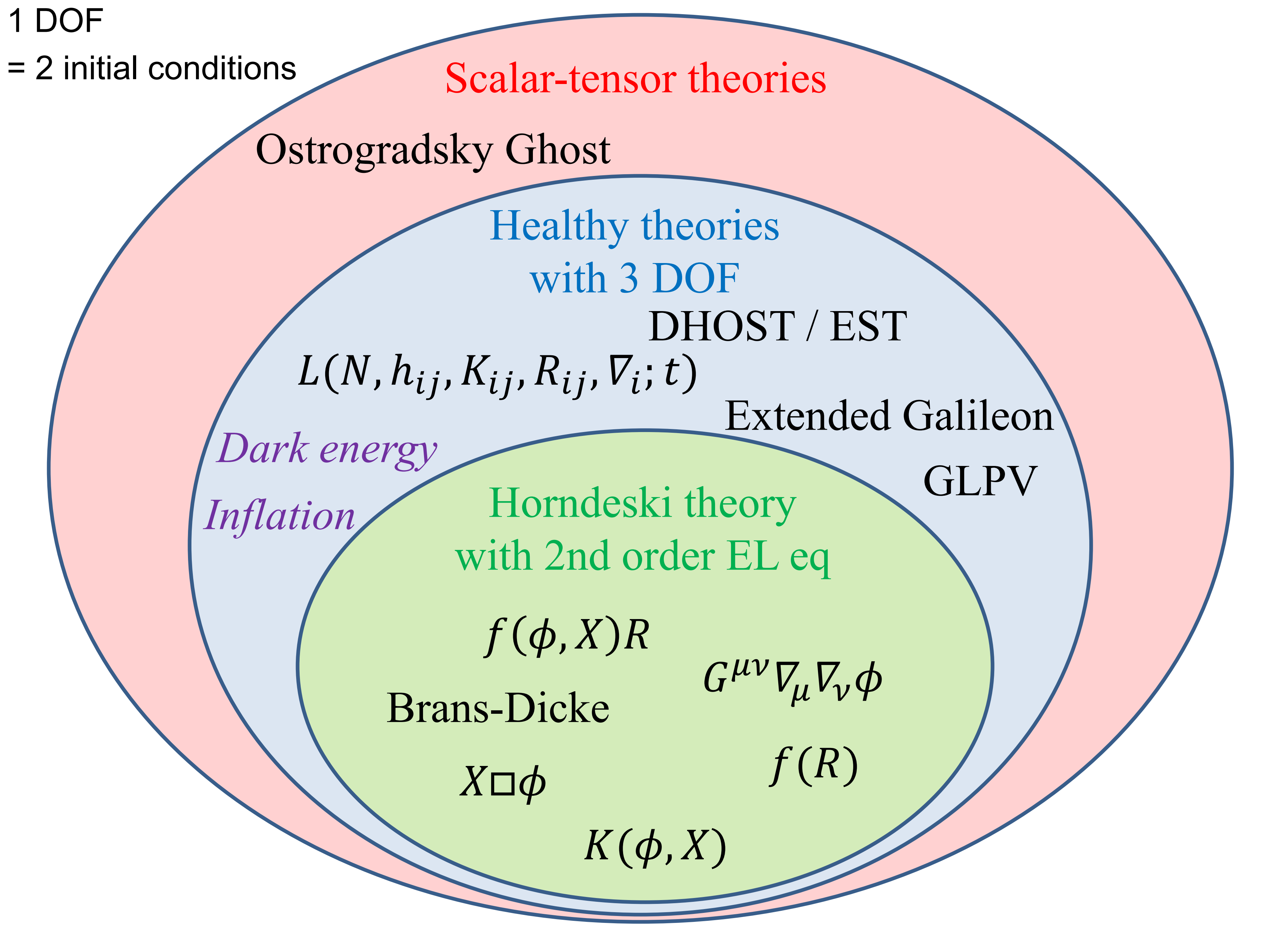
Degenerate theories with higher derivatives

Hayato Motohashi

IFIC, University of Valencia

2017.03.02 Workshop on gravity & cosmology for young researchers, YITP


1 DOF
= 2 initial conditions
Scalar-tensor theories
Ostrogradsky Ghost
Healthy theories
with 3 DOF
DHOST / EST
$L(N, h_{ij}, K_{ij}, R_{ij}, \nabla_i; t)$
Extended Galileon
Dark energy
GLPV
Inflation
Horndeski theory
with 2nd order EL eq
$f(\phi, X)R$
$G^{\mu\nu}\nabla_\mu\nabla_\nu\phi$
Brans-Dicke
$f(R)$
$X\Box\phi$
$K(\phi, X)$

Outline

Purpose

- To clarify how to construct theories evading Ostrogradsky ghost.

