* Instabilities in Garnular Shear Flows

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Instabilities in freely cooling state

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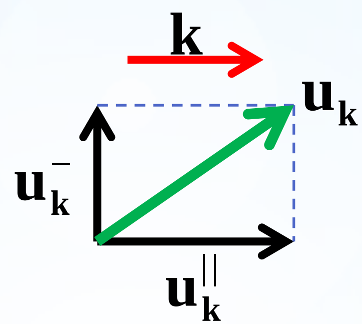
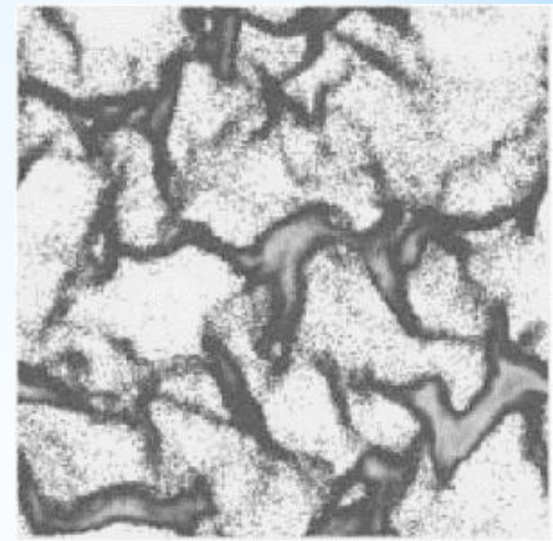
Hydrodynamic mode

$$\nu_{\mathbf{k}},\ \theta_{\mathbf{k}},\ \mathbf{u}_{\mathbf{k}} = \mathbf{u}_{\mathbf{k}}^{||} + \mathbf{u}_{\mathbf{k}}^{-}$$





Growth rate

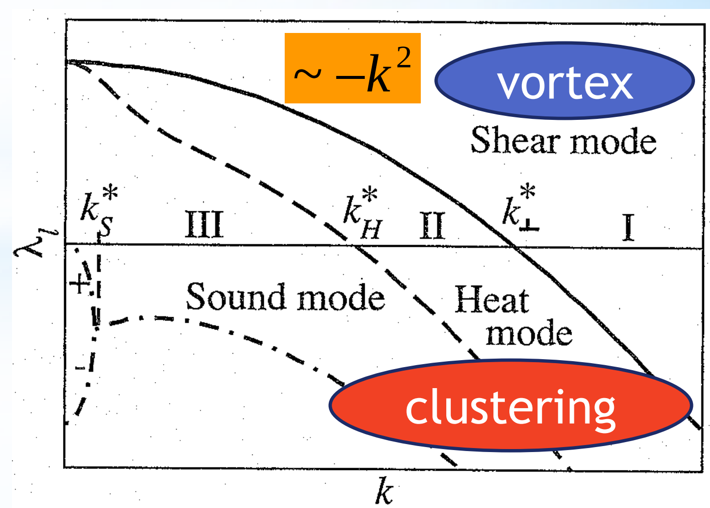
$$\sigma(k) = \lambda(k) + i\omega(k)$$

Shear mode $\lambda_{-}(k),\ \ \omega_{-}(k) = 0$

Heat mode $\lambda_{\mathrm{H}}(k),\ \ \omega_{\mathrm{H}}(k) = 0$

Sound mode $\lambda_{\mathrm{S}}(k) < 0,\ \ \omega_{\mathrm{S}}(k) \neq 0$

$$k_{-}^{*},\ k_{\mathrm{H}}^{*}\ \propto\ \varepsilon \equiv \sqrt{1-e^{2}}$$



Instabilities in granular shear flows

Finite-size systems

Linear stability analysis

M. Alam & P. R. Nott, J. Fluid Mech. 377 (1998) 99

Weakly nonlinear analysis

P. Shukla & M. Alam, Phys. Rev. Lett. 103 (2009) 068001
P. Shukla & M. Alam, J. Fluid Mech. 666 (2011) 204

Hydrodynamic limit

Weakly nonlinear analysis (Ginzburg-Landau equation)

