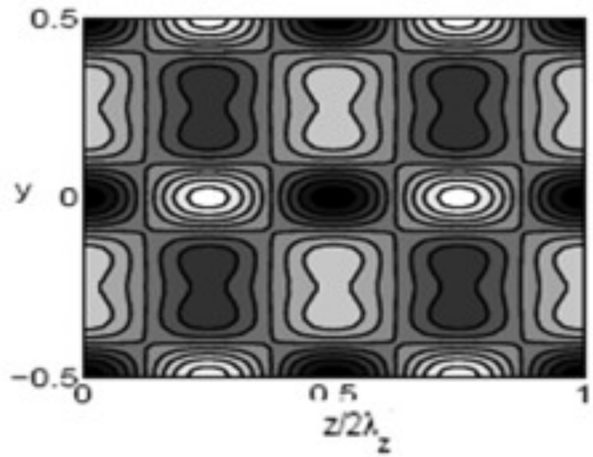
Linear



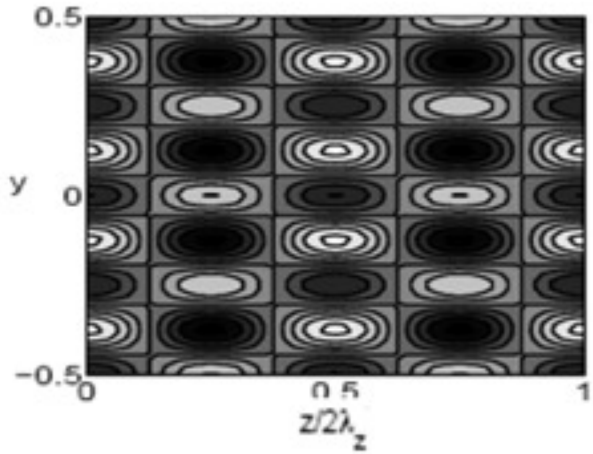
Unstable, subcritical patterns

$H = 140, \phi^0 = 0.5,$
$e = 0.8, k_z = 2$

Nonlinear



Linear



- ➢Patterns exists in streamwise and gradient direction.
- ➢Nonlinear pattern looks very different from linear patterns.

Conclusions

- In dilute limit finite amplitude solutions occur via supercritical bifurcation for large wavenumbers and via subcritical bifurcation for small wavenumbers.

- Transition from supercritical to subcritical in moderate-to-dense limit.

- The finite amplitude nonlinear patterns look very different from its linear analogue.

Shukla & Alam (2013c, preprint for PoF)

Theory for Spatially Modulated Patterns

Complex Ginzburg Landau Equation (CGLE)