* Ostrogradsky ghost – Ghost (carrying negative energy) associated with general higher derivative theory.

Result

- We derived no-ghost condition (degeneracy condition) for various types of Lagrangians.

Simple example: $L = \frac{1}{2}\ddot{x}^2$

- 4th order EL eq

$$\ddddot{x} = 0$$

requires 4 initial conditions = 2 DOF.

$$z = \frac{x+y}{\sqrt{2}}, w = \frac{x-y}{\sqrt{2}}$$

- Hamiltonian is unbounded

$$L = \ddot{x}y - \frac{1}{2}y^2 = -\dot{x}\dot{y} - \frac{1}{2}y^2$$

$$= \frac{1}{2}\dot{w}^2 - \frac{1}{2}\dot{z}^2 - \frac{1}{4}(w-z)^2$$

$$H = \frac{1}{2}p_w^2 - \frac{1}{2}p_z^2 + \frac{1}{4}(w-z)^2$$

2 DOF = 1 healthy + 1 ghost

Problematic when coupled to a normal system

Ostrogradsky theorem for $L(\ddot{\phi}, \dot{\phi}, \phi)$

For Lagrangian $L(\ddot{\phi}, \dot{\phi}, \phi)$ with $\phi = \phi(t)$,

$\partial^2 L/\partial\ddot{\phi}^2 \neq 0 \Longrightarrow H$ is unbounded.

'L is nondegenerate'.

Woodard, 1506.02210

- EL eq

$$\frac{\partial^2 L}{\partial\ddot{\phi}^2}\ddddot{\phi} + \frac{d}{dt}\left(\frac{\partial^2 L}{\partial\ddot{\phi}^2}\right)\dddot{\phi} = (\text{terms up to } \ddot{\phi})$$

$\partial^2 L/\partial\ddot{\phi}^2 \neq 0 \Longrightarrow$ 4th order system = 2 DOF

Ostrogradsky theorem for $L(\ddot{\phi}, \dot{\phi}, \phi)$

For Lagrangian $L(\ddot{\phi}, \dot{\phi}, \phi)$ with $\phi = \phi(t)$,

$\partial^2 L/\partial\ddot{\phi}^2 \neq 0 \Longrightarrow H$ is unbounded.

- Hamiltonian analysis

Dirac, "Lectures on Quantum Mechanics"

Henneaux, Teitelboim, "Quantization of gauge systems"

Consider an equivalent form

$$L\big(\dot{Q}, Q, \phi\big) + \lambda(Q - \dot{\phi})$$

6 ini. conds.

Canonical momenta for ϕ, λ, Q:

$$p = -\lambda, \qquad \pi = 0, \qquad P = \partial L/\partial \dot{Q}$$

2 primary constraints

$\partial^2 L/\partial\ddot{\phi}^2 \neq 0$

$\Longrightarrow \dot{Q} = \dot{Q}(P, Q, \phi)$

$\{p + \lambda, \pi\} = 1 \Longrightarrow$ second class, no secondary constraints

$\Longrightarrow (6 - 2)/2 = 2$ DOF

Ostrogradsky theorem for $L(\ddot{\phi}, \dot{\phi}, \phi)$

For Lagrangian $L(\ddot{\phi}, \dot{\phi}, \phi)$ with $\phi = \phi(t)$,

$\partial^2 L / \partial \ddot{\phi}^2 \neq 0 \Longrightarrow H$ is unbounded.

- Hamiltonian analysis

Consider an equivalent form

$$L\big(\dot{Q}, Q, \phi\big) + \lambda(Q - \dot{\phi})$$

Canonical momenta for ϕ, λ, Q:

$$p = -\lambda, \qquad \pi = 0, \qquad P = \partial L / \partial \dot{Q}$$

Total Hamiltonian

$$H_T = pQ + P\dot{Q}(P, Q, \phi) - L(\dot{Q}(P, Q, \phi), Q, \phi)$$

p shows up only linearly.

H is unbounded.