K. Saitoh & H. Hayakawa, Granular Matter 13 (2011) 697

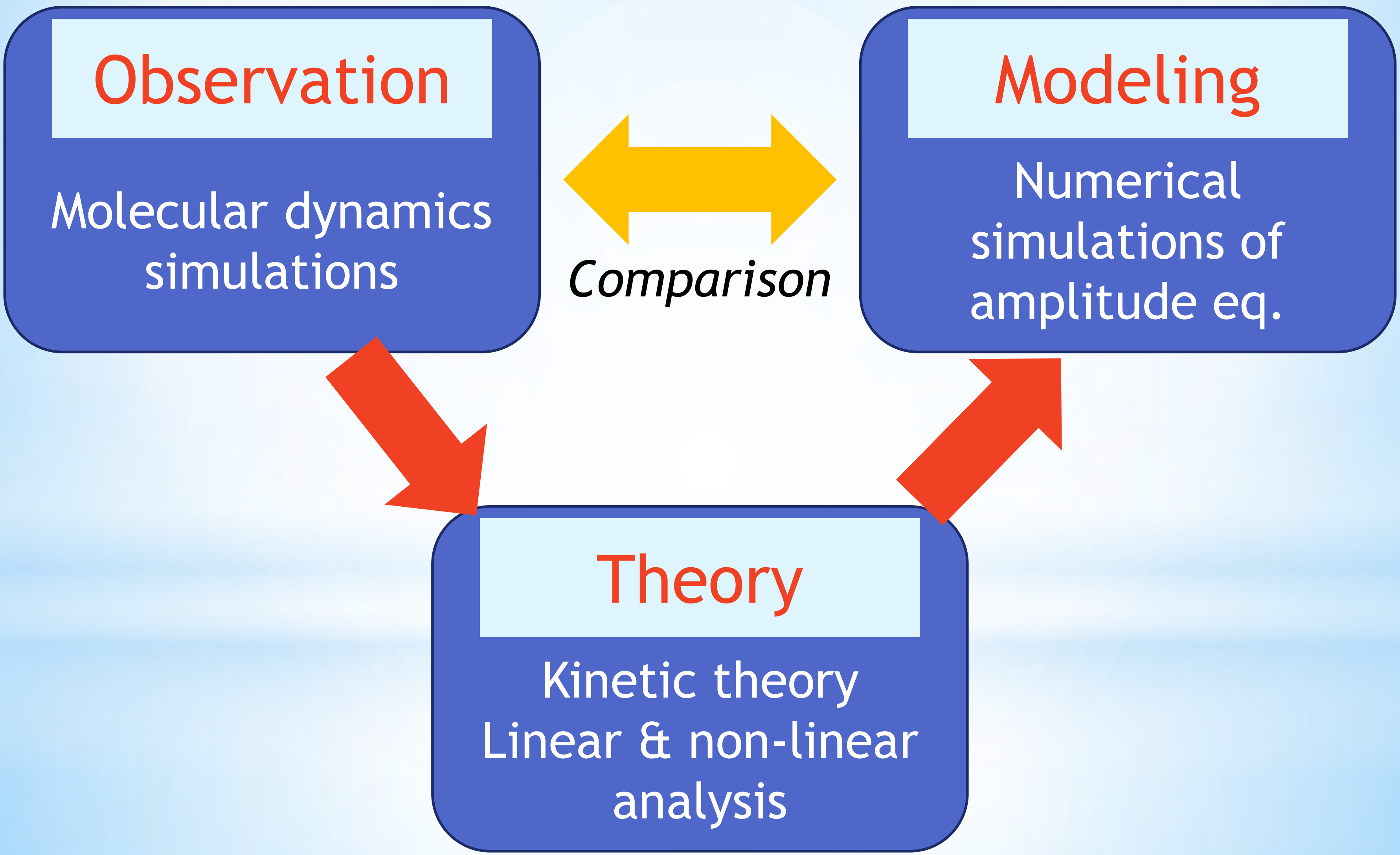
Numerical solution of the Ginzburg-Landau equation

K. Saitoh & H. Hayakawa, AIP. Conf. Proc. 1501 (2012) 1001
K. Saitoh & H. Hayakawa, Phys. Fluids (2013) in press

Molecular dynamics simulation

K. Saitoh & H. Hayakawa, Phys. Rev. E 75 (2007) 021302

Strategy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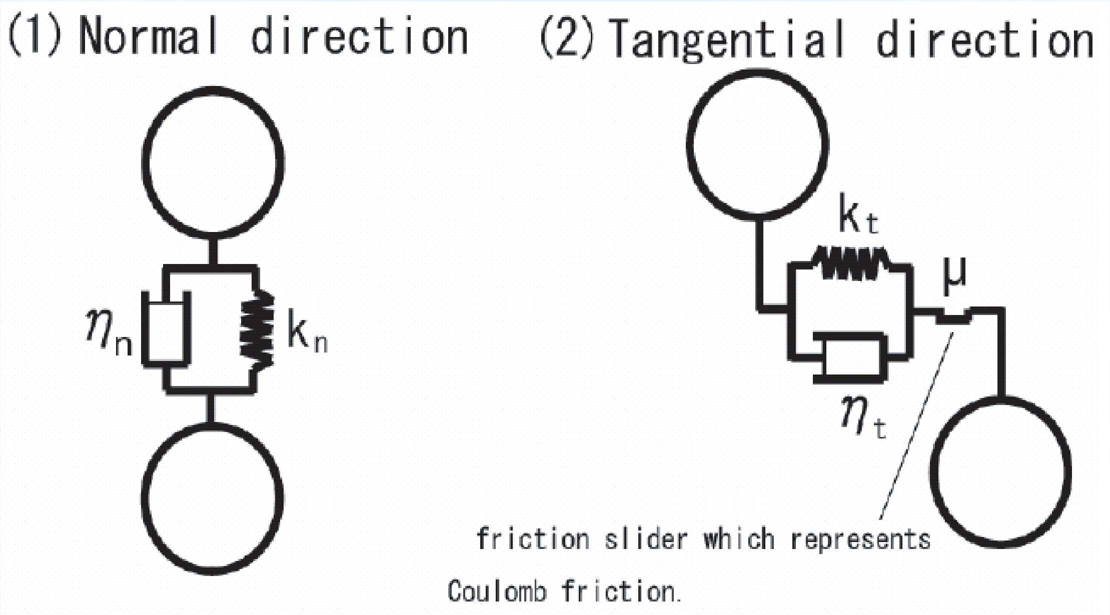
Molecular dynamics simulations
Comparison
Modeling
Numerical simulations of amplitude eq.
Theory
Kinetic theory
Linear & non-linear analysis

Molecular dynamics simulations

Model



2-dimensional frictional granular particles

$$e = 0.85 \qquad \mu = 0.2$$

System size

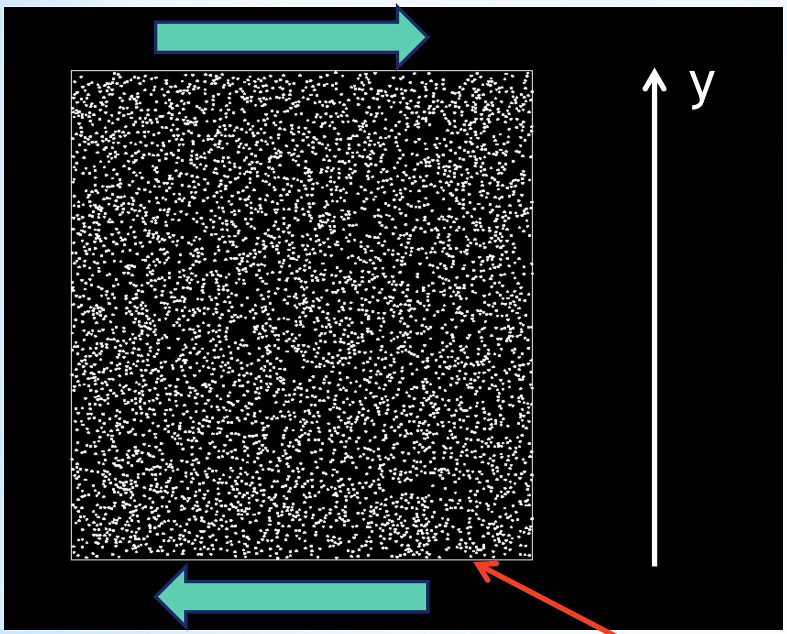
$$N = 5000 \qquad L \times L = 180d \times 180d$$

