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古典および量子力学の厳密に解ける模型と 時空の隠れた対称性

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時空の対称性

様々な物理を理解するための強力な解析手法を与える

時空の性質

時空の定性的な理解・分類

物質の振舞い

粒子や場の方程式の可積分性

他分野との関連

ハミルトン力学への応用
微分幾何学への還元
(幾何学化)

Plan of this talk

1. Review

– Hidden symmetry of Kerr black holes –

2. Integrable systems in classical mechanics and spacetime symmetries

3. Exactly solvable systems in quantum mechanics and spacetime symmetries

4. Summary & Outlook

1. Review

– Hidden symmetry of Kerr black holes –

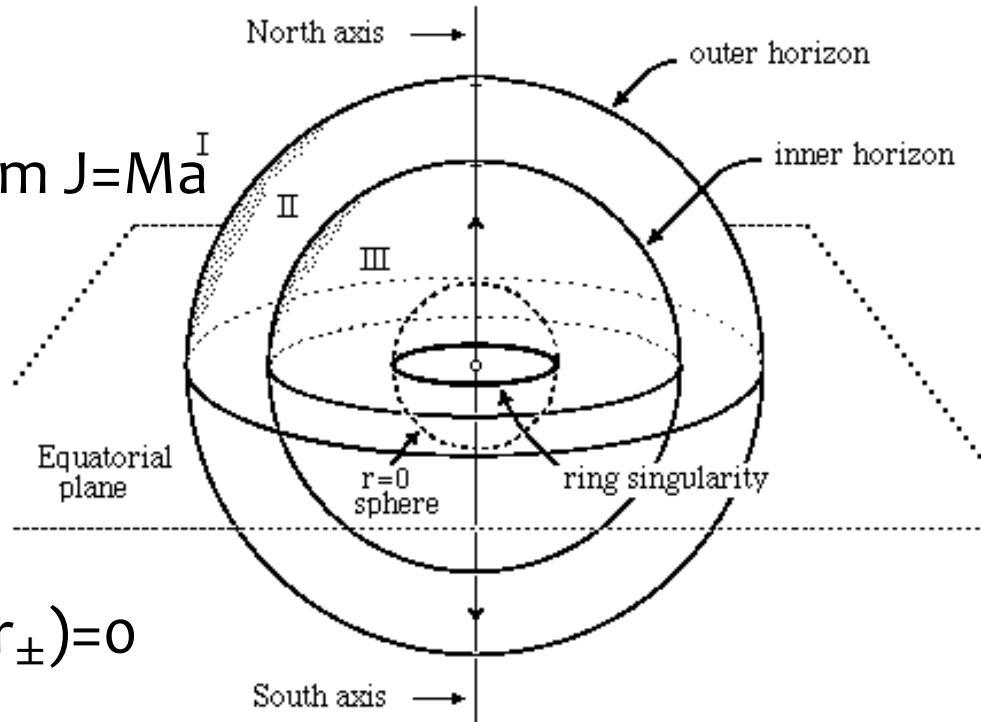
Geometry of Kerr spacetime

Kerr's metric Kerr (1963)

$$ds^2 = -\frac{\Delta}{\Sigma} \left(dt - a \sin^2 \theta d\phi \right)^2 + \frac{\sin^2 \theta}{\Sigma} \left(a dt - (r^2 + a^2) d\phi \right)^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2$$

where $\Delta = r^2 - 2Mr + a^2$, $\Sigma = r^2 + a^2 \cos^2 \theta$

- Two parameters
mass M , angular momentum $J=Ma$
- Two isometries
time translation $\partial/\partial t$
rotation $\partial/\partial \Phi$
- Ring singularity at $\Sigma=0$, i.e.,
 $r=0, \theta=\pi/2$
- Two horizons at $r=r_{\pm}$ s.t. $\Delta(r_{\pm})=0$



Separation of variables in various equations for the Kerr metric

- Hamilton-Jacobi equation for geodesics

$$\partial_\lambda S + g^{ab} \partial_a S \partial_b S = 0$$

- Klein-Gordon equation

$$(\nabla^2 - m^2)\Phi = 0$$

Carter (1968)

- Maxwell equation

$$\nabla_\mu F^{\mu\nu} = 0$$

- Linearized Einstein's equation

$$\delta G_{\mu\nu} = 0$$

Teukolsky (1972)

- Neutrino equation

$$\gamma^\mu (\partial_\mu + \Gamma_\mu) \psi = 0$$

Teukolsky (1973), Unruh (1973)

- Dirac equation

$$(\gamma^\mu \nabla_\mu + m) \Psi = 0$$

Chandrasekhar (1976), Page (1976)

Hidden symmetries

Generalizations of Killing symmetry has been studied since 1970s in order to give an account of such separabilities in the Kerr spacetime.

vector	Killing vector	conformal Killing vector
symmetric	Killing-Stackel (KS) Stackel (1895)	conformal Killing-Stackel (CKS)
anti-symmetric	Killing-Yano (KY) Yano (1952)	conformal Killing-Yano (CKY) Tachibana (1969), Kashiwada (1968)

Geodesic equations as Hamilton's equations

Geodesic equation

$$\frac{dx^\mu}{d\tau} = \frac{\partial H}{\partial p_\mu}, \quad \frac{dp_\mu}{d\tau} = -\frac{\partial H}{\partial x^\mu}, \quad \text{for } H = \frac{1}{2}g^{ab}p_ap_b.$$

$F(x, p)$: a constant of motion \Leftrightarrow

$$0 = \frac{dF}{d\tau} = \frac{\partial F}{\partial x^\mu} \frac{\partial H}{\partial p_\mu} - \frac{\partial F}{\partial p_\mu} \frac{\partial H}{\partial x^\mu} =: \{F, H\}_P$$

Poisson's bracket

Constants of motion and Killing tensors

Assume $C_K = K^{a_1 \dots a_n} p_{a_1} \dots p_{a_n}$ $H = \frac{1}{2} g^{ab} p_a p_b$

$$\{H, C_K\}_P = 0$$

$$\Leftrightarrow \frac{\nabla^{(a_1} K^{a_2 \dots a_{n+1})}}{= 0 ; \text{Killing equation}} p_{a_1} p_{a_2} \dots p_{a_{n+1}} = 0$$

Def. *Killing-Stackel tensor (KS)* is a rank- n symmetric tensor \mathbf{K} obeying the Killing equation. Stackel (1895)

Hamilton-Jacobi approach

For a D -dimensional manifold (M^D, g) , a local coordinate system x^a is called a separable coordinate system if a Hamilton-Jacobi equation in these coordinates

$$H(x^a, p_a) = \kappa_0, \quad p_a = \frac{\partial S}{\partial x^a}$$

where κ_0 is a constant, is completely integrable by (additive) separation of variables, i.e.,

$$S = S_1(x^1, c) + S_2(x^2, c) + \cdots + S_D(x^D, c)$$

where $S_a(x^a, c)$ depends only on the corresponding coordinate x^a and includes D constants $c = (c_1, \dots, c_D)$.

Benenti's S_r -separability structure

Theor. A D -dimensional manifold (M^D, g) admits separability of H-J equation for geodesics if and only if

1. There exist r indep. commuting Killing vectors $X_{(i)}$:

$$[X_{(i)}, X_{(j)}] = 0,$$

2. There exist $D-r$ indep. rank-2 Killing tensors $K_{(\mu)}$, which satisfy

$$[K_{(\mu)}, K_{(\nu)}] = 0, \quad [X_{(i)}, K_{(\mu)}] = 0,$$

3. The Killing tensors $K_{(\mu)}$ have in common $D-r$ eigenvectors $X_{(\mu)}$ s.t.

$$[X_{(\mu)}, X_{(\nu)}] = 0, \quad [X_{(i)}, X_{(\mu)}], \quad g(X_{(i)}, X_{(\mu)}) = 0.$$

Benenti-Francaviglia (1979)

Comments:

- Some examples which are **not** separable **but** integrable are known.

cf.) Gibbons-TH-Kubiznak-Warnick (2011)

Hidden symmetry of Kerr spacetime I

Kerr spacetime admits a rank-2 irreducible Killing tensor .

Walker-Penrose (1970)

$$K_{ab} = K_{(ab)} , \quad \nabla_{(c} K_{ab)} = 0$$

Comments:

- Kerr spacetime has 4 independent and mutually commuting constants of geodesic motion, which are corresponding to 2 **Killing vectors** and 2 **rank-2 Killing tensors**.

$$(\partial_t)^a : \quad E = (\partial_t)^a p_a \quad g_{ab} : \quad \kappa_0 = g^{ab} p_a p_b$$

$$(\partial_\phi)^a : \quad L = (\partial_\phi)^a p_a \quad K_{ab} : \quad \kappa = K^{ab} p_a p_b$$

- One also finds that this Killing tensor admits the Benenti's S₂-separability structure of the H-J equation for geodesics.