$(6 - 2)/2 = 2$ DOF

$= 1$ healthy $+ 1$ ghost

Eliminating Ostrogradsky ghost

For Lagrangian $L(\ddot{\phi}, \dot{\phi}, \phi)$ with $\phi = \phi(t)$,

$\partial^2 L/\partial\ddot{\phi}^2 \neq 0 \Longrightarrow H$ is unbounded.

Let us impose $\partial^2 L/\partial\ddot{\phi}^2 = 0$

✓ Removes $\ddddot{\phi}$ and $\dddot{\phi}$ from EOM

✓ EL eq is 2nd order as

$$\frac{\partial^2 L}{\partial\ddot{\phi}^2}\ddddot{\phi} + \frac{d}{dt}\left(\frac{\partial^2 L}{\partial\ddot{\phi}^2}\right)\dddot{\phi} = (\text{terms up to } \ddot{\phi})$$

✓ Hamiltonian analysis

$P = \partial L/\partial\dot{Q}$ becomes additional 1 primary constraint

$\Longrightarrow$ Additional 1 secondary constraint

$\Longrightarrow (6-2-2)/2 = 1$ DOF. H is bounded.

Eliminating Ostrogradsky ghost

For Lagrangian $L(\ddot{\phi}, \dot{\phi}, \phi)$ with $\phi = \phi(t)$,

$\partial^2 L / \partial \ddot{\phi}^2 \neq 0 \Longrightarrow H$ is unbounded.

$\partial^2 L / \partial \ddot{\phi}^2 = 0$:The degeneracy (ghost-free) condition.

✓ EL eq is 2nd order $\Longrightarrow$ 1 DOF

✓ H is bounded $\Longrightarrow$ healthy

The most general ghost-free Lagrangian is

$$L = \ddot{\phi} f(\dot{\phi}, \phi) + g(\dot{\phi}, \phi) = G(\dot{\phi}, \phi)$$

so long as we consider $L(\ddot{\phi}, \dot{\phi}, \phi)$.

Ostrogradsky ghost for $L(\ddot{\phi}^a, \dot{\phi}^a, \phi^a)$

For $L(\ddot{\phi}^a, \dot{\phi}^a, \phi^a)$ with $\phi^a = \phi^a(t)$ and $a = 1, \cdots n$,

$$\det K \neq 0 \Longrightarrow H \text{ is unbounded,}$$

where $K_{ab} = \dfrac{\partial^2 L}{\partial \ddot{\phi}^a \partial \ddot{\phi}^b}$.

'kinetic matrix'

- EL eq

$$K_{ab} \ddddot{\phi}^b = \left(\text{terms up to } \dddot{\phi}^a\right)$$

  $\det K \neq 0 \Longrightarrow$ 4th order system.

- H is unbounded.

  n healthy + n ghost DOF.

Eliminating Ostrogradsky ghost

For $L(\ddot{\phi}^a, \dot{\phi}^a, \phi^a)$ with $\phi^a = \phi^a(t)$ and $a = 1, \cdots n$,

$\det K \neq 0 \Longrightarrow H$ is unbounded,

where $K_{ab} = \dfrac{\partial^2 L}{\partial \ddot{\phi}^a \partial \ddot{\phi}^b}$.

Let us impose $K_{ab} = 0$

HM, Suyama, 1411.3721

$$M_{ab} = \frac{\partial^2 L}{\partial \ddot{\phi}^a \partial \dot{\phi}^b} - \frac{\partial^2 L}{\partial \ddot{\phi}^b \partial \dot{\phi}^a}$$

- EL eq

$$K_{ab} \ddddot{\phi}^b + \left(\dot{K}_{ab} + M_{ab}\right) \dddot{\phi}^b = \left(\text{terms up to } \ddot{\phi}^a\right)$$

Clearly, imposing $K_{ab} = 0$ is no more sufficient.

$K_{ab} = 0$, $\det M \neq 0$ (n: even) $\Longrightarrow$ 3rd order system

- H is unbounded. $3n$ ini. conds. are required.

Eliminating Ostrogradsky ghost

For $L(\ddot{\phi}^a, \dot{\phi}^a, \phi^a)$ with $\phi^a = \phi^a(t)$ and $a = 1, \cdots n$,

$\det K \neq 0$ or $\det M \neq 0$ $\Longrightarrow$ H is unbounded,

where $K_{ab} = \dfrac{\partial^2 L}{\partial \ddot{\phi}^a \partial \ddot{\phi}^b}$ and $M_{ab} = \dfrac{\partial^2 L}{\partial \ddot{\phi}^a \partial \dot{\phi}^b} - \dfrac{\partial^2 L}{\partial \ddot{\phi}^b \partial \dot{\phi}^a}$.