Mean area fraction

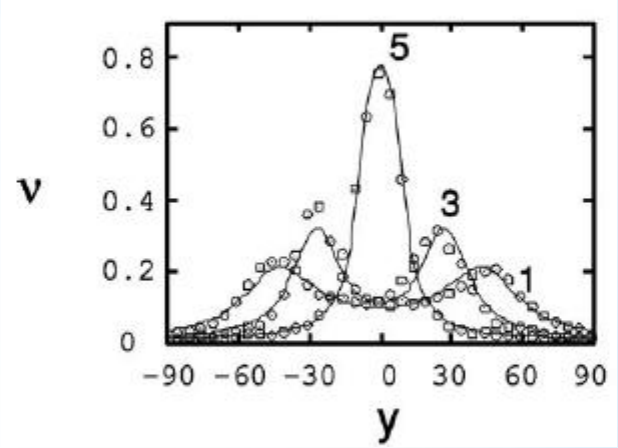
$$\nu_0 = 0.12$$

Molecular dynamics simulations

Dense plug formation

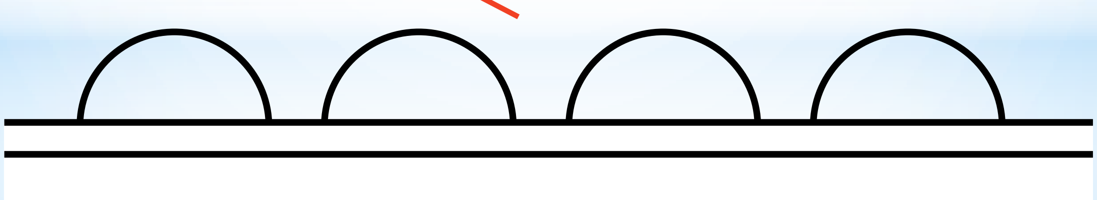


Time development of the area fraction



(Saitoh & Hayakawa, 2007)

Bumpy wall



We also observed a similar plug formation of frictionless granular particles under the Lees-Edwards boundary condition (Saitoh & Hayakawa, 2013)

Kinetic theory of granular gases

Granular hydrodynamic equations

Continuum equation $$\left(\frac{\partial}{\partial t}+u_j\nabla_j\right)\nu=-\nu\nabla_j u_j$$

Equation of motion $$\nu\left(\frac{\partial}{\partial t}+u_j\nabla_j\right)u_i=-\nabla_j P_{ij}$$

Equation of energy $$\frac{\nu}{2}\left(\frac{\partial}{\partial t}+u_j\nabla_j\right)\theta=-P_{ij}\nabla_i u_j-\nabla_j q_j-\chi$$

a. Area fraction, velocity fields, and granular temperature $\nu, u_i\, and\, \theta$.

b. Heat flux $q_i=-\kappa\nabla_i\theta-\lambda\nabla_i\nu$

c. Energy-sink term $\chi=\left(1-e^2\right)\left[\chi_1(\nu)\,\theta^{3/2}-\chi_2(\nu)\,\theta\,\nabla_j u_j\right]$

d. Hydrostatic pressure & transport coefficients are the functions of $\nu\ and\ \theta$.

Jenkins & Richman (1985)

Linear stability analysis

Granular hydrodynamic equations

Jenkins & Richman (1985)
2-dimensional frictionless disks
The Lees-Edwards boundary conditions

Scaling units

Mass	m
Length	d
Time	$t_0 \equiv d/U$

Shear rate

$$\dot{\gamma} = U/L = (d/L)\tau_0^{-1} \equiv \varepsilon \tau_0^{-1}$$

ε : the ratio of particle's diameter to gap

Hydrodynamic limit

$$\varepsilon << 1$$

Linear stability analysis

Homogeneous state

$$\nu=\nu_0,\quad \mathbf{u}=(\varepsilon y,0),\quad \theta=\theta_0 \propto \varepsilon^2/(1-e^2)$$

y : non-dimensionalized coordinate

Finite temperature approximation

$$\theta_0 \sim O(1) \quad \text{i.e.} \quad 1-e^2=\varepsilon^2$$

Hydrodynamic fields

$$\phi=(\nu,\ u,\ w,\ \theta)^{\mathrm{T}}$$

$$\phi_0=(\nu_0,\ \varepsilon y,\ 0,\ \theta_0)^{\mathrm{T}}$$

$$\delta\phi=(\delta\nu,\ \delta u,\ \delta w,\ \delta\theta)^{\mathrm{T}}$$

$$\phi(\mathbf{r},t)=\phi_0+\delta\phi(\mathbf{r},t)$$

Kelvin mode $\mathbf{k}(t)\equiv(k_x,\ k_y-\varepsilon t k_x)$

$$\delta\phi(\mathbf{r},t)=A^{\mathrm{L}}\sum_{k_y}\varphi_{k_y}^{\mathrm{L}}e^{ik_y y}+A^{\mathrm{NL}}\sum_{k_x\neq 0}\sum_{k_y}\varphi_{\mathbf{k}(t)}^{\mathrm{NL}}e^{i\mathbf{k}(t)\cdot\mathbf{r}}$$