Landau Equation

$$\frac{dA}{dt} = c^{(0)}A + c^{(2)}A\,|\,A\,|^2$$

Ordinary differential equation

Holds for **spatially periodic** patterns

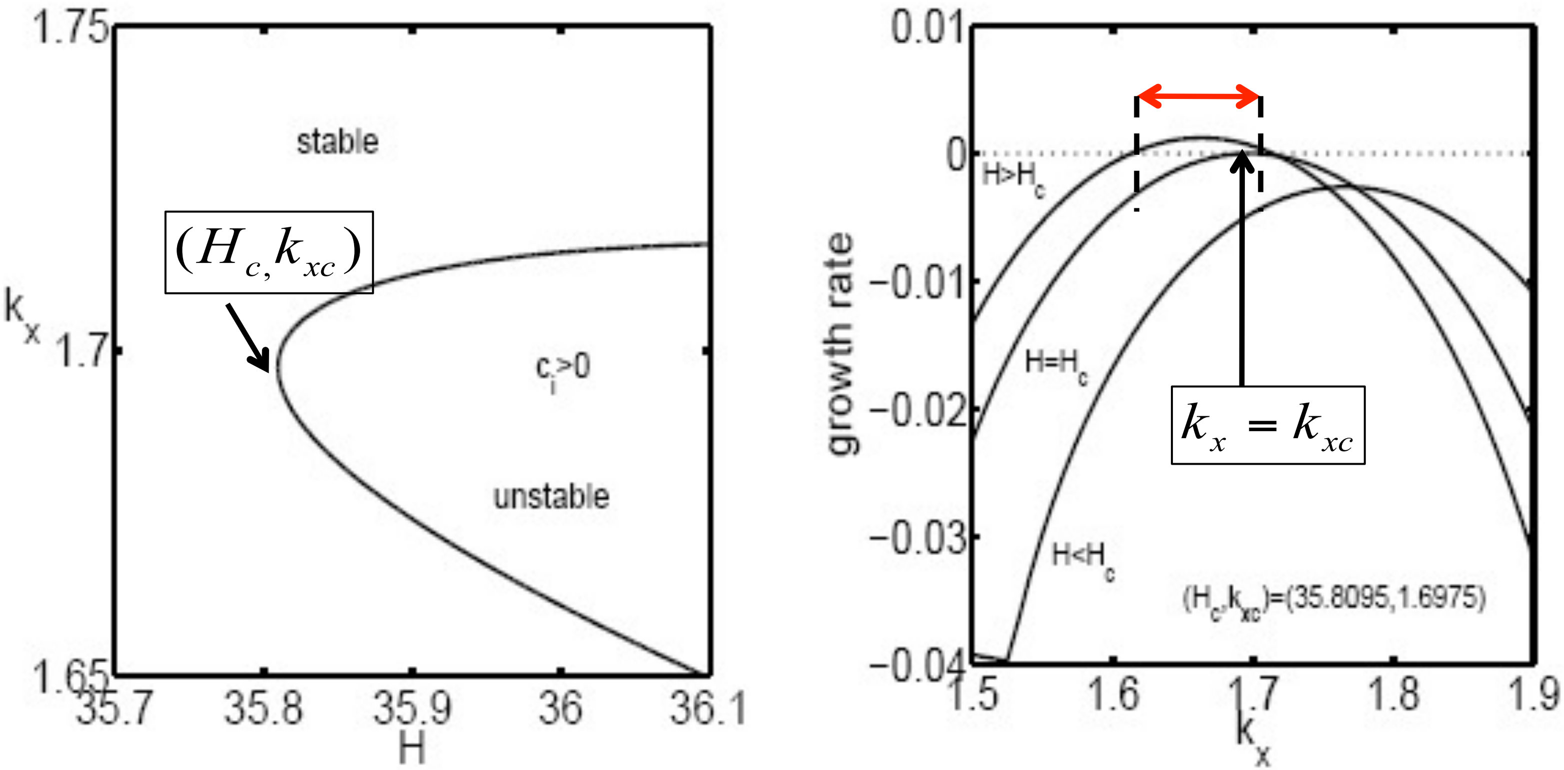
Complex Ginzburg Landau Equation

$$\frac{\partial A}{\partial t} = \varepsilon^2 A + a_2\frac{\partial^2 A}{\partial X^2} + c^{(2)}A\,|\,A\,|^2$$

Partial differential equation

Holds for spatially modulated patterns

Under which condition CGLE arises?



For $H < H_c$ all modes are decaying : Homogeneous state is stable,

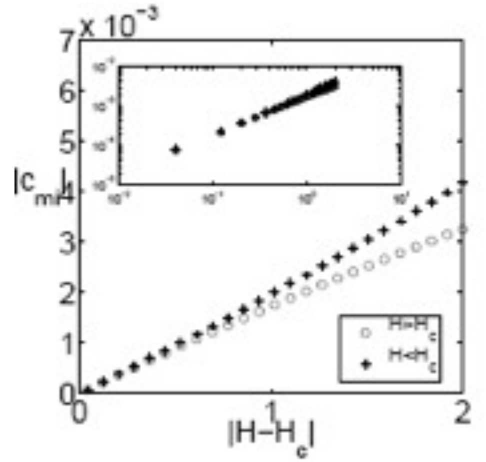
$H = H_c$ at $k_x = k_{xc}$ a critical wave number gains neutral stability,

$H > H_c$ there is a narrow band of wavenumbers around the critical value where the growth rate is slightly positive.

width of the unstable wavenumbers: $\propto (H - H_c)^{1/2}$

Theory (Multiple scale analysis)

$$\left(\frac{\partial}{\partial t} - L\right) X'(x,y,t) = \sum_{i=2}^{\infty} N_i$$



Growth rate is of order $H - H_c$ Stewartson & Stuart (1971)