Ghost-free condition

$$K_{ab} = 0 \quad \text{and} \quad M_{ab} = 0$$

✓ Removes $\ddddot{\phi}^a$

✓ Removes $\dddot{\phi}^a$

✓ EL eq $\Longrightarrow$ 2nd order system

✓ H is bounded.

✓ It is possible to generalize the same logic to $L(\phi^{a(m)}, \phi^{a(m-1)}, \dots, \phi^a)$

Evading Ostrogradsky ghost

- If $L(\ddot{\phi}^a, \dot{\phi}^a, \phi^a)$ has higher-order EL eq

$\Longrightarrow$ Ghost DOF.

- Does it hold for system with multiple variables with different order of derivatives? ($\supset$ ST theory)

$\Longrightarrow$ Not always.

- If a set of higher-order EL eqs can be reduced to 2nd order system, the system does not suffer from Ostrogradsky ghost.

$\Longrightarrow$ There exists a healthy class of scalar-tensor theory beyond Horndeski

Horndeski theory

3 DOF = 2 for $g_{\mu\nu}$ + 1 for ϕ

Most general theory for 2nd order EL eq is Horndeski theory or generalized Galileon

Horndeski, 1974

Deffayet, Gao, Steer, Zahariade, 1103.3260

Kobayashi, Yamaguchi, Yokoyama, 1105.5723

$$L_2 = G_2(\phi, X) \qquad X \equiv \nabla_\mu \phi \nabla^\mu \phi$$

$$L_3 = G_3(\phi, X) \Box \phi$$

$$L_4 = G_4(\phi, X) R - 2G_{4X}(\phi, X)\left[(\Box\phi)^2 - \left(\nabla_\mu \nabla_\nu \phi\right)^2\right]$$

$$L_5 = G_5(\phi, X) G^{\mu\nu} \nabla_\mu \nabla_\nu \phi + \frac{1}{3} G_{5X}(\phi, X)\left[(\Box\phi)^3 - 3\Box\phi\left(\nabla_\mu \nabla_\nu \phi\right)^2 + \left(\nabla_\mu \nabla_\nu \phi\right)^3\right]$$

✓ 2nd order EL eq $\Longrightarrow$ Ostrogradsky ghost-free

beyond Horndeski: GLPV

In ADM formalism, Horndeski Lagrangian is written as

$$L = A_2(\phi,X) + A_3(\phi,X)K$$
$$+A_4(\phi,X)\left(K^2 - K_{ij}^2\right) + B_4(\phi,X)\,{}^{(3)}R$$
$$+A_5(\phi,X)(K^3 - 3KK_{ij}^2 + 2K_{ij}^3) + B_5(\phi,X)(U - K\,{}^{(3)}R/2)$$

with $A_4 = 2XB_{4,X} - B_4,\ A_5 = -XB_{5,X}/3$.

GLPV theory: same L without the two conditions

Gleyzes, Langlois, Piazza, Vernizzi, 1404.6495, 1408.1952

✓ 3 DOF

Domenech, Mukohyama, Namba, Naruko, Saitou, Watanabe, 1507.05390

✓ Higher order EL eq is reducible to 2nd order system.

Deffayet, Esposito-Farese, Steer, 1506.01974

- Includes subclass related to Horndeski through disformal transformation.

$$g_{\mu\nu} \to A(\phi,X)g_{\mu\nu} + B(\phi,X)\nabla_\mu\phi\nabla_\nu\phi$$

Example: $L = \frac{1}{2}(\dot{x} - \ddot{y})^2 + \frac{1}{2}\dot{y}^2$

- EL eq is a priori 4th order

$$\frac{d}{dt}(\dot{x} - \ddot{y}) = 0, \qquad \frac{d^2}{dt^2}(\dot{x} - \ddot{y}) + \ddot{y} = 0$$