Layering mode $(k_x=0)$

Non-layering mode $(k_x\neq 0)$

Linear stability analysis

Linearized granular hydrodynamic equation

$$\frac{d}{dt}\varphi^{\mathrm{I}} = L(t)\varphi^{\mathrm{I}}$$

$$L(t) = L_0(k_x, k_y) + tk_x L_0(k_x, k_y) + (tk_x)^2 L_2(k_y)$$

Layering mode $\left(k_x = 0 \right)$

Matrix $L(t) = L_0(0, k_y)$ *: independent of time*

Growth rate $\varphi^{\mathrm{L}}_{k_y} \propto e^{\sigma t}$

Eigenvalue problem

$$L_0(0, k_y)\varphi^{\mathrm{L}}_{k_y} = \sigma\, \varphi^{\mathrm{L}}_{k_y}$$

Linear stability analysis

Perturbative calculations

Eigenvalue problem $L_0\,\varphi = \sigma\,\varphi$

Wave vector $k_y \equiv \varepsilon q$

cf.) Clustering instabilities

$$k_-^*,\ k_{\mathrm{H}}^* \ \propto\ \varepsilon$$

Matrix $L_0 = \varepsilon\,M_1 + \varepsilon^2 M_2 + \cdots$

Eigenvalue $\sigma = \varepsilon\,\sigma_1 + \varepsilon^2 \sigma_2 + \cdots$

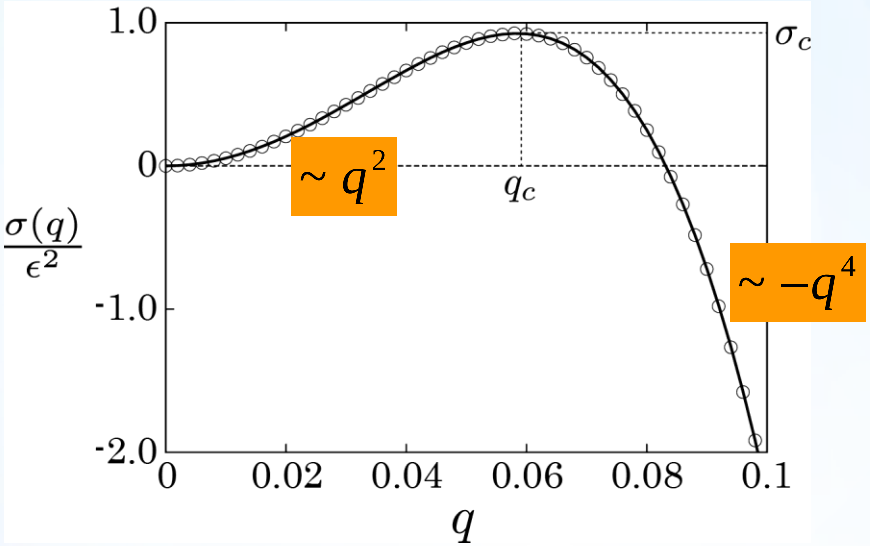
Right eigenvector $\varphi = \varphi_0 + \varepsilon\,\varphi_1 + \cdots$

Left eigenvector $\tilde{\varphi} = \tilde{\varphi}_0 + \varepsilon\,\tilde{\varphi}_1 + \cdots$

1*) We omit the superscript "L" and subscript "*ky*".
2*) We have 4 eigenvalues and 4 eigenvectors.

Linear stability analysis

Dispersion relation



Open circles
Numerical solution

Solid line

$$\lambda(q) = a_2 q^2 - a_4 q^4$$

The eigenvalue with the maximum real part

$$\sigma_1 = 0, \quad \sigma_2 = a_2 q^2 - a_4 q^4 + \cdots$$

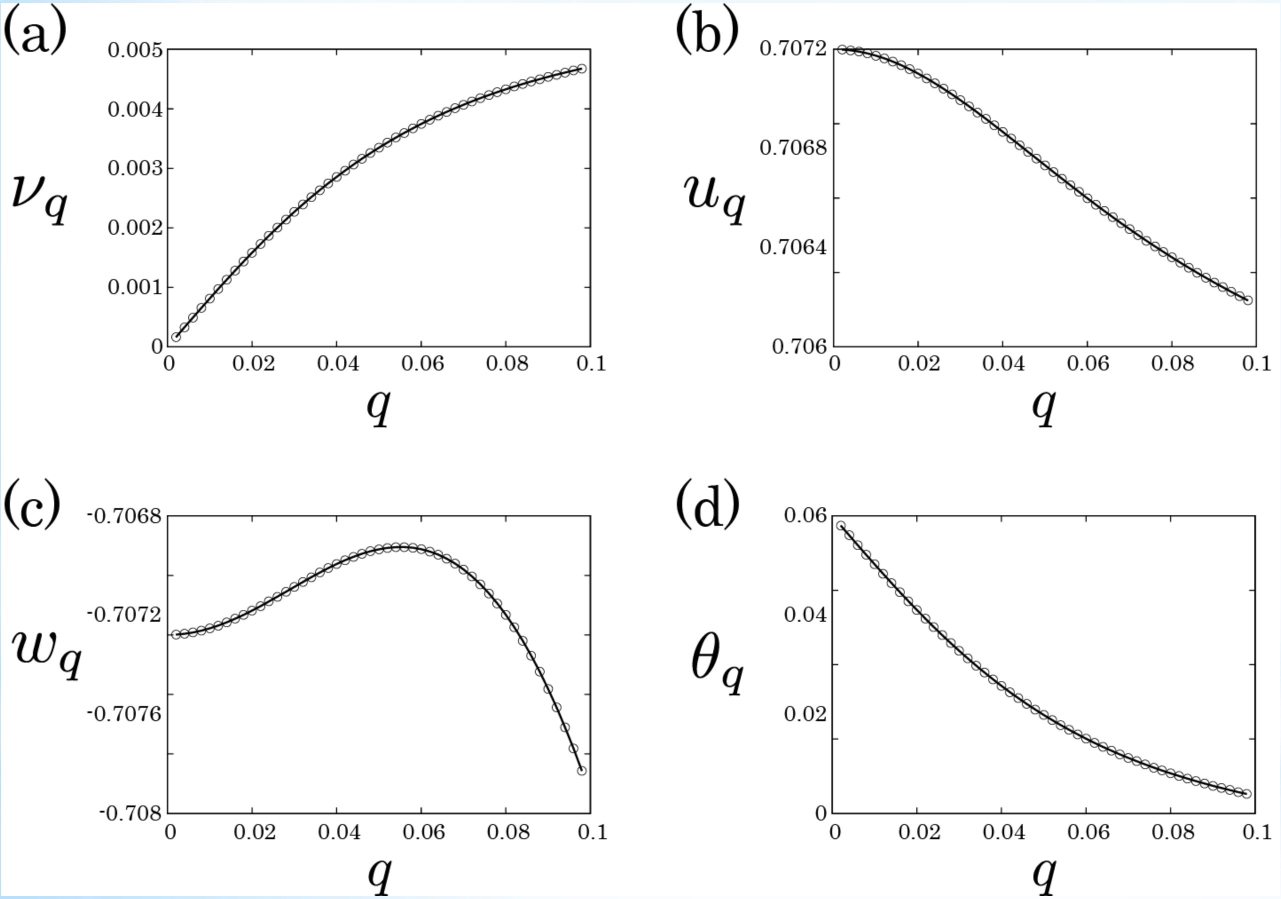
$$\therefore \sigma(q) = \varepsilon^2 \left(a_2 q^2 - a_4 q^4 + \cdots\right)$$