Hidden symmetry of Kerr spacetime II

The Killing tensor K can be written as the square of a rank-2 Killing-Yano tensor f . Penrose-Floyd (1973)

$$\exists f \text{ s.t. } K_{ab} = f_a{}^c f_{bc}, \quad f_{ba} = -f_{ab}, \quad \nabla_{(a} f_{b)c} = 0$$

↑ rank-2 KY equation

Comments:

- Killing-Yano tensor (KY) is a rank- p anti-symmetric tensor f obeying $\nabla_{(a} f_{b_1)b_2 \dots b_p} = 0$. Yano (1952)
- Having a Killing-Yano tensor, one can **always** construct the corresponding Killing tensor. On the other hand, **not every** Killing tensor can be decomposed in terms of a Killing-Yano tensor. Collinson (1976), Stephani (1978)

Hidden symmetry of Kerr spacetime III

The Killing-Yano tensor f in the Kerr spacetime generates two Killing vectors. Hughston-Sommers (1973)

$$\xi^a \equiv (\partial_t)^a = \frac{1}{3} \nabla_b (*f)^{ba}$$

$$\eta^a \equiv -a^2(\partial_t)^a - a(\partial_\phi)^a = K^a{}_b \xi^b$$

In the end, all the symmetries necessary for complete integrability and separability of the H-J equation for geodesics can be generated by a single rank-2 Killing-Yano tensor.

$$g_{ab}, \quad f_{ab} \quad \begin{array}{c} \xrightarrow{\hspace{2cm}} \\[-1ex] \searrow \quad \nearrow \end{array} \quad K_{ab} = f_a{}^c f_{bc} \quad \begin{array}{c} \xrightarrow{\hspace{2cm}} \\[-1ex] \searrow \end{array} \quad \xi^a \quad \begin{array}{c} \xrightarrow{\hspace{2cm}} \\[-1ex] \searrow \end{array} \quad \eta^a$$

Hidden symmetry of Kerr spacetime IV

Klein-Gordon equation

\mathcal{O} : a sym. op. for $D \Leftrightarrow [\mathcal{O}, D] = 0$

For the scalar Laplacian \square for the Kerr metric,

$$\hat{\eta}^{(j)} = \eta^{(j)a} \nabla_a, \quad \hat{K}^{(j)} = \nabla_a K^{(j)ab} \nabla_b,$$

are symmetry operators, i.e.,

$$[\hat{\eta}^{(j)}, \square] = [\hat{K}^{(j)}, \square] = 0 \quad \text{Carter (1977)}$$

Dirac equation

For the Dirac operator D , the operator

$$\hat{f} \equiv i\gamma_5 \gamma^a \left(f_a{}^b \nabla_b - \frac{1}{6} \gamma^b \gamma^c \nabla_c f_{ab} \right)$$

is a symmetry operator whenever f is a Killing-Yano tensor.

Carter-McLenaghan (1979)

Hidden symmetry of Kerr spacetime V

The Killing-Yano tensor is derived from a 1-form potential \mathbf{b} ,

$$f = *db$$

Carter (1987)

Comments:

- Obviously, $h = *f$ is closed 2-form.
- One finds that h is a conformal Killing-Yano tensor (CKY) of rank-2, i.e., it follows

$$\nabla_a h_{bc} + \nabla_b h_{ac} = 2g_{ab}\xi_c - g_{ac}\xi_b - g_{bc}\xi_a$$

where

$$\xi^a = \frac{1}{3}\nabla_b h^{ba}$$

Tachibana (1969)

Separability structures for Kerr black hole

Algebraic type of curvature is type-D.

Geodesic motion is completely integrable.

Carter (1968)

Hamilton-Jacobi equation is separable.

Klein-Gordon equation is separable.

Carter (1968)

K-G symmetry operators exist.

Carter (1977)

Dirac equation is separable.

Chandrasekhar (1976)

Dirac symmetry operators exist.

Carter-McLenaghan (1979)

A closed CKY 2-form exists.

Carter (1987)

Carter's class

Kerr's metric

$$ds^2 = -\frac{\Delta}{\Sigma} \left(dt - a \sin^2 \theta d\phi \right)^2 + \frac{\sin^2 \theta}{\Sigma} \left(a dt - (r^2 + a^2) d\phi \right)^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2$$

where $\Delta = r^2 - 2Mr + a^2$, $\Sigma = r^2 + a^2 \cos^2 \theta$

↓ coord. trasf. $p = a \cos \theta$, $\tau = t - a\phi$, $\sigma = -\frac{\phi}{a}$

$$ds^2 = -\frac{Q}{r^2 + p^2} (d\tau - p^2 d\sigma)^2 + \frac{P}{r^2 + p^2} (d\tau + r^2 d\sigma)^2 + \frac{r^2 + p^2}{Q} dr^2 + \frac{r^2 + p^2}{P} dp^2 \quad (\text{Boyer's coordinates})$$

where $Q = r^2 - 2Mr + a^2$, $P = -p^2 + a^2$ Carter (1968)

The “off-shell” metric with Q and P replaced by arbitrary functions $Q(r)$ and $P(p)$ is said to be of **Carter's class**.

Spacetimes admitting a Killing-Yano tensor

Theor. A spacetime (M^4, g) admits a rank-2 Killing-Yano tensor if and only if the metric is of Carter's class, i.e.,

$$\begin{aligned} ds^2 = & -\frac{Q(r)}{r^2 + p^2}(d\tau - p^2 d\sigma)^2 \\ & + \frac{P(p)}{r^2 + p^2}(d\tau + r^2 d\sigma)^2 + \frac{r^2 + p^2}{Q(r)}dr^2 + \frac{r^2 + p^2}{P(p)}dp^2 \end{aligned}$$

Dietz-Rudiger (1982), Taxiarchis (1985)

Carter's metric in Einstein-Maxwell theory

The Carter's metric

$$ds^2 = -\frac{Q(r)}{r^2 + p^2}(d\tau - p^2 d\sigma)^2 + \frac{P(p)}{r^2 + p^2}(d\tau + r^2 d\sigma)^2 + \frac{r^2 + p^2}{Q(r)}dr^2 + \frac{r^2 + p^2}{P(p)}dp^2$$

obeys the Einstein-Maxwell equations provided that the metric functions take the form

$$Q = -\frac{\lambda}{3}r^4 + \epsilon r^2 - 2mr + k + e^2 + g^2$$
$$P = -\frac{\lambda}{3}p^4 - \epsilon p^2 + 2np + k$$

and the vector potential is given by

$$A = -\frac{1}{r^2 + p^2}[er(d\tau - p^2 d\sigma) + gp(d\tau + r^2 d\sigma)]$$

This metric has **six independent parameters**.

Plebanski-Demianski metric

The Plebanski-Demianski family is represented by the metric

Plebanski-Demianski (1976)

$$ds^2 = \frac{1}{(1 - pr)^2} \left\{ -\frac{Q}{r^2 + p^2} (d\tau - p^2 d\sigma)^2 + \frac{P}{r^2 + p^2} (d\tau + r^2 d\sigma)^2 + \frac{r^2 + p^2}{Q} dr^2 + \frac{r^2 + p^2}{P} dp^2 \right\}$$

This metric obeys the Einstein-Maxwell equations provided that the metric functions take the form

$$\begin{aligned} Q &= -(k + \lambda/3)r^4 - 2nr^3 + \epsilon r^2 - 2mr + k + e^2 + g^2 \\ P &= -(k + e^2 + g^2 + \lambda/3)p^4 + 2mp^3 - \epsilon p^2 + 2np + k \end{aligned}$$

with **seven parameters** and the vector potential is given by

$$A = -\frac{1}{r^2 + p^2} [er(d\tau - p^2 d\sigma) + gp(d\tau + r^2 d\sigma)]$$

TABLE I

Known exact solutions of the Einstein and Einstein-Maxwell Equations of type D

$m + in, a, e + ig, \lambda$	$m + in, a + ib, e + ig, \lambda$	$m + in, b, e + ig, \lambda$
Plebanski [3] 1975	Kinnersley [2] 1975	
$m + in, a, e\lambda$	$m + in, a, e + ig$	$m + in, b, e, \lambda$
Carter [11] 1968	Demianski, Newman [12] 1966	Carter [11] 1968
$m + in, a, \lambda$	$m + in, a + ib$	$m + in, b, e$
Frolov [22] 1973	Kinnersley [13] 1969	Levi-Civita [4] 1918
m, a, λ	$m + in, a$	Newman, Tamburino [5] 1961
Demianski [14] 1973	Demianski [15] 1966	Robinson, Trautman [6] 1962
	Kramer, Neugebauer [24] 1968	Ehlers, Kundt [5] 1962
	Robinson, J. Robinson Zünd [27] 1969	
m, λ	m, a	$m + in, \lambda$
Kottler [17] 1918	Reissner, Nordstrom [18] 1916	Demianski [16] Frolov [22] 1973
λ	m, e	
de Sitter [30] 1917	Kerr [9] 1963	
$m + in$	$m + in, e$	e, b
	Brill [23] 1964	Bertotti [28] 1959
		Robinson [29] 1959
		Taub [21] 1951
m		m
		Schwarzschild [20] 1916

Exact solutions – vacuum sols for black holes

vacuum Einstein's Eq.

$$Ric(g) = \lambda g$$

with S^n horizon topology

Four dimensions

	mass,	NUT,	rotation,	λ
Schwarzschild (1916)	○			
Kerr (1963)	○		○	
Carter (1968)	○	○	○	○

Higher dimensions

	mass,	NUTs,	rotations,	λ
Tangherlini (1916)	○			
Myers-Perry (1986)	○		○	
Gibbons-Lu-Page-Pope (2004)	○		○	○
5-dim. Hawking, et al. (1998)				
Chen-Lu-Pope (2006)	○	○	○	○

↑ The most general known solution of this class
= higher-dimensional Kerr-NUT-(A)dS