The timescale at which nonlinear interaction affects the evolution of fundamental mode is of order 1/(growth rate)

$$\varepsilon^2 = d_{1r}\,|H - H_c|$$

$\tau = \varepsilon^2 t$ → Slow time scale

$\xi = \varepsilon(x - c_g t)$ → Slow length scale

↓

Group velocity

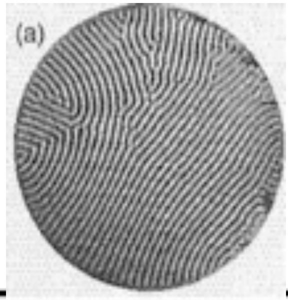
$$(\omega I - L_{k_{xc}}) X_{13} = \frac{1}{d_{1r}}\frac{\partial L_{k_x}}{\partial H} A X_1 - \frac{\partial A}{\partial \tau} X_1 + G_{13}|A|^2 A + \frac{\partial^2 A}{\partial \xi^2}\left[\left(c_g + \frac{1}{i}\left[\frac{\partial L_{k_x}}{\partial k_x}\right]\right) X^{[1;2]} - \frac{1}{2}\left[\frac{\partial^2 L_{k_x}}{\partial k_x^2}\right] X_1\right]$$

$$\frac{\partial A(\xi,\tau)}{\partial \tau} = \frac{d_1}{d_{1r}} A + a_2 \frac{\partial^2 A}{\partial \xi^2} + c^{(2)}|A|^2 A$$

$$\frac{\partial A}{\partial t} = \varepsilon^2 A + a_2 \frac{\partial^2 A}{\partial X^2} + c^{(2)} A\,|A|^2$$

$$X = x - c_g t$$

Patterns in Vibrated Bed



Patterns in Vibrated bed can be predicted by the complex Ginzburg LE
(Tsimring and Aranson 1997, Blair et. al. 2000)

$$\frac{\partial \psi}{\partial t} = \gamma\psi^* - (1 - i\omega)\psi + (1 + ib)\nabla^2\psi - |\psi|^2\psi - \rho\psi$$

Recent work of Saitoh and Hayakawa (Granular Matter 2011) on TDGL in ``*unbounded*'' shear flow.

Conclusions

- Complex Ginzburg Landau equation has been derived that describes spatio-temporal patterns in a ``*bounded*'' sheared granular fluid.
- Numerical results awaited...

Summary

- Landau-type order parameter theory for the gradient banding in gPCF has been developed using center manifold reduction. **Ref: PRL, vol. 103, 068001, (2009)**

- Analytical solution for the shearbanding instability, comparison with numerics & bifurcation scenario have been obtained. **Ref: JFM, vol. 666, 204-253, (2011a)**

- The order parameter theory for the 2D and 3D gPCF has been developed. Nonlinear patterns and bifurcations have been studied. **Ref: JFM, vol. 672, 147-195 (2011b)**