The most unstable mode

$$q_c \cong \sqrt{a_2 / 2a_4}$$

Linear stability analysis

Eigenvectors (layering mode)



Open circles
Numerical solutions

Solid line

$$\varphi(q) = \varphi_0 + \varepsilon\varphi_1$$

1*) We also confirm good agreements of $\tilde{\varphi}$

Linear stability analysis

Eigenvectors (non-layering mode)

$$\frac{d}{dt}\varphi^{\mathrm{NL}} = L(t)\varphi^{\mathrm{NL}}$$

$$v^{\mathrm{NL}}_{\mathbf{q}(t)} = -\frac{p_0}{\theta_0 J}E_2(t) + \frac{\nu_0}{J}E_3(t)\cos\omega(t)$$

$$u^{\mathrm{NL}}_{\mathbf{q}(t)} = -\frac{\varepsilon t}{\sqrt{1+(\varepsilon t)^2}}E_1(t) - \frac{1}{\sqrt{1+(\varepsilon t)^2}}E_3(t)\sin\omega(t)$$

$$w^{\mathrm{NL}}_{\mathbf{q}(t)} = -\frac{1}{\sqrt{1+(\varepsilon t)^2}}E_1(t) + \frac{\varepsilon t}{\sqrt{1+(\varepsilon t)^2}}E_3(t)\sin\omega(t)$$

$$\theta^{\mathrm{NL}}_{\mathbf{q}(t)} = \frac{p'_0}{J}E_2(t) + \frac{2p_0}{J}E_3(t)\cos\omega(t)$$

$$E_1(t),\quad E_2(t),\quad E_3(t) \sim e^{-\varepsilon^2 t}$$

Weakly nonlinear analysis

Long length & long time scales

Scaling of the eigenvalue and wave number

$$\sigma \approx \varepsilon^2 \lambda \qquad k_y = \varepsilon q$$

Note:

$$\sigma t = \lambda \tau \qquad k_y y = q \zeta$$

Long time scale $\tau = \varepsilon^2 t \longrightarrow \partial_t = \varepsilon^2 \partial_\tau$

Long length scale $\zeta = \varepsilon y \longrightarrow \partial_y = \varepsilon \partial_\zeta$

The most unstable solution

$$\delta\phi_{\mathrm{m}} = A^{\mathrm{L}}(\tau, \zeta)\varphi_{q_c}^{\mathrm{L}} e^{iq_c\zeta} + \text{ c.c.}$$

1*) Other modes can be suppressed.

Weakly nonlinear analysis

Perturbative expansions

$$A = \varepsilon A_1 + \varepsilon^2 A_2 + \cdots$$

$$L_0 = \varepsilon M_1 + \varepsilon^2 M_2 + \cdots$$

Perturbative calculations

$O(1)$, $O(\varepsilon)$: absent

consistent

$O(\varepsilon^2)$ $\quad M_1 \varphi^{\mathrm{L}}_{q_c} = 0 \longleftrightarrow \because \tilde{\varphi}^{\mathrm{L}}_{q_c} M_1 \varphi^{\mathrm{L}}_{q_c} = \sigma_1 = 0$

$O(\varepsilon^3)$ $\quad \varphi^{\mathrm{L}}_{q_c} \partial_\tau A^{\mathrm{L}}_1 = M_2 \varphi^{\mathrm{L}}_{q_c} A^{\mathrm{L}}_1 + D \partial^2_\zeta A^{\mathrm{L}}_1 + N_3 A^{\mathrm{L}}_1 | A^{\mathrm{L}}_1 |^2$

1D TDGL eq. $\quad \therefore \partial_\tau A^{\mathrm{L}}_1 = \sigma_2 A^{\mathrm{L}}_1 + d\, \partial^2_\zeta A^{\mathrm{L}}_1 + \beta A^{\mathrm{L}}_1 | A^{\mathrm{L}}_1 |^2$

$$\because \tilde{\varphi}^{\mathrm{L}}_{q_c} \varphi^{\mathrm{L}}_{q_c} = 1, \quad \tilde{\varphi}^{\mathrm{L}}_{q_c} M_2 \varphi^{\mathrm{L}}_{q_c} = \sigma_2, \quad d \equiv \tilde{\varphi}^{\mathrm{L}}_{q_c} D, \quad \beta \equiv \tilde{\varphi}^{\mathrm{L}}_{q_c} N_3$$

Weakly nonlinear analysis

Higher order calculations

$$O(\varepsilon^3) \quad \partial_\tau A_1^L = \sigma_2 A_1^L + d\,\partial_\zeta^2 A_1^L + \beta A_1^L |A_1^L|^2$$

$$O(\varepsilon^4) \quad \varepsilon\,\partial_\tau A_2^L = \varepsilon\,\sigma_2 A_2^L + d\,\partial_\zeta^2 A_2^L + \beta\left(A_1^{L\,2} A_2^{L\,*} + 2|A_1^L|^2 A_2^L\right)$$

$$O(\varepsilon^5) \quad \varepsilon^2\partial_\tau A_3^L = \varepsilon^2\sigma_2 A_3^L + d\,\partial_\zeta^2 A_3^L + \beta\left(A_1^L A_2^{L\,2^*} + \cdots\right) + \gamma A_1^L |A_1^L|^4$$