D-dimensional Kerr-NUT-(A)dS metric

$$D = 2n + \varepsilon \quad (\varepsilon = 0 \text{ or } 1)$$

$$ds^2 = \sum_{\mu=1}^n \frac{dx_\mu^2}{Q_\mu} + \sum_{\mu=1}^n Q_\mu \left[\sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k \right]^2 + \varepsilon \frac{c}{A^{(n)}} \left[\sum_{k=0}^n A^{(k)} d\psi_k \right]^2$$

where

Chen-Lu-Pope (2006)

$$Q_\mu = \frac{X_\mu}{U_\mu}, \quad U_\mu = \prod_{\substack{\nu=1 \\ \nu \neq \mu}}^n (x_\mu^2 - x_\nu^2), \quad X_\mu = X_\mu(x_\mu),$$

$$A_\mu^{(k)} = \sum_{\substack{1 \leq \nu_1 < \nu_2 < \dots < \nu_k \leq n \\ \nu_i \neq \mu}} x_{\nu_1}^2 x_{\nu_2}^2 \cdots x_{\nu_k}^2, \quad A^{(k)} = \sum_{1 \leq \nu_1 < \nu_2 < \dots < \nu_k \leq n} x_{\nu_1}^2 x_{\nu_2}^2 \cdots x_{\nu_k}^2,$$

$$A_\mu^{(0)} = A^{(0)} = 1, \quad c = \text{const.} .$$

This metric satisfies $R_{ab} = -(D - 1)c_n g_{ab}$ provided that

$$D=2n \quad X_\mu = \sum_{k=0}^n c_{2k} x_\mu^{2k} + b_\mu x_\mu$$

$$D=2n+1 \quad X_\mu = \sum_{k=1}^n c_{2k} x_\mu^{2k} + b_\mu + \frac{(-1)^n c}{x_\mu^2}$$

How about higher dimensions?

- Higher dim. Kerr-NUT-(A)dS -

A closed CKY 2-form exists.

Kubiznak-Frolov (2007)

Geodesic motion is completely integrable.

Page-Kubiznak-Vasudevan-Krtous (2007)

Algebraic type of curvature is type-D.

Hamamoto-TH-Oota-Yasui (2007)

Hamilton-Jacobi equation is separable.

Klein-Gordon equation is separable.

Frolov-Krtous-Kubiznak (2007)

K-G symmetry operators exist.

Sergyeyev, Krtous (2008)

Dirac equation is separable.

Oota-Yasui (2008)

Dirac symmetry operators exist.

Benn-Charlton (1996), Wu (2009)

How about higher dimensions?

- Higher dim. Kerr-NUT-(A)dS -

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TH-Oota-Yasui (2008)

TH-Kubiznak-WarnickYasui (2010)

Manifolds admitting a CCKY 2-form

Theor. Suppose a Riemannian manifold (M^D, g) admits a **non-degenerate** closed CKY 2-form h . Then the metric takes the form

$$g = \sum_{\mu=1}^n \frac{dx_\mu^2}{Q_\mu} + \sum_{\mu=1}^n Q_\mu \left[\sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k \right]^2 + \varepsilon S \left[\sum_{k=0}^n A^{(k)} d\psi_k \right]^2 ,$$

where

$$Q_\mu = \frac{X_\mu}{U_\mu} , \quad U_\mu = \prod_{\substack{\nu=1 \\ \nu \neq \mu}}^n (x_\mu^2 - x_\nu^2) , \quad X_\mu = X_\mu(x_\mu) , \quad S = \frac{c}{A^{(n)}} , \quad A_\mu^{(0)} = A^{(0)} = 1 ,$$

$$A_\mu^{(k)} = \sum_{\substack{1 \leq \nu_1 < \nu_2 < \dots < \nu_k \leq n \\ \nu_i \neq \mu}} x_{\nu_1}^2 x_{\nu_2}^2 \cdots x_{\nu_k}^2 , \quad A^{(k)} = \sum_{1 \leq \nu_1 < \nu_2 < \dots < \nu_k \leq n} x_{\nu_1}^2 x_{\nu_2}^2 \cdots x_{\nu_k}^2 .$$

TH-Oota-Yasui (2007), Krtous-Frolov-Kubiznak (2008)

Recent developments

- Attempts to exact solutions in various theories
- Remarkably, conformal Killing-Yano symmetry in the presence of **torsion** seems suitable for solutions in supergravity theories.
- Applications to
 - Exactly solvable systems in classical and quantum mechanics.
(Key words: First integrals, symmetry and ladder operators)
 - Special geometries
(Key words: Special holonomy)

2. Integrable systems in classical mechanics and spacetime symmetries

Equations of motion in classical mechanics

Equations of motion

$$\frac{dx^\mu}{d\tau} = \frac{\partial H}{\partial p_\mu}, \quad \frac{dp_\mu}{d\tau} = -\frac{\partial H}{\partial x^\mu},$$

$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu \quad ; \text{ Geodesic Hamiltonian}$$

$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu + U \quad ; \text{ Natural Hamiltonian}$$

$F(x, p)$: a constant of motion \Leftrightarrow

$$0 = \frac{dF}{d\tau} = \frac{\partial F}{\partial x^\mu} \frac{\partial H}{\partial p_\mu} - \frac{\partial F}{\partial p_\mu} \frac{\partial H}{\partial x^\mu} =: \{F, H\}_P$$

Poisson's bracket

Polynomial first integrals and Killing tensors

Assume $C_K = K^{a_1 \dots a_n} p_{a_1} \dots p_{a_n}$ $H = \frac{1}{2} g^{ab} p_a p_b$

$$\{H, C_K\}_P = 0$$

$$\Leftrightarrow \frac{\nabla^{(a_1} K^{a_2 \dots a_{n+1})}}{= 0 ; \text{Killing equation}}$$

Def. *Killing-Stackel tensor (KS)* is a rank- n symmetric tensor \mathbf{K} obeying the Killing equation. Stackel (1895)

Geometrization

$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu + U \quad \rightarrow \quad \bar{H} = \frac{1}{2} G^{\mu\nu} p_\mu p_\nu$$

- Maupertuis' principle
- Canonical transformation
- Eisenhart lift

□ Eisenhart lift

$$\bar{H} = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu + U \mathbf{p}_s^2$$

□ Eisenhart-Duval lift

$$\bar{H} = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu + U \mathbf{p}_s^2 + p_t p_s$$

□ Generalized Eisenhart lift

$$\bar{H} = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu + U (\mathbf{p}_s^2 + p_t^2)$$

Eisenhart-Duval lift

Original metric

$$dz^2 = g_{\mu\nu} dx^\mu dx^\nu$$

Hamilton's equation with

$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu + U$$

$$p_t = 0, p_s = 1$$

Bergmann metric

$$d\bar{z}^2 = -2U dt^2 + 2dt ds + g_{\mu\nu} dx^\mu dx^\nu$$

Hamilton's equation with

$$\bar{H} = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu + U \mathbf{p}_s^2 + \mathbf{p}_t \mathbf{p}_s$$

$$\exists F \text{ s.t. } \{F, H\}_{Poisson} = 0$$

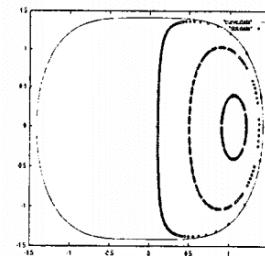
$$\exists \bar{F} \text{ s.t. } \{\bar{F}, \bar{H}\}_{Poisson} = 0$$

Polynomial first integrals can convert to each other

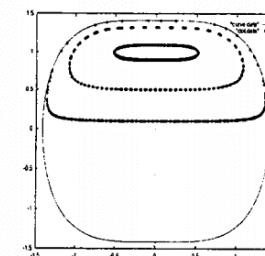
Several approaches to (non-)integrability

Ex. $H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{4}(x^4 + y^4) + \frac{\epsilon}{2}x^2y^2$

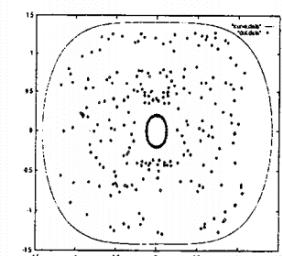
- Poincare section



$$\epsilon = 1$$



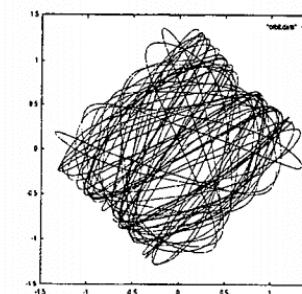
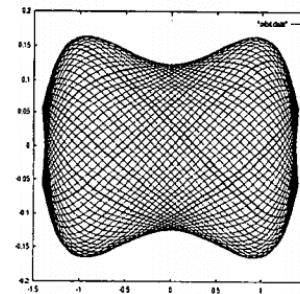
$$\epsilon = 3$$



$$\epsilon = 5$$

- Lyapunov exponent

$$|\delta(t)| \approx |\delta(0)|e^{\lambda t}$$



- Kowalevskaya's singularity analysis
- Differential Galois theory
- Painleve analysis

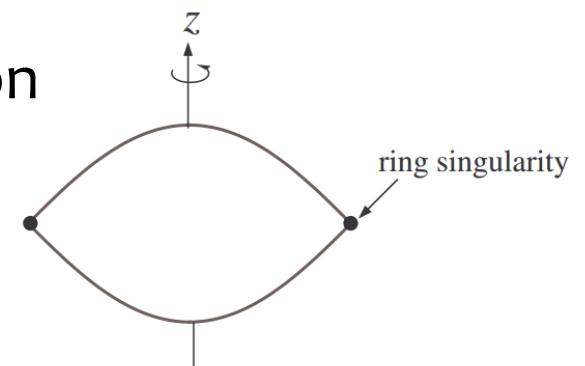
\Rightarrow nonintegrable except for $\epsilon = 0, 1, 3$