but can be rearranged to 2nd order system

$$\ddot{x} = 0, \qquad \ddot{y} = 0$$

- Invertible transformation

$$X = x - \dot{y}, \qquad Y = y$$
$$(x = X + \dot{Y}, \qquad y = Y)$$

leads Lagrangian to

$$L = \frac{1}{2}\dot{X}^2 + \frac{1}{2}\dot{Y}^2$$

✓ General invertible transformation in field theory

$\Longrightarrow$ Kazufumi's poster Takahashi, HM, Suyama, Kobayashi, 1702.01849

Example: $L = \frac{1}{2}\frac{\dot{x}^2}{1+\ddot{y}} + \frac{1}{2}\dot{y}^2$

Gabadadze, Hinterbichler, Khoury, Pirtskhalava, Trodden, 1208.5773

- EL eq is a priori 4th order

$$\frac{d}{dt}\left(\frac{\dot{x}}{1+\ddot{y}}\right) = 0, \qquad \frac{d^2}{dt^2}\left(\frac{\dot{x}^2}{(1+\ddot{y})^2}\right) + \ddot{y} = 0$$

but can be rearranged to 2nd order system

$$\ddot{x} = 0, \qquad \ddot{y} = 0$$

- Is it equivalent to $L = \frac{1}{2}\dot{X}^2 + \frac{1}{2}\dot{Y}^2$?

  At least, $X = x + a\dot{y},\ Y = y$ cannot transform it.

- Is there some other transformation?

  Not clear.

Healthy scalar-tensor theory

Langlois, Noui, 1510.06930, 1512.06820
Crisostomi, Koyama, Tasinato, 1602.03119
Achour, Langlois, Noui, 1602.08398

quadratic DHOST / EST

- Write down all possible derivative terms with coefficient functions at $(\nabla\nabla\phi)^2, (\partial\partial g)^2$ order

$$S = \int d^4x\sqrt{-g}\left[F_2(\phi,X)R + \sum_{i=1}^{5} A_i(\phi,X)L_i^{(2)}\right]$$

$$L_1^{(2)} = \left(\phi_{;\mu\nu}\right)^2, L_2^{(2)} = (\Box\phi)^2, L_3^{(2)} = (\Box\phi)\phi^{;\mu}\phi_{;\mu\nu}\phi^{;\nu},$$

$$L_4^{(2)} = \phi^{;\mu}\phi_{;\mu\nu}\phi^{;\nu\rho}\phi_{;\rho}, L_5^{(2)} = \left(\phi^{;\mu}\phi_{;\mu\nu}\phi^{;\nu}\right)^2.$$

- Find degenerate condition on F_2 and A_i.

Healthy scalar-tensor theory

quadratic DHOST / EST

- Parallel to what we saw previously, but needs to deal with mixed orders of derivatives.

- Toy model was first studied

$$L = \frac{1}{2}a\ddot{\phi}^2 + \frac{1}{2}k_0\dot{\phi}^2 + \frac{1}{2}k_{ij}\dot{q}^i\dot{q}^j + b_i\ddot{\phi}\dot{q}^i + c_i\dot{\phi}\dot{q}^i - V(\phi, q)$$

which applies to only $(\nabla\nabla\phi)^2$ order.

- More general analysis is required beyond $(\nabla\nabla\phi)^2$.

$L(\ddot{\phi}, \dot{\phi}, \phi; \dot{q}, q)$

HM, Noui, Suyama, Yamaguchi, Langlois, 1603.09355

$\phi(t), q(t)$ have different orders of derivatives.

Equivalent form

$$L(\dot{Q}, Q, \phi; \dot{q}, q) + \lambda(\dot{\phi} - Q).$$

8 ini. conds.

Canonical momenta for ϕ and λ

$\Longrightarrow$ 2 primary constraints

From $P = L_{\dot{Q}}$ and $p = L_{\dot{q}}$ $\quad (L_X \equiv \partial L / \partial X)$

$$\begin{pmatrix} \delta P \\ \delta p \end{pmatrix} = \underbrace{\begin{pmatrix} L_{\dot{Q}\dot{Q}} & L_{\dot{q}\dot{Q}} \\ L_{\dot{q}\dot{Q}} & L_{\dot{q}\dot{q}} \end{pmatrix}}_{\text{kinetic matrix } K} \begin{pmatrix} \delta \dot{Q} \\ \delta \dot{q} \end{pmatrix}.$$

If $\det K \neq 0 \Longrightarrow$ No further primary constraints.

We thus impose $\boxed{\det K = 0}$.

$$L(\ddot{\phi}, \dot{\phi}, \phi; \dot{q}, q)$$

Several cases are possible. Let us consider the case

$$L_{\dot{q}\dot{q}} \neq 0, \quad \text{and} \quad L_{\dot{Q}\dot{Q}} - L_{\dot{q}\dot{Q}}^2 / L_{\dot{q}\dot{q}} = 0,$$

as q-system is nondegenerate and we focus on how the degeneracy cures ϕ-system.