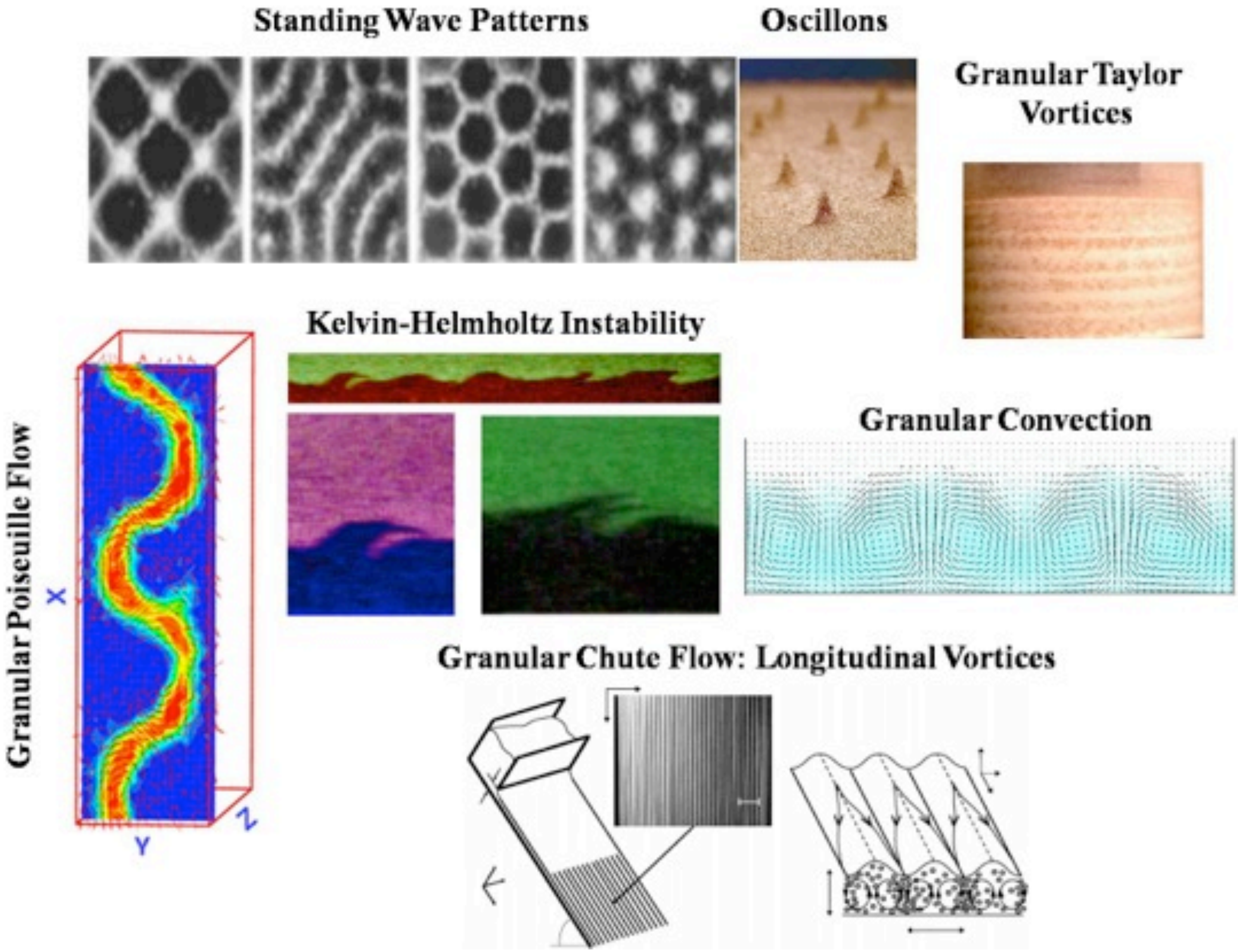
- Nonlinear states and bistability for vorticity banding have been analysed. **Ref: JFM, vol. 718 (2013b)**

- Coupled Landau equations for resonating and non-resonating cases have been derived. **Preprint**

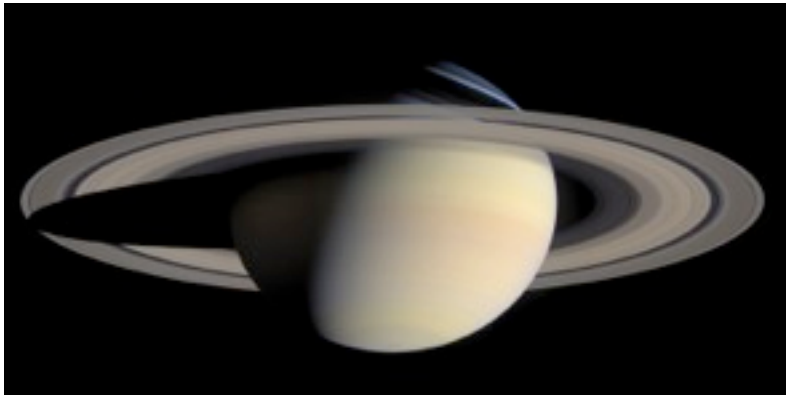
- Complex Ginzburg Landau equation has been derived for bounded shear flow. **Preprint**

Outlook

Present order-parameter theory can be modified for other pattern forming problems, e.g. granular convection, granular Taylor-Couette flow, inclined Chute flow, etc.



Revisit nonlinear theory of Saturn's Ring



- ➢Non-isothermal model with spin, stress anisotropy & self-gravity ...??
- ➢ Spatially modulated waves ...
- ➢ Wave interactions ...
- ➢ Secondary instability, ...