Zipoy-Voorhees spacetime

Zipoy-Voorhees metric

$$ds^2 = - \left(\frac{x-1}{x+1} \right)^\delta dt^2 + \left(\frac{x+1}{x-1} \right)^\delta \left[(x^2 - y^2) \left(\frac{x^2 - 1}{x^2 - y^2} \right)^{\delta^2} \left(\frac{dx^2}{x^2 - 1} + \frac{dy^2}{1 - y^2} \right) + (x^2 - 1)(1 - y^2) dz^2 \right]$$

- A static, axially symmetric vacuum solution
- A static limit of the Tomimatsu-Sato solution
- Special cases: flat ($\delta = 0$), Schwarzschild ($\delta = 1$), Curzon ($\delta \rightarrow \infty$)
- Otherwise, naked singularity
- For $\delta = 2, 3, \geq 4$, degenerate Killing horizons sharing ring singularity



Kodama-Hikida (2003)

Geodesics in the Zipoy-Voorhees spacetime

- Numerical investigations suggest that geodesics are integrable (?)
Sota-Suzuki-Maeda (1996), Brink (2008)

“We have found that for any bound orbit both in the ZV spacetime and in the Curzon spacetime, the sectional curvature is always negative and no chaotic behavior of the orbit is seen at least from our analysis by use of the Lyapunov exponent and the Poincare map.”

- Recently, signs of chaos were found numerically
Lukes-Gerakopoulos (2012)
- Non-existence of irreducible KS tensors yielding the Liouville integrability with two Killing vectors, up to rank 11
Kruglikov-Matveev (2012), Vollmer (2016)
- Non-existence of irreducible KS tensors of rank 2

Structure of Killing vector equations

$$\nabla_{(\mu} \xi_{\nu)} = 0 \quad \rightarrow \quad \begin{aligned}\nabla_\mu \xi_\nu &= \nabla_{[\mu} \xi_{\nu]} \\ \nabla_\mu \nabla_{[\nu} \xi_{\rho]} &= -R_{\nu\rho\mu}{}^\sigma \xi_\sigma\end{aligned}$$

Structure of Killing vector equations

$$\nabla_{(\mu} \xi_{\nu)} = 0 \quad \rightarrow \quad \begin{aligned} \nabla_\mu \xi_\nu &= \nabla_{[\mu} \xi_{\nu]} \\ \nabla_\mu \nabla_{[\nu} \xi_{\rho]} &= -R_{\nu\rho\mu}{}^\sigma \xi_\sigma \end{aligned}$$

Structure of Killing vector equations

$$\nabla_{(\mu} \xi_{\nu)} = 0 \quad \rightarrow \quad \begin{aligned} \nabla_\mu \xi_\nu &= \nabla_{[\mu} \xi_{\nu]} \\ \nabla_\mu \nabla_{[\nu} \xi_{\rho]} &= -R_{\nu\rho\mu}{}^\sigma \xi_\sigma \end{aligned}$$

- Prolongation bundle $E^1 = \Lambda^1(M) \oplus \Lambda^2(M)$
- Killing connection $D_\mu \hat{\xi}_A = 0$
- Integrability conditions

$$R_{\mu\nu A}{}^B \hat{\xi}_B = 0, \quad (\nabla_\mu R_{\nu\rho A}{}^B) \hat{\xi}_B = 0, \quad \dots$$

Tensor fields

Symmetric

Killing-Stackel (KS)

$$\nabla_{(\mu} K_{\nu_1 \dots \nu_p)} = 0$$

Symmetric
homothetic (SH)



$$\nabla_{(\mu} K_{\nu_1 \dots \nu_p)} = g_{(\mu \nu_1} \Phi_{\nu_2 \dots \nu_p)}$$

Conformal Killing-Stackel (CKS)



$$\nabla_\mu \nabla_{(\nu} K_{\rho_1 \dots \rho_p)} = 0$$

Symmetric affine (SA)

Anti-symmetric

Killing-Yano (KY)

$$\nabla_{(\mu} f_{\nu_1 \nu_2 \dots \nu_p)} = 0$$

Anti-symmetric
homothetic (AH)



$$\nabla_{(\mu} f_{\nu_1 \nu_2 \dots \nu_p)} = p g_{\mu [\nu_1} F_{\nu_2 \dots \nu_p]}$$

Conformal Killing-Yano (CKY)



$$\nabla_\mu \nabla_{(\nu} f_{\rho_1 \rho_2 \dots \rho_p)} = 0$$

Anti-symmetric affine
(AA)

Structure of Killing-Yano equations

$$\nabla_{(\mu} f_{\nu_1 \dots \nu_p)} = 0 \quad \rightarrow \quad \begin{aligned} \nabla_\mu f_{\nu_1 \dots \nu_p} &= \nabla_{[\mu} f_{\nu_1 \dots \nu_p]} \\ \nabla_\mu (\nabla_{[\nu} f_{\rho_1 \dots \rho_p]}) &= -R_{[\nu \rho_1 | \mu}{}^\alpha f_{\alpha | \rho_2 \dots \rho_p]} \end{aligned}$$

- Prolongation bundle $E^p = \Lambda^p(M) \oplus \Lambda^{p+1}(M)$

$$N = \text{rank}(E^p) = \binom{n+1}{p+1}$$

- Killing connection $D_\mu \hat{\xi}_A = 0$

Semmelmann 2002

- Curvature conditions

TH-Yasui 2014

$$R_{\mu\nu A}{}^B \hat{\xi}_B = 0, \quad (\nabla_\mu R_{\nu\rho A}{}^B) \hat{\xi}_B = 0, \quad \dots$$

Our Mathematica package

INPUT:

- Metric data

FUNCTIONS:

- Compute the Killing curvature
- Solve the integrability conditions

*available at the URL: http://www.research.kobe-u.ac.jp/fsci-pacos/KY_upperbound/

KY tensors in 4D type-D vacuum solutions

TH-Yasui 2014

	$p = 1$	$p = 2$	$p = 3$
Maximally symmetric	10	10	5
Case I ex) Schwarzschild, Taub-NUT	4	1	0
Case II ex) Kerr	2	1	0
Case III ex) Plebanski-Demianski	2	0	0
Case IV	4	1	0

KY tensors in 4D spacetimes

TH-Yasui 2014

	$p = 1$	$p = 2$	$p = 3$
Maximally symmetric	10	10	5
Plebanski-Demianski	2	0	0
Kerr	2	1	0
Schwazschild	4	1	0
FLRW	6	4	1
Self-dual Taub-NUT	4	4	0
Eguchi-Hanson	4	3	0

KY tensors in 5D spacetimes

TH-Yasui 2014

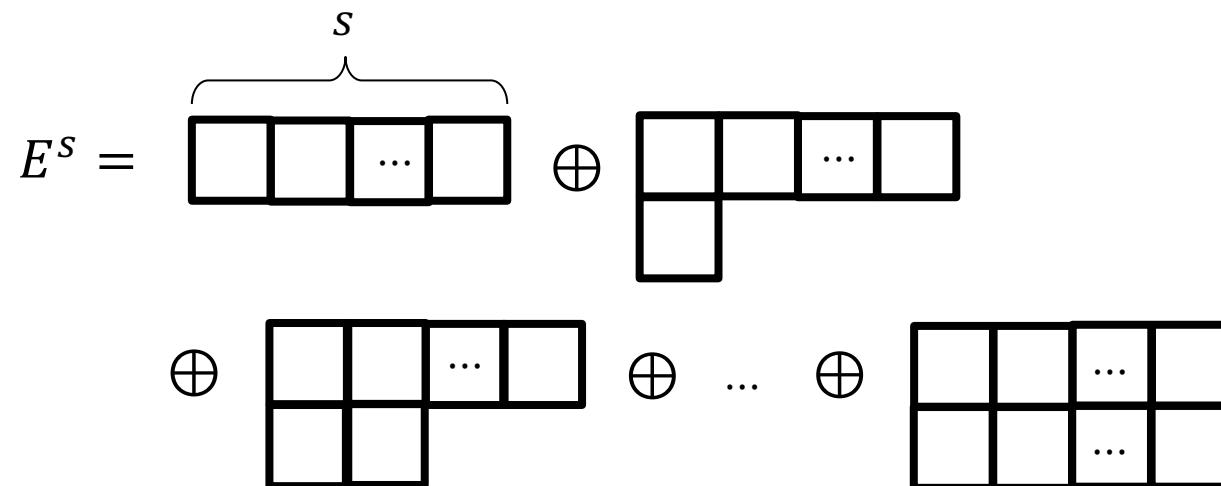
	$p = 1$	$p = 2$	$p = 3$	$p = 4$
Maximally symmetric	15	20	15	6
Myers-Perry	3	0	1	0
Emparan-Reall	3	0	0	0
Kerr string	3	1	0	1