Diagonalization

$$K = \begin{pmatrix} L_{\dot{q}\dot{Q}}^2 / L_{\dot{q}\dot{q}} & L_{\dot{q}\dot{Q}} \\ L_{\dot{q}\dot{Q}} & L_{\dot{q}\dot{q}} \end{pmatrix} = O^{-1} \begin{pmatrix} 0 & 0 \\ 0 & L_{\dot{q}\dot{q}}(L_{\dot{q}\dot{Q}}^2 / L_{\dot{q}\dot{q}}^2 + 1) \end{pmatrix} O$$

Thus,

$$\delta P = \frac{L_{\dot{q}\dot{Q}}}{L_{\dot{q}\dot{q}}} \delta p, \qquad \delta p = L_{\dot{q}\dot{Q}} \delta \dot{Q} + L_{\dot{q}\dot{q}} \delta \dot{q}$$

$$\Longrightarrow \; P = F(p, q, Q, \phi)$$

$L(\ddot{\phi}, \dot{\phi}, \phi; \dot{q}, q)$

The condition

$$L_{\dot{q}\dot{q}} \neq 0, \quad \text{and} \quad L_{\dot{Q}\dot{Q}} - L^2_{\dot{q}\dot{Q}}/L_{\dot{q}\dot{q}} = 0,$$

is equivalent to 1 primary constraint $P = F(p, q, Q, \phi)$

$\Longrightarrow$ Additional 1 secondary constraint

$\Longrightarrow (8 - 2 - 2)/2 = 2$ DOF.

✓ 2 DOF, H is bounded.

✓ EL eq can be reducible to 2nd order system.

$$L(\ddot{\phi},\dot{\phi},\phi;\dot{q},q) \sim L(\dot{Q},Q,\phi;\dot{q},q)$$

✓ EL eq

$$\left\{\begin{array}{l} \left.\begin{array}{l} L_q - \dfrac{dL_{\dot{q}}}{dt} = 0, \\ L_Q - \dfrac{dL_{\dot{Q}}}{dt} = \lambda, \end{array}\right\} \Longrightarrow \quad {\color{blue}K}\begin{pmatrix}\ddot{Q}\\ \ddot{q}\end{pmatrix} = \begin{pmatrix}V-\lambda\\ v\end{pmatrix} \\ L_\phi = \dot{\lambda}, \\ \dot{\phi} = Q, \end{array}\right.$$

$$\Longrightarrow \quad \lambda = V - F_p v, \qquad \boxed{L_{\dot{q}\dot{q}}(\ddot{q}+F_p\ddot{Q}) = v}$$

$$\Longrightarrow \quad \boxed{(L_{\dot{q}Q}-L_{\dot{Q}q})(\ddot{q}+F_p\ddot{Q}) = w}$$

$$\boxed{\mathcal{E}_1 \equiv L_{\dot{q}\dot{q}}w - (L_{\dot{q}Q}-L_{\dot{Q}q})v = 0}$$

$$\Downarrow$$

From $\dot{\mathcal{E}}_1 = 0$ we can express $\dddot{\phi}$ in terms of 1st order derivatives $\Longrightarrow$ $\boxed{\mathcal{E}_2 = 0}$

2nd order eqs

$L(\ddot{\phi}, \dot{\phi}, \phi; \dot{q}^i, q^i)$

$\phi(t), q^i(t)$ with $i = 1, \cdots m$.

Equivalent form

$$L(\dot{Q}, Q, \phi; \dot{q}^i, q^i) + \lambda(\dot{\phi} - Q).$$

Canonical momenta for ϕ and λ

$2m + 6$ ini. conds.

$\Longrightarrow$ 2 primary constraints

From $P = L_{\dot{Q}}$ and $p_i = L_{\dot{q}}$ $\quad (L_X \equiv \partial L / \partial X)$

$$\begin{pmatrix} \delta P \\ \delta p_i \end{pmatrix} = \begin{pmatrix} L_{\dot{Q}\dot{Q}} & L_{\dot{q}^j\dot{Q}} \\ L_{\dot{q}^i\dot{Q}} & L_{\dot{q}^i\dot{q}^j} \end{pmatrix} \begin{pmatrix} \delta \dot{Q} \\ \delta \dot{q}^j \end{pmatrix}.$$

kinetic matrix K

If $\det K \neq 0 \Longrightarrow$ No further primary constraints.