Sum up

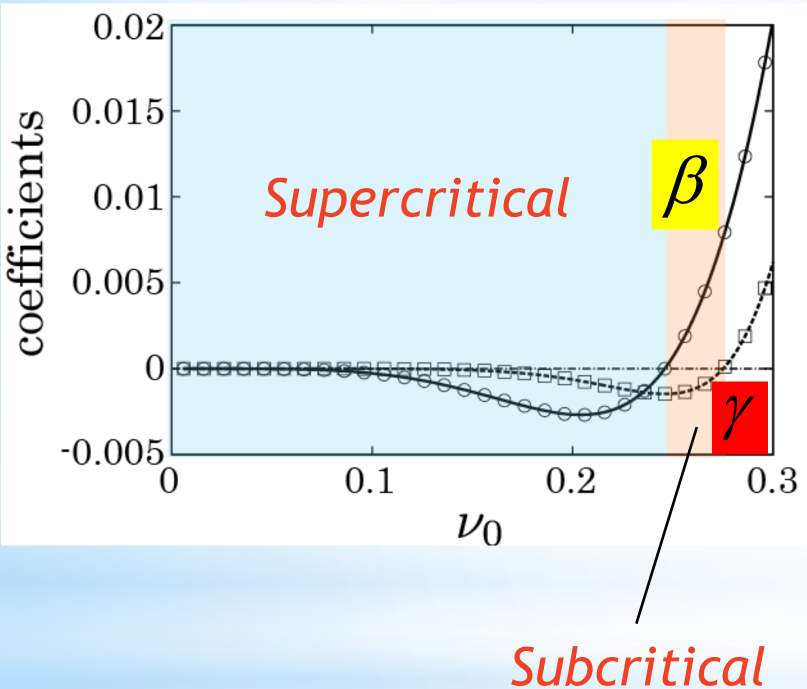
$$\partial_\tau \tilde{A}^L = \sigma_2 \tilde{A}^L + d\,\partial_\zeta^2 \tilde{A}^L + \beta \tilde{A}^L |\tilde{A}^L|^2 + \gamma \tilde{A}^L |\tilde{A}^L|^4$$

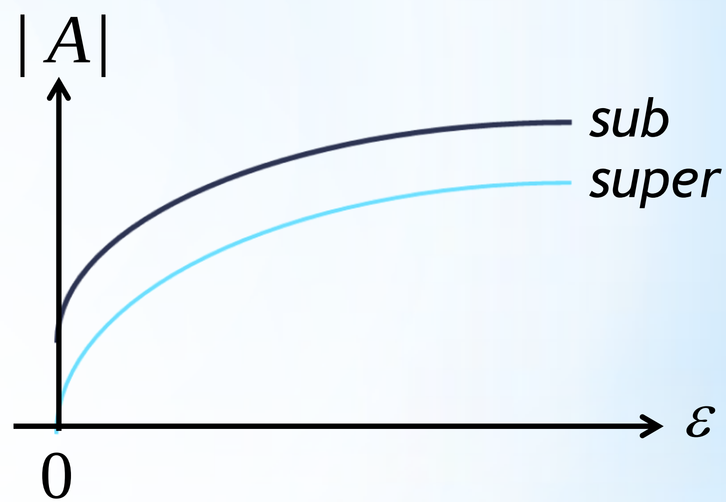
Envelop function

$$\tilde{A}^L \equiv A_1^L + \varepsilon A_2^L + \varepsilon^2 A_3^L$$

Weakly nonlinear analysis

Bifurcation analysis

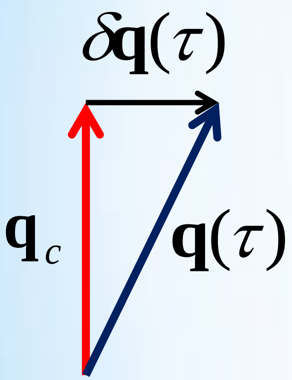




Elastic & non-shear limit
i.e. Equilibrium gas $\varepsilon = 0$

Weakly nonlinear analysis

Hybrid approach



Small deviation around the most unstable mode

$$\mathbf{q}(\tau) = \mathbf{q}_c + \delta\mathbf{q}(\tau)$$

The most unstable solution

$$\delta\phi_{\mathrm{m}} \approx A\varphi^{\mathrm{L}}_{q_c} e^{i\mathbf{q}(\tau)\cdot\mathbf{z}} + \text{c.c.}$$

$$\delta\phi_{\mathrm{d}} = A\varphi^{\mathrm{NL}}_{\mathbf{q}(\tau)} e^{i\mathbf{q}(\tau)\cdot\mathbf{z}} + \text{c.c.}$$

"Deviated" solution

$$\delta\phi_{\mathrm{h}} \approx A(\tau,\xi,\zeta)\left[\varphi^{\mathrm{L}}_{q_c} + \varphi^{\mathrm{NL}}_{\mathbf{q}(\tau)}\right] e^{i\mathbf{q}(\tau)\cdot\mathbf{z}} + \text{c.c.}$$

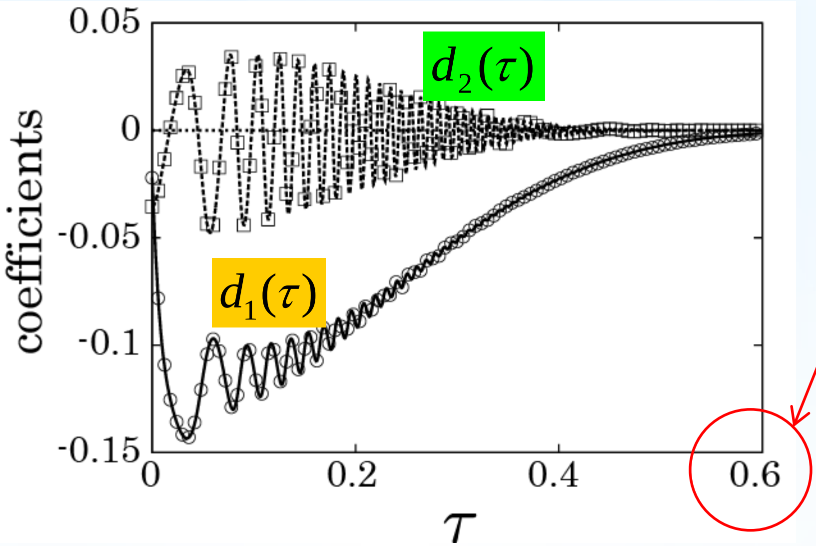
2D TDGL eq.