Structure of Killing-Stackel equations

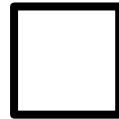
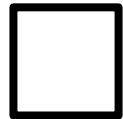
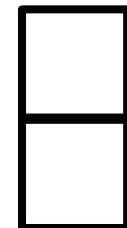
A rank-s Killing-Stackel tensor \Leftrightarrow A parallel section of E^s

$$\nabla_{(\mu} K_{\nu_1 \dots \nu_s)} = 0$$

$$D_\mu \sigma^A = 0$$



Young diagram

 TM T^a  $S^2 TM$ T^{ab} $T^{ab} = T^{(ab)}$  $\Lambda^2 TM$ $T^{ab} = T^{[ab]}$

Young symmetrizer

$$T^{abcdef}$$



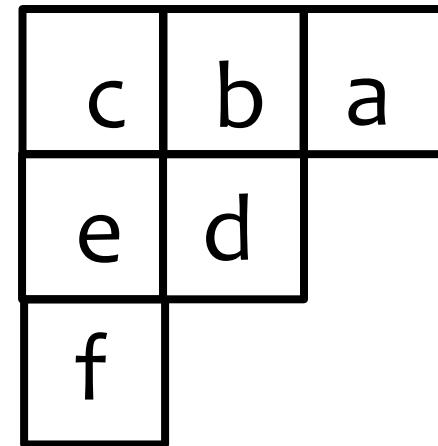
$$T_1^{abcdef} = T^{(abc)(de)f}$$



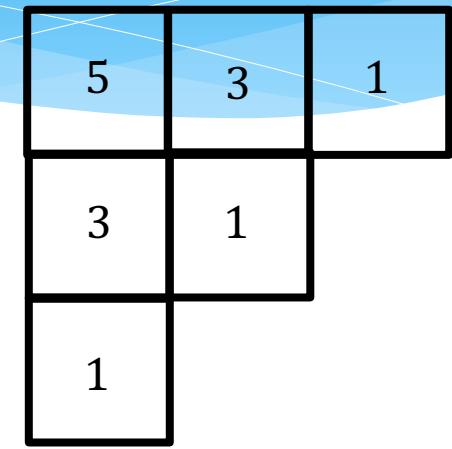
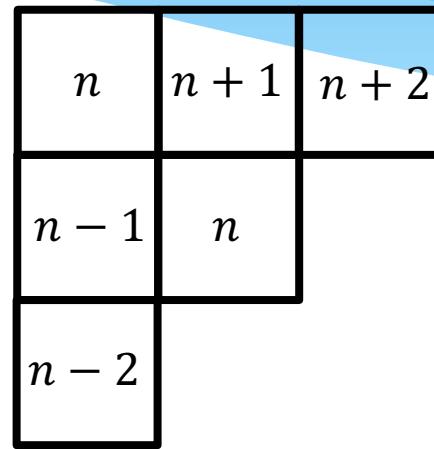
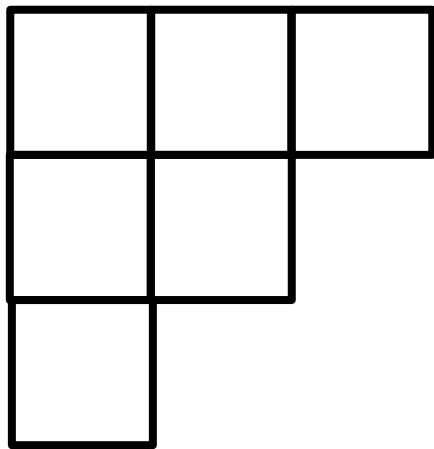
$$T_2^{abcdef} = T_1^{ab[c|d|ef]}$$



$$T_3^{abcdef} = T_2^{a[b|c|d]ef}$$



Dimension formula

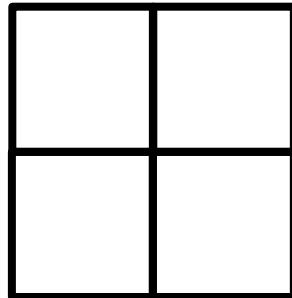


Hooke length

$$\text{ランク} = \frac{(n+2)(n+1)n^2(n-1)(n-2)}{5 \cdot 3^2 \cdot 1^3}$$

Ex) Riemann tensor

$$R_{abcd} = -R_{bacd} \quad R_{abcd} = -R_{bacd}$$
$$R_{a[bcd]} = 0 \quad R_{abcd} = R_{cdab}$$



n	$n + 1$
$n - 1$	n

3	2
2	1

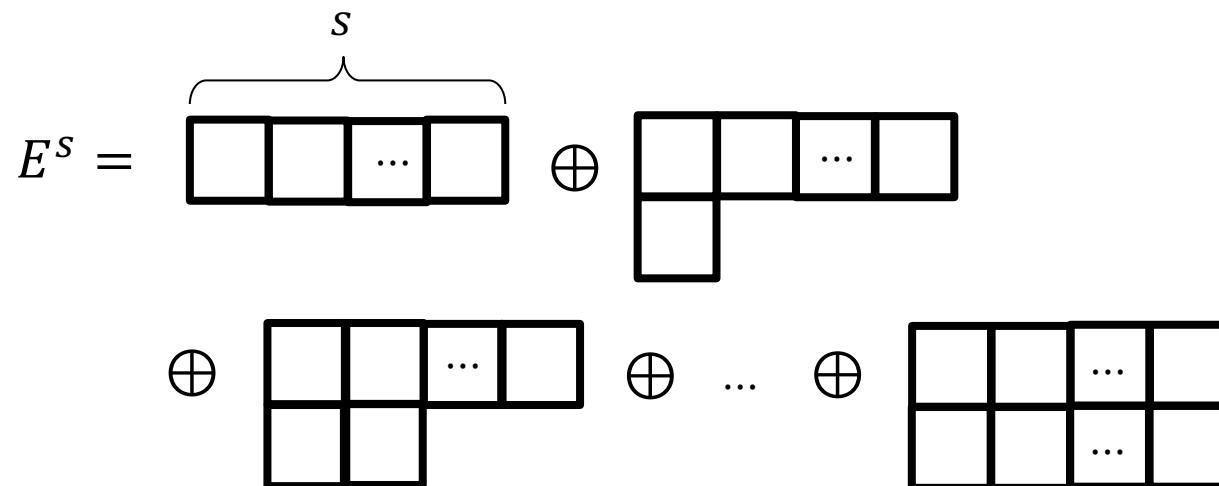
$$\frac{(n+1)n^2(n-1)}{3 \cdot 2^2 \cdot 1} = \frac{n^2(n^2-1)}{12}$$

Structure of Killing-Stackel equations

A rank-s Killing-Stackel tensor \Leftrightarrow A parallel section of E^s

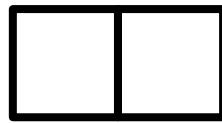
$$\nabla_{(\mu} K_{\nu_1 \dots \nu_s)} = 0$$

$$D_\mu \sigma^A = 0$$



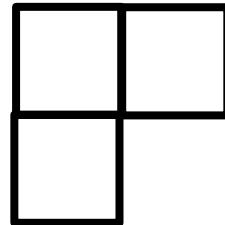
Prolongated variables of KS equations

- K_{ab}



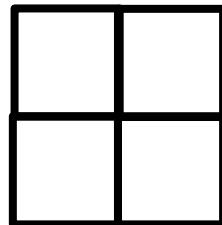
$$N_1 = \frac{(n+1)n}{2 \cdot 1}$$

- $\psi_{abc} \equiv \nabla_{[a} K_{b]c}$



$$N_2 = \frac{(n+1)n(n-1)}{3 \cdot 1^2}$$

- $\phi_{abcd} \equiv \nabla_{[a} \nabla_{|[b} K_{c]d]}$



$$N_3 = \frac{(n+1)n^2(n-1)}{3 \cdot 2^2 \cdot 1}$$

KS tensors in 4D spacetimes

4D metrics	rank-2		
	KV	KST	KYT
Maximally symmetric	10	50	10
Schwarzschild	4	11	1
Kerr	2	5 (1)	1
Reissner-Nordstrom	4	11	1
extreme-	4	11	1
Zipoy-Voorhees ($\delta = 2 \sim 11$)	2	4	0

3. Exactly solvable systems in quantum mechanics and spacetime symmetries

Classical

Hamilton's equation

$$\frac{dx^\mu}{d\tau} = \frac{\partial H}{\partial p_\mu}, \quad \frac{dp_\mu}{d\tau} = -\frac{\partial H}{\partial x^\mu}$$

$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu + U \quad \xrightarrow{p_\mu \rightarrow -i\nabla_\mu}$$

First integral

$$\{H, F\} = 0$$

Quantum

Schrodinger's equation

$$H\psi = E\psi$$

$$H = -\frac{1}{2} g^{\mu\nu} \nabla_\mu \nabla_\nu + U$$

Symmetry operator

$$[H, D] = 0$$

Ladder operator

$$[H, D] = \varepsilon D$$

Original metric

$$dz^2 = g_{\mu\nu} dx^\mu dx^\nu$$

Schrodinger's equation

$$H\psi = E\psi$$

$$\text{where } H = -\frac{1}{2m} \Delta_g + U$$

Symmetry operator

$$[H, D] = 0$$

Ladder operator

$$[H, D] = \varepsilon D$$

Bergmann metric

$$d\bar{z}^2 = -2U dt^2 + 2dt ds + g_{\mu\nu} dx^\mu dx^\nu$$

Laplace equation

$$\Delta_E \Phi = 0$$

$$\text{where } \Phi = e^{-iEt+ims} \psi(x)$$

Symmetry operator

$$[\Delta_E, \tilde{D}] \propto \Delta_E$$

$$\Delta_E \Phi = 0 \Rightarrow \Delta_E(\tilde{D}\Phi) = 0$$

$(Q \in C^\infty(M))$

$$\nabla_\mu \xi_\nu = Q g_{\mu\nu}$$

Closed conformal Killing (CCKV)

$$\nabla_{[\mu} \xi_{\nu]} = 0$$

$(Q \in C^\infty(M))$

$$\nabla_{(\mu} \xi_{\nu)} = Q g_{\mu\nu}$$

Conformal Killing (CKV)

$$\lambda = 0$$

Killing (KV) \subset Homothetic (HV)

$$\nabla_{(\mu} \xi_{\nu)} = 0$$

$$\nabla_{(\mu} \xi_{\nu)} = \lambda g_{\mu\nu}$$

$(\lambda \in R)$

$$\nabla_\mu Q = 0$$

(M, g_{ab}) : Riem. mfd.