We thus impose $\boxed{\det K = 0}$.

$L(\ddot{\phi}, \dot{\phi}, \phi; \dot{q}^i, q^i)$

To avoid removing healthy q^i DOF, let us assume $\det L_{\dot{q}^i \dot{q}^j} \neq 0$, under which $\det K = 0$ reads

$$L_{\dot{Q}\dot{Q}} - L_{\dot{q}^i \dot{Q}} k^{ij} L_{\dot{q}^j \dot{Q}} = 0,$$

where $k^{ij} L_{\dot{q}^j \dot{q}^k} = \delta^i_k$.

The condition is equivalent to additional 1 primary constraint

$$P = F\left(p_i, q^i, Q, \phi\right)$$

$\Longrightarrow$ Additional 1 secondary constraint

$\Longrightarrow (2m + 6 - 2 - 2)/2 = m + 1$ DOF.

✓ $m + 1$ DOF, H is bounded.

✓ EL eq is reducible to 2nd order system.

$$L(\ddot{\phi}^a, \dot{\phi}^a, \phi^a; \dot{q}^i, q^i)$$

$\phi^a(t), q^i(t)$ with $a = 1, \cdots n$ and $i = 1, \cdots m$.

An equivalent form

$$L(\dot{Q}^a, Q^a, \phi^a; \dot{q}^i, q^i) + \lambda_a(\dot{\phi}^a - Q^a).$$

Canonical momenta for ϕ^a and λ_a

$2m + 6n$ ini. conds.

$\Longrightarrow$ $2n$ primary constraints

As a natural generalization, we assume $\det L_{\dot{q}^i \dot{q}^j} \neq 0$,

and impose $\det K = 0$, i.e.

$$L_{\dot{Q}^a \dot{Q}^b} - L_{\dot{q}^i \dot{Q}^a} k^{ij} L_{\dot{q}^j \dot{Q}^b} = 0,$$

✓ Removes $\ddddot{\phi}^a$

which is equivalent to additional n primary constraints

$$\Xi_a \equiv P_a - F_a(p_i, q^i, Q^b, \phi^b) = 0.$$

In this case, the number of constraints is not sufficient.

$$L(\ddot{\phi}^a, \dot{\phi}^a, \phi^a; \dot{q}^i, q^i)$$

From the consistency condition

$$\dot{\Xi}_a = \{\Xi_a, H\} + \xi^b\{\Xi_a, \Xi_b\} = 0.$$

If $\{\Xi_a, \Xi_b\}$ is invertible, the Lagrange multipliers ξ^b are determined and there are no secondary constraints.

We thus impose

$$\boxed{\{\Xi_a, \Xi_b\} = 0,} \longleftarrow \checkmark \text{ Removes } \dddot{\phi}^a$$

or $$\frac{\partial^2 L}{\partial \dot{Q}^{[a}\partial \dot{\phi}^{b]}} + \frac{\partial^2 L}{\partial \dot{Q}^{[a}\partial \dot{q}^i} k^{ij}\left(-\frac{\partial^2 L}{\partial \dot{q}^j \partial \dot{\phi}^{b]}} + \frac{\partial^2 L}{\partial q^j \partial \dot{Q}^{b]}} + \frac{\partial^2 L}{\partial \dot{q}^j \partial q^m} k^{mn} \frac{\partial^2 L}{\partial \dot{q}^n \partial \dot{Q}^{b]}}\right) = 0$$

$\Longrightarrow$ n secondary constraints: $\{\Xi_a, H\} = 0$.

$\Longrightarrow$ $(2m + 6n - 2n - n - n)/2 = m + n$ DOF.

✓ $m + n$ DOF, H is bounded.

✓ EL eq is reducible into 2nd order system.

Healthy scalar-tensor theory

cubic DHOST / EST

Achour, Crisostomi, Koyama, Langlois, Noui, Tasinato, 1608.08135

- At $(\nabla\nabla\phi)^3, (\partial\partial g)^2\nabla\nabla\phi$ order

$$S = \int d^4x\sqrt{-g}\left[F_3(\phi,X)R + \sum_{i=1}^{10} B_i(\phi,X)L_i^{(3)}\right]$$

$$L_1^{(3)} = (\Box\phi)^3, L_2^{(3)} = \Box\phi\left(\phi_{;\mu\nu}\right)^2, L_3^{(3)} = \phi_{;\mu\nu}\phi^{;\nu\rho}\phi_{;\rho}^{;\mu},$$
$$L_4^{(3)} = (\Box\phi)^2\phi^{;\mu}\phi_{;\mu\nu}\phi^{;\nu}, L_5^{(3)} = \cdots.$$

- Find degenerate condition on F_3 and B_i.