$$\partial_\tau A = \sigma_2 A + d_1(\tau)\,\partial_\xi^2 A + d_2(\tau)\,\partial_\xi \partial_\zeta A + d\,\partial_\zeta^2 A + \beta A|A|^2$$

New terms

Weakly nonlinear analysis

Time dependent diffusion coefficients



Decay to zero before $\tau = 1$

$$\because \varphi^{\mathrm{NL}}_{\mathbf{q}(\tau)} \sim e^{-\tau}$$

2D TDGL eq.

$$\partial_\tau A = \sigma_2 A + d_1(\tau)\,\partial_\xi^2 A + d_2(\tau)\,\partial_\xi \partial_\zeta A + d\,\partial_\zeta^2 A + \beta\, A|A|^2$$

→ *Zero*

1D TDGL eq.

$$\partial_\tau A = \sigma_2 A + d\,\partial_\zeta^2 A + \beta\, A|A|^2$$

Numerical solutions

Scheme

$$\partial_\tau A = \sigma_2 A + d_1(\tau)\,\partial_\xi^2 A + d_2(\tau)\,\partial_\xi \partial_\zeta A + d\,\partial_\zeta^2 A + \beta\, A|A|^2$$

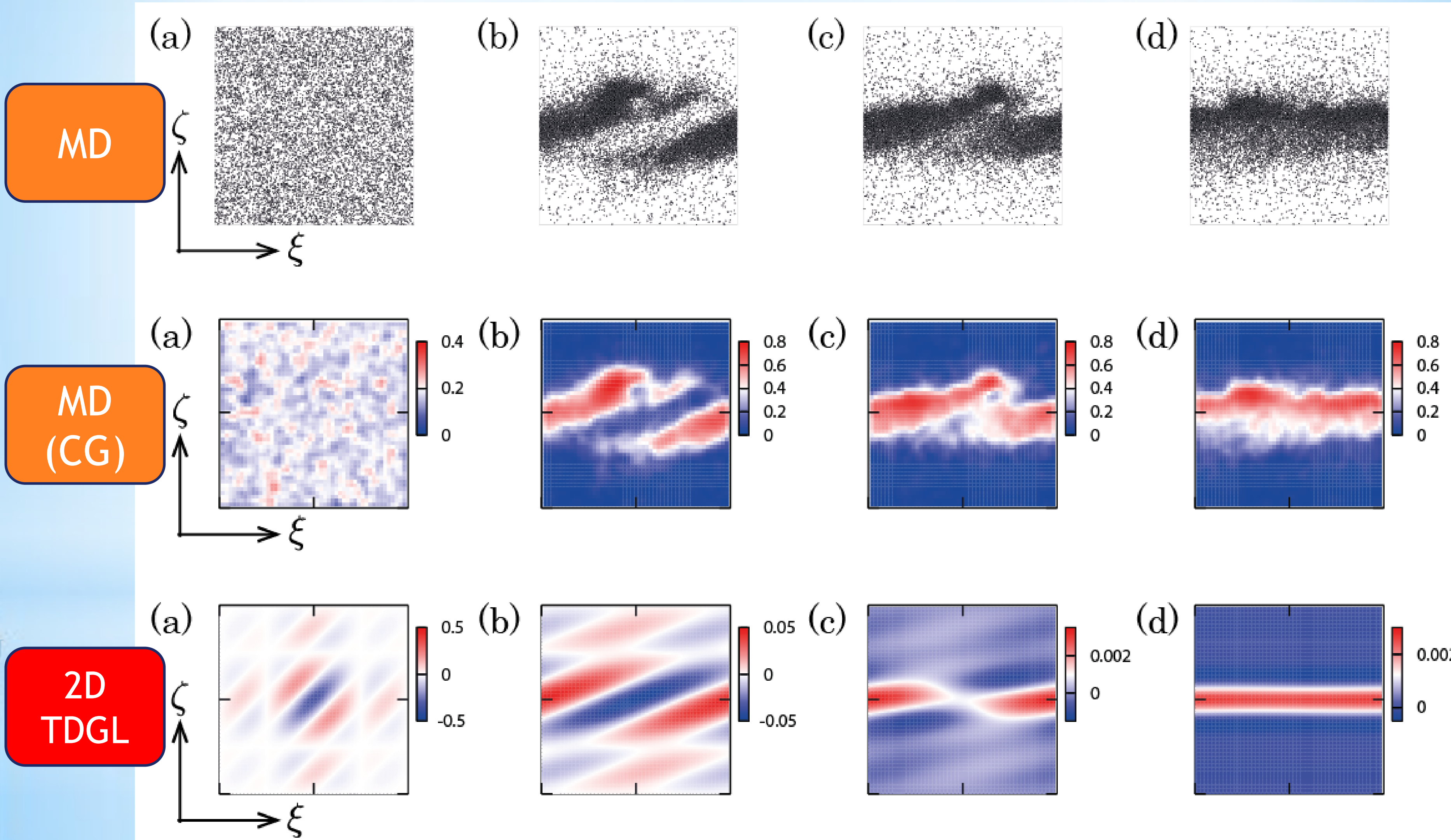
The 4th order Runge-Kutta method for the time-integration