$\Delta \equiv g^{ab} \nabla_a \nabla_b$: the (scalar) Laplacian of g on M

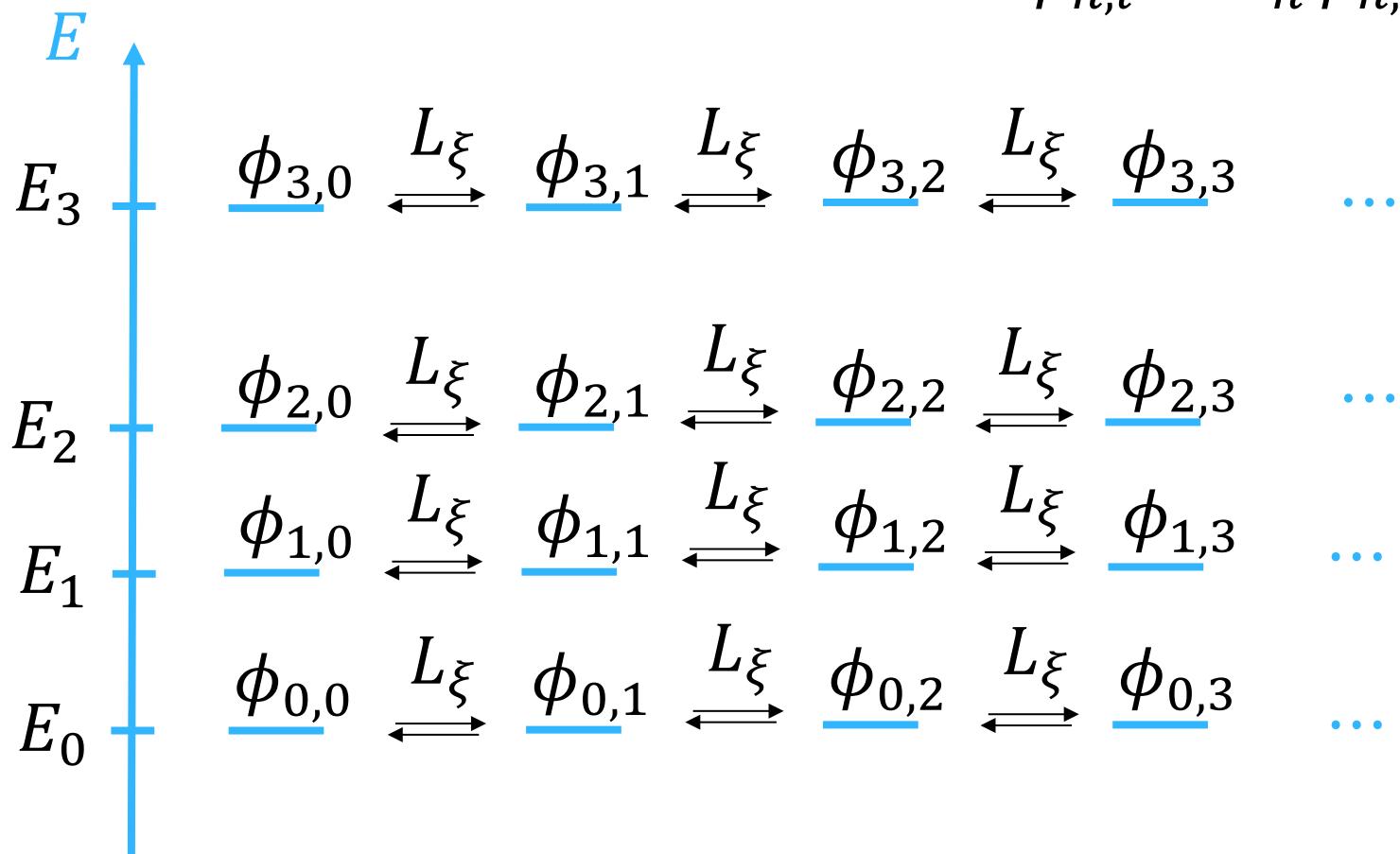
$$\begin{aligned}\xi^\mu &: \text{KV, i.e., } \nabla_{(\mu} \xi_{\nu)} = 0 \\ \Rightarrow [\Delta, L_\xi] &= 0\end{aligned}$$

For ϕ_n s.t. $\Delta \phi_n = E_n \phi_n$,

$$\Delta(L_\xi \phi_n) = [\Delta, L_\xi] \phi_n + L_\xi \Delta \phi_n = E(L_\xi \phi_n)$$

$$\xi^\mu : \text{KV} \quad \Rightarrow \quad [\Delta, L_\xi] = 0$$

$$\Delta\phi_{n,l} = E_n \phi_{n,l}$$



ζ^μ : CKV, i.e., $\nabla_{(\mu}\zeta_{\nu)} = Qg_{\mu\nu}$

$$\Rightarrow [\Delta, L_\zeta] = (n - 2)(\nabla^\mu Q)\nabla_\mu + 2Q\Delta$$

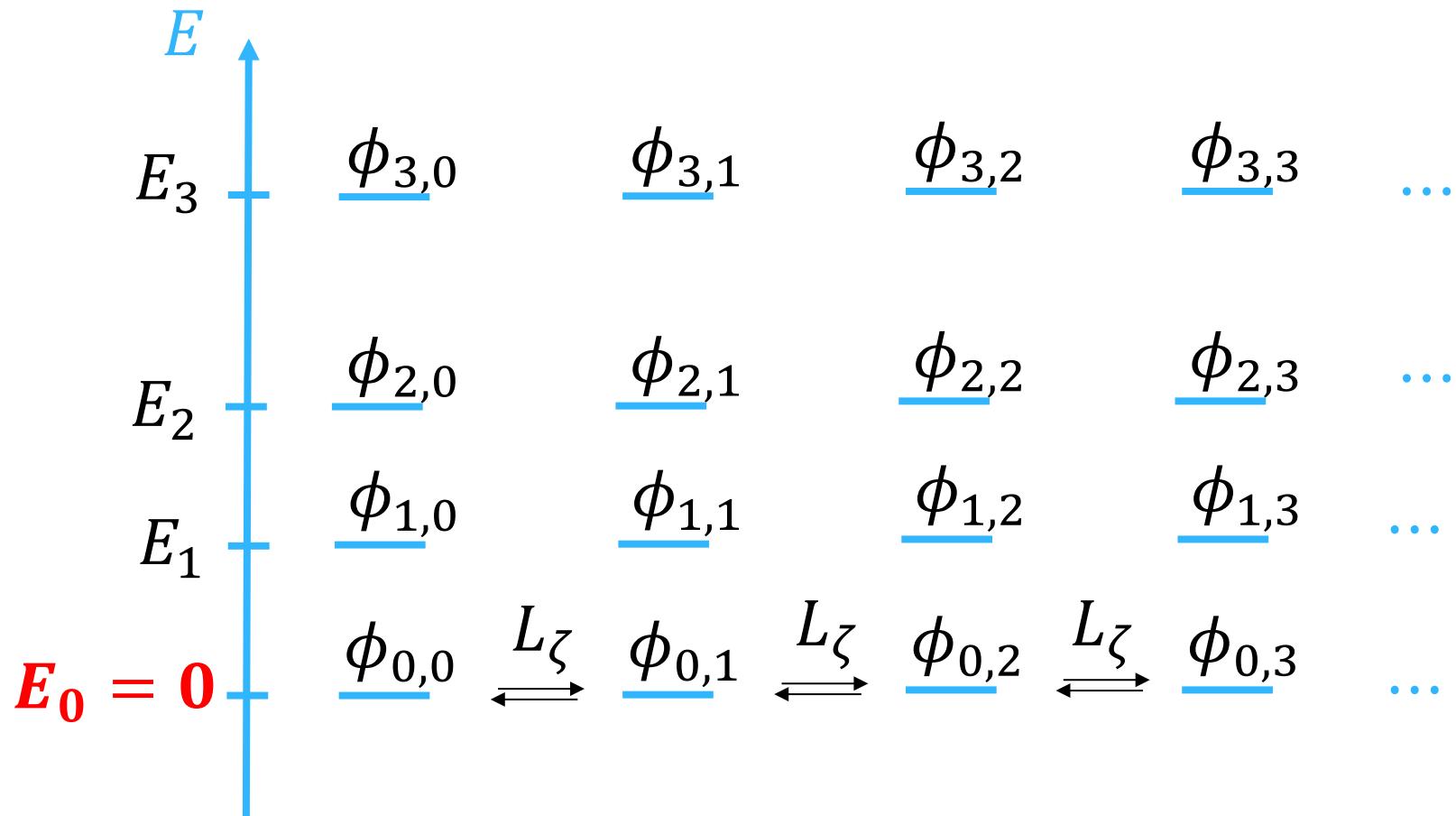
If $n = 2$ or Q is constant, then

$$[\Delta, L_\zeta] = 2Q\Delta$$

For ϕ_0 s.t. $\Delta\phi_0 = 0$,

$$\Delta(L_\zeta\phi_0) = [\Delta, L_\zeta]\phi_0 + L_\zeta\Delta\phi_0 = 0$$

i) ζ^μ : CKV in 2dim.
ii) ζ^μ : HV in n dim. $\Rightarrow [\Delta, L_\zeta] = 2Q\Delta$