Healthy scalar-tensor theory

quadratic & cubic DHOST / EST

✓ Ghost-free Lagrangian for $(\nabla\nabla\phi)^2$ and $(\nabla\nabla\phi)^3$ is constructed using the above approach with ADM decomposition.

Open questions

cf. HM, Suyama, Takahashi, 1608.00071

- DOF counting differs in the unitary gauge (nondegenerate vs degenerate).
- Can we check whether $\exists$ redefinition of fields for given higher-order theory?
- Higher powers? Third order derivatives? Multi-field?

HM, Suyama, Yamaguchi, in progress

$L(\dddot{\psi}, \ddot{\psi}, \dot{\psi}, \psi; \dot{q}, q)$

Equivalent form

$$L(\dot{Q}_2, Q_2, Q_1, \psi; \dot{q}, q) + \lambda_1(\dot{\psi} - Q_1) + \lambda_2(\dot{Q}_1 - Q_2).$$

Canonical momenta for $(q, \psi, \lambda_1, \lambda_2)$

12 ini. conds.

$\Longrightarrow$ 4 primary constraints

Need more constraints from (P_1, P_2)

Impose $\det K = 0$ with $\det L_{\dot{q}\dot{q}} \neq 0$, i.e.

$$L_{\dot{Q}_2\dot{Q}_2} - L^2_{\dot{Q}_2\dot{q}}/L_{\dot{q}\dot{q}} = 0,$$

which is equivalent to additional 1 primary constraint

$$\Phi_5 \equiv P_2 - F(p, Q_2, Q_1, \psi, q) = 0,$$

Generates 1 secondary constraint from $\dot{\Phi}_5 = 0$

$$\Phi_6 = 0.$$

✓ Removes $\psi^{(6)}$ and $\psi^{(5)}$

$$L(\dddot{\psi}, \ddot{\psi}, \dot{\psi}, \psi; \dot{q}, q)$$

Additional condition to obtain 1 secondary constraint

$$\{\Phi_6, \Phi_5\} = 0 \implies \Phi_7 = 0$$

Then $\{\Phi_7, \Phi_5\} = 0$ identically holds and generates 1 secondary constraint $\Phi_8 = 0$ from.

✓ Removes $\psi^{(4)}$ and $\psi^{(3)}$

Finally we impose

$$\{\Phi_7, \Phi_6\} \neq 0$$

to make Dirac algorithm end with Φ_8 and $(12 - 4 - 2 - 2)/2 = 2$ DOF.

✓ Hamiltonian analysis

✓ EL eqs should be reducible to 2nd order system

– explicitly checked removing up to $\psi^{(6)}, \psi^{(5)}, \psi^{(4)}$.

Summary

- Ostrogradsky ghost can be eliminated by imposing constraints, or degeneracy conditions.

- They play a crucial role for construction of healthy scalar-tensor theories, and allowed one to construct healthy theories with $(\nabla\nabla\phi)^2$ and $(\nabla\nabla\phi)^3$.

- It may be possible to construct more general healthy theories e.g. with $\nabla\nabla\nabla\phi$.

Summary

We derived the degeneracy condition for various Lagrangians:

HM, Suyama, 1411.3721

HM, Noui, Suyama, Yamaguchi, Langlois, 1603.09355

HM, Suyama, Yamaguchi, in progress

- ✓ $L(\ddot{\phi}^a, \dot{\phi}^a, \phi^a)$
- ✓ $L(\ddot{\phi}, \dot{\phi}, \phi; \dot{q}, q)$
- ✓ $L(\ddot{\phi}, \dot{\phi}, \phi; \dot{q}^i, q^i) \ \sim \ \phi + g_{\mu\nu}$
- ✓ $L(\ddot{\phi}^a, \dot{\phi}^a, \phi^a; \dot{q}^i, q^i) \ \sim \ \phi^a + g_{\mu\nu}$
- ✓ $L(\dddot{\psi}, \ddot{\psi}, \dot{\psi}, \psi; \dot{q}, q)$
- ✓ $L(\dddot{\psi}, \ddot{\psi}, \dot{\psi}, \psi; \dot{q}^i, q^i)$

… Healthy theory with 3rd order derivative!