The central difference method for the diffusion terms

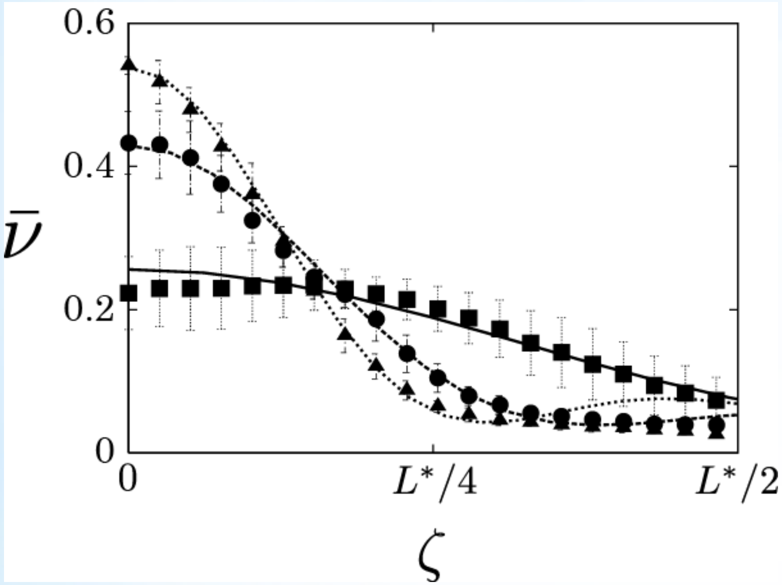
Periodic boundary conditions in the sheared frame

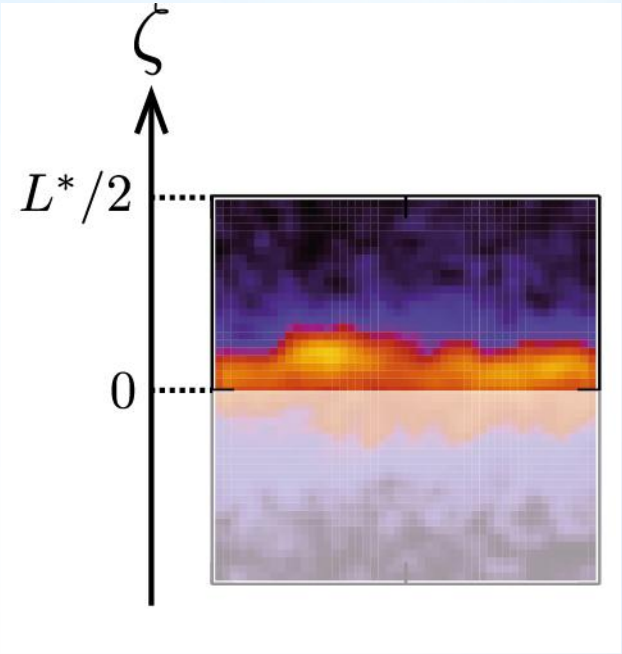
Small perturbations with randomly chosen wave numbers

Numerical solutions



Numerical solutions





Area fraction $\nu(\zeta,\tau) = \nu_0 + 2\nu_{q_c} A(\zeta,\tau)\cos(q(\tau)\zeta)$

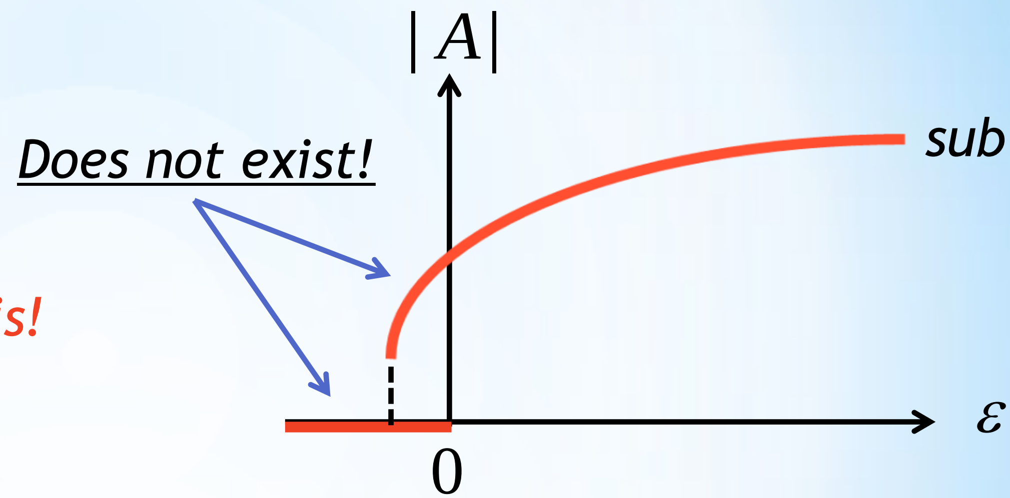
Scaling function $\bar{\nu}(\zeta,\tau) \equiv a^* \nu(\zeta/\zeta^*(\tau), \tau/\tau^*)$

Discussion

Hysteresis loop

By definition $\varepsilon > 0$

We couldn't discuss hysteresis!



Finite-size systems

The ratio of a particle's diameter to gap $\varepsilon = d/L$ *does not need to be small.*

Problem 1 → The mean granular temperature diverges in the elastic limit (e=1) $\theta_0 \propto \dfrac{\varepsilon^2}{1-e^2}$

Problem 2 → e is independent of ε and becomes another parameter. What is a small parameter for the nonlinear analysis?

Problem 3 → The Fourier transformations could be questionable.

Conclusion

Observation

- **Dense plug formations** in 2-dimensional granular shear flows are observed in both the bumpy & Lees-Edwards boundaries.

Theory

- **Granular hydrodynamic equations** derived by the kinetic theory can describe the dynamics of dense plug formations.
- We perturbatively solved the linearized granular hydrodynamic equations and found the **hydrodynamic modes** and **eigenvalues**.
- From the weakly nonlinear analysis, we derived the **1D TDGL equation** and discussed the **bifurcations** of steady amplitude.

Modeling

- By taking the "hybrid" approach, we also introduced the **2D TDGL equation** with the time dependent diffusion coefficients.
- The 2D TDGL equation "qualitatively" describes the **plug formations.**