- $\xi^\mu : KV \Rightarrow [\Delta, L_\xi] = 0$
- $\zeta^\mu : HV \Rightarrow [\Delta, L_\zeta] = \frac{2}{n} (\nabla_\mu \zeta^\mu) \Delta$
- $\zeta^\mu : CKV \text{ in 2 dim.} \Rightarrow [\Delta, L_\zeta] = \frac{2}{n} (\nabla_\mu \zeta^\mu) \Delta$
- $\zeta^\mu : CKV \text{ on scalar-flat spacetime in } n \text{ dim.}$
 $\Rightarrow [\Delta, D] = \frac{2}{n} (\nabla_\mu \zeta^\mu) \Delta$
 $D \equiv L_\zeta + \frac{n-2}{2n} (\nabla_\mu \zeta^\mu)$

Ex) Eisenhart-Duval lift of a 1-dim harmonic oscillator

$$\left(-\frac{1}{2m} \frac{d^2}{dx^2} + \frac{m}{2} \omega^2 x^2 \right) \psi = E\psi$$

Bergmann metric $ds^2 = -\omega^2 x^2 dt^2 + 2dtds + dx^2$

- 3-dim spacetime
- Laplace Eq. $\Delta_E \Phi = 0$ with $\Phi = e^{-iEt+ims} \psi(x)$

$$\Rightarrow \left(-\frac{1}{2m} \Delta + \frac{m}{2} \omega^2 x^2 \right) \psi = E\psi$$

- Conformally flat \Rightarrow 10 CKVs (including 4 KVs)
- Scalar-flat \Rightarrow 10 symmetry operators

$$\mathbf{KV} \quad \zeta_1 = \frac{\partial}{\partial s}, \quad Q_1 = 0,$$

$$\mathbf{KV} \quad \zeta_2 = \frac{\partial}{\partial t}, \quad Q_2 = 0,$$

$$\nabla_{(\mu} \zeta_{\nu)} = Q g_{\mu\nu}$$

$$\mathbf{KV} \quad \zeta_3 = \omega x \cos(\omega t) \frac{\partial}{\partial s} - \sin(\omega t) \frac{\partial}{\partial x}, \quad Q_3 = 0,$$

$$\mathbf{KV} \quad \zeta_4 = \omega x \sin(\omega t) \frac{\partial}{\partial s} + \cos(\omega t) \frac{\partial}{\partial x}, \quad Q_4 = 0,$$

$$\mathbf{HV} \quad \zeta_5 = 2s \frac{\partial}{\partial s} + x \frac{\partial}{\partial x}, \quad Q_5 = 1,$$

$$\zeta_6 = \cos^2(\omega t) \frac{\partial}{\partial t} + \frac{1}{2} \omega^2 x^2 \cos(2\omega t) \frac{\partial}{\partial s} - \frac{1}{2} \omega x \sin(2\omega t) \frac{\partial}{\partial x}, \quad Q_6 = -\frac{\omega}{2} \sin(2\omega t),$$

$$\zeta'_6 = \sin^2(\omega t) \frac{\partial}{\partial t} - \frac{1}{2} \omega^2 x^2 \cos(2\omega t) \frac{\partial}{\partial s} + \frac{1}{2} \omega x \sin(2\omega t) \frac{\partial}{\partial x}, \quad Q'_6 = \frac{1}{2} \omega \sin(2\omega t),$$

$$\zeta_7 = \sin(2\omega t) \frac{\partial}{\partial t} + \omega^2 x^2 \sin(2\omega t) \frac{\partial}{\partial s} + \omega x \cos(2\omega t) \frac{\partial}{\partial x}, \quad Q_7 = \omega \cos(2\omega t),$$

$$\zeta'_7 = \cos(2\omega t) \frac{\partial}{\partial t} + \omega^2 x^2 \cos(2\omega t) \frac{\partial}{\partial s} - \omega x \sin(2\omega t) \frac{\partial}{\partial x}, \quad Q'_7 = -\omega \sin(2\omega t),$$

$$\zeta_8 = -\frac{1}{2} x^2 \frac{\partial}{\partial t} + (s^2 - \frac{1}{4} \omega^2 x^4) \frac{\partial}{\partial s} + sx \frac{\partial}{\partial x}, \quad Q_8 = s,$$

$$\zeta_9 = x \cos(\omega t) \frac{\partial}{\partial t} + \omega x \left(\frac{1}{2} \omega x^2 \cos(\omega t) - s \sin(\omega t) \right) \frac{\partial}{\partial s} \\ - \left(\frac{1}{2} \omega x^2 \sin(\omega t) + s \cos(\omega t) \right) \frac{\partial}{\partial x}, \quad Q_9 = -\omega x \sin(\omega t),$$

$$\zeta_{10} = x \sin(\omega t) \frac{\partial}{\partial t} + \omega x \left(\frac{1}{2} \omega x^2 \sin(\omega t) + s \cos(\omega t) \right) \frac{\partial}{\partial s}$$

$$+ \left(\frac{1}{2} \omega x^2 \cos(\omega t) - s \sin(\omega t) \right) \frac{\partial}{\partial x}, \quad Q_{10} = \omega x \cos(\omega t),$$

$$D_1^\pm \equiv \zeta_4 \mp i\zeta_3 = e^{\pm i\omega t} \left(\frac{\partial}{\partial x} \mp i\omega x \frac{\partial}{\partial s} \right)$$

$$D_2^\pm \equiv \frac{1}{\omega} (\zeta_7 \mp i\zeta'_7) + \frac{1}{2} e^{\pm 2i\omega t} = e^{\pm 2i\omega t} \left(x \frac{\partial}{\partial x} \mp i\omega x^2 \frac{\partial}{\partial s} \mp \frac{i}{\omega} \frac{\partial}{\partial t} + \frac{1}{2} \right)$$

$$\Phi = e^{(iEt - ims)} \psi(x)$$

$$D_1^\pm \Phi = e^{(i(E \pm \omega)t - ims)} \left(\frac{d}{dx} \mp m\omega x \right) \psi(x)$$

\Rightarrow

$$\overline{\equiv \mathbf{a}_\pm}$$

$$D_2^\pm \Phi = e^{(i(E \pm 2\omega)t - ims)} \left(x \frac{d}{dx} \mp m\omega x^2 \pm \frac{E}{\omega} + \frac{1}{2} \right) \psi(x)$$

$$\overline{\equiv \mathbf{b}_\pm}$$

Ex) Eisenhart-Duval lift of a 1-dim harmonic oscillator

$$\left(-\frac{1}{2m} \frac{d^2}{dx^2} + \frac{m}{2} \omega^2 x^2 \right) \psi = E\psi$$

Bergmann metric $ds^2 = -\omega^2 x^2 dt^2 + 2dtdx + dx^2$

- Ladder operators 1

$$a_{\pm} = \frac{d}{dx} \mp m\omega x$$

$$\begin{aligned}\psi_n &\rightarrow \psi_{n+1} \\ E_n &\rightarrow E_{n+1} = E_n + \omega\end{aligned}$$

- Ladder operators 2

$$b_{\pm} = x \frac{d}{dx} \mp m\omega x^2 \pm \left(n + \frac{1}{2}\right) + \frac{1}{2}$$

$$\begin{aligned}\psi_n &\rightarrow \psi_{n+2} \\ E_n &\rightarrow E_{n+2} = E_n + 2\omega\end{aligned}$$

Deformation for ladder operators and intertwining operators

with Kimura with Sakamoto, Tatsumi

Carter, Eastwood

Higher-order symmetries

$$K = \nabla_{\mu_1} \dots \nabla_{\mu_{s-1}} K^{\mu_1 \dots \nu_s} \nabla_{\nu_s} \quad \text{s.t.}$$

$$[\Delta, K] = 0$$

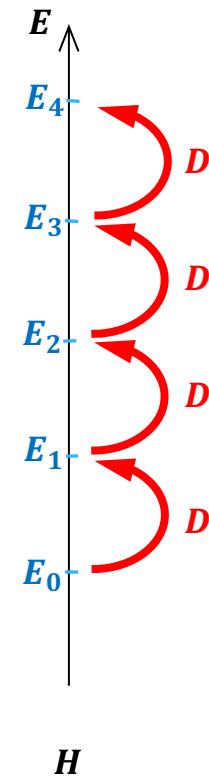
$$[\Delta, K] \propto \Delta$$

$$[\Delta, K] \propto K$$

Other lifts Gibbons et al, Kleinert et al, Cariglia

$$[H, D] = \varepsilon D$$

$$\Rightarrow H D\psi_n = (E_n + \varepsilon)D\psi_n$$

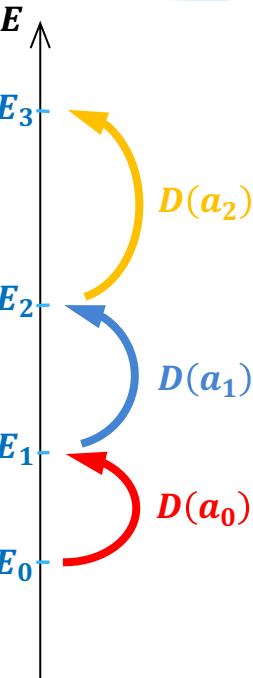


$$[H, D(a)] = \varepsilon(a)D(a) + Q(a)(H - \lambda(a))$$

$$\Rightarrow H D(a_n)\psi_n = (E_n + \varepsilon(a_n))D(a_n)\psi_n$$

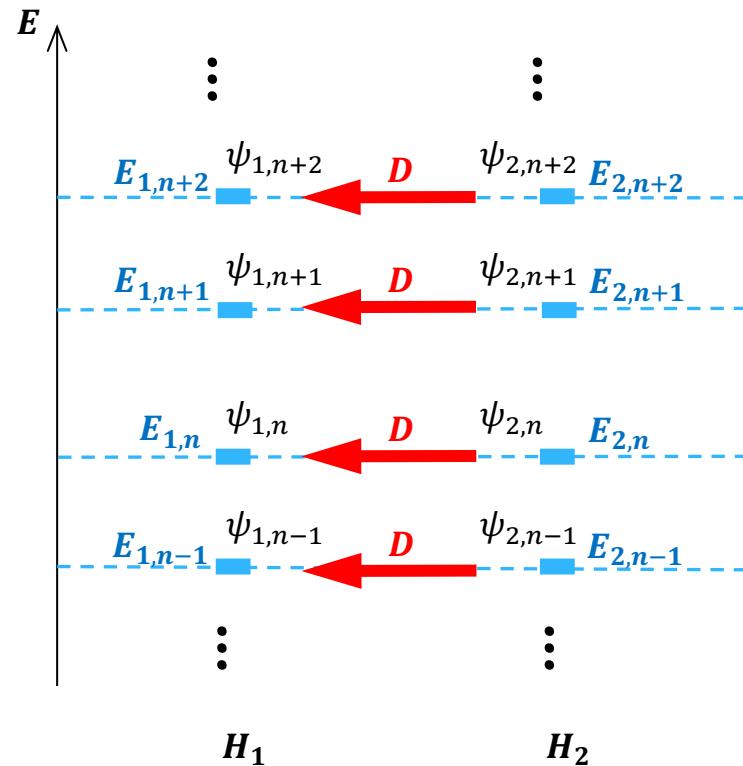
$$\text{where } \lambda(a_n) = E_n$$

with Kimura



$$H_1 D = D H_2$$

$$\Rightarrow H_1 (D\psi_{2,n}) = E_{2,n} (D\psi_{2,n})$$



with Sakamoto, Tatsumi

$$H(\nu)D(\nu) = D(\nu) \left(H(f(\nu)) + \varepsilon(\nu) \right)$$

$$\Rightarrow H(\nu)D(\nu)\psi_n(f(\nu)) = \left(E_n(f(\nu)) + \varepsilon(\nu) \right) D(\nu)\psi_n(f(\nu))$$

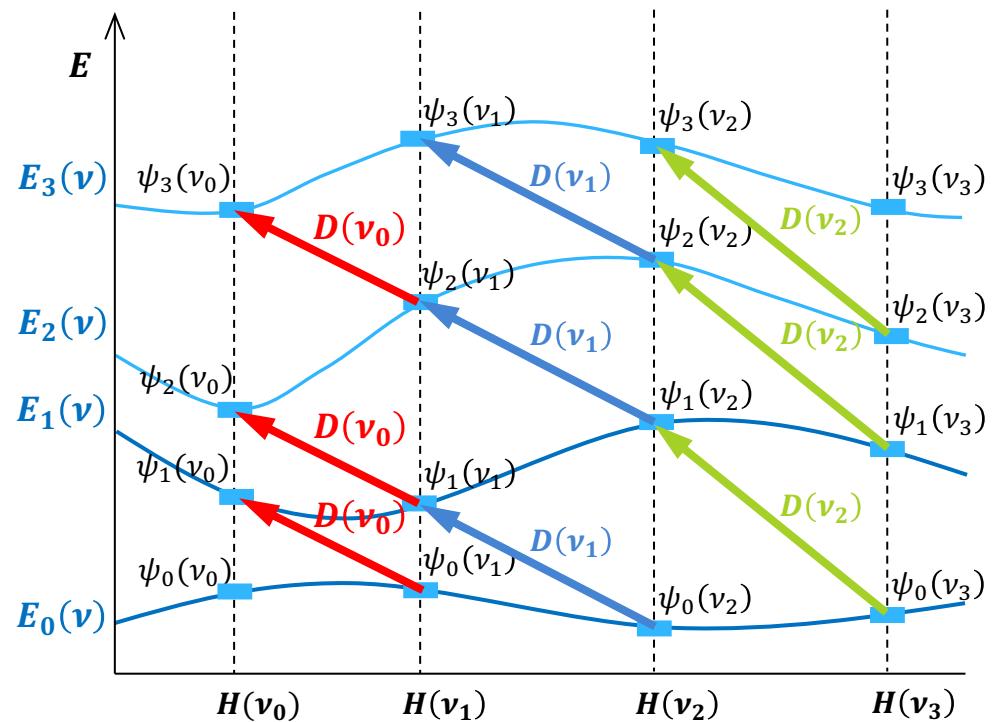
Ex) Radial part of 3D Coulomb potential

$$\frac{1}{2m} \left(-\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) + \frac{\ell(\ell+1)}{r^2} - \frac{g}{r} \right) \psi = E\psi$$

Ladder operator

$$D_{\pm}(\ell) \equiv \frac{1}{r} \left(\pm \frac{d}{dr} - \frac{\ell+1}{r} + \frac{g}{\ell+1} + 1 \right)$$

$$f(\ell) = \ell + 1$$



$$H(\nu)D(\nu, a) = D(\nu, a) \left(H(f(\nu, a)) + \varepsilon(f(\nu, a), a) \right) \\ + Q(\nu, a) \left(H(f(\nu, a)) - \lambda(f(\nu, a), a) \right)$$

$$\Rightarrow H(\nu)D(\nu, a_n)\psi_n(f(\nu, a_n)) \\ = (E_n(f(\nu, a_n)) + \varepsilon(f(\nu, a_n), a_n))D(\nu, a_n)\psi_n(f(\nu, a_n))$$

$$\text{where } \lambda(f(\nu, a_n), a_n) = E_n(f(\nu, a_n))$$

with Sakamoto, Tatsumi



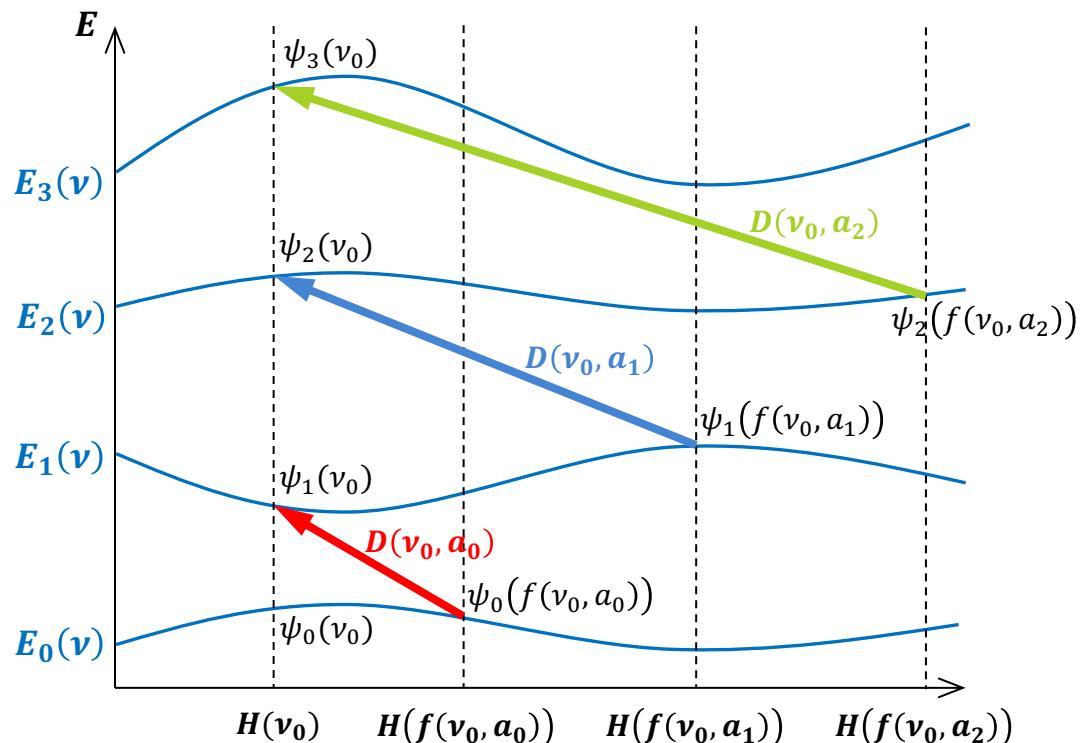
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$$\frac{1}{2m} \left(-\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) + \frac{\ell(\ell+1)}{r^2} - \frac{g}{r} \right) \psi = E\psi$$

Ladder operator

$$D(g, n) \equiv r \frac{d}{dr} - \frac{rg}{2(n+1)} + n + 1$$

$$f(g, n) = \frac{ng}{n+1}$$



時空の対称性

様々な物理を理解するための強力な解析手法を与える

時空の性質

時空の定性的な理解・分類

物質の振舞い

粒子や場の方程式の可積分性

他分野との関連

ハミルトン力学への応用
微分幾何学への還元
(幾何学化)