Relation between invertible transformations and physical DOFs in scalar-tensor theories

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Based on "General invertible transformation and physical degrees of freedom" (arXiv:1702.01849)
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Introduction

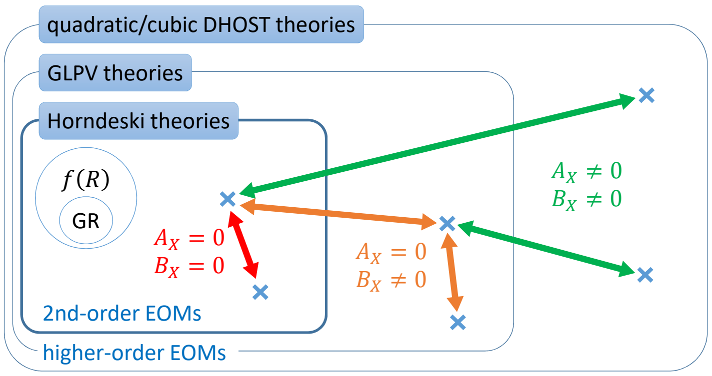
- Disformal transformation

$$\tilde{g}_{\mu\nu} = A(\phi, X) g_{\mu\nu} + B(\phi, X)\nabla_\mu\phi\nabla_\nu\phi, \qquad X \equiv -\frac{1}{2} g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$$

- Invertible (= one-to-one correspondence between $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$) if

$$A(A - XA_X + 2X^2 B_X) \neq 0$$

- Healthy scalar-tensor theories with 2+1 DOFs



- In general, the Euler-Lagrange (EL) equations in the new frame contain derivatives of order higher than in the original frame due to the derivative of ϕ in the transformation law.

Natural questions

- Why does the number of DOFs remain unchanged even with the higher-order EL equations?
- What if we perform a more generic invertible transformation?

➡ We showed that the number of DOFs is not changed by any invertible transformation that depends on field and their derivatives.

Main theorem

- General field theory in D-dimensional spacetime

$$S = \int d^D x\, L(\phi^i, \partial_\mu\phi^i, \cdots, \partial_{(n)}\phi^i), \qquad \partial_{(n)} \equiv \partial_{\mu_1}\cdots\partial_{\mu_n}$$

EL equations for ϕ^i are $(2n)$th-order differential equations:

$$\mathcal{E}_i^{(\phi)} \equiv \sum_{q=0}^{n} (-1)^q\, \partial_{(q)} \frac{\partial L[\phi]}{\partial(\partial_{(q)}\phi^i)} = 0.$$

- Transformation to a new set of fields ψ^i

$$\phi^i = f^i(\psi^j, \partial_\mu\psi^j, \cdots, \partial_{(m)}\psi^j)$$

Invertibility: $\exists g^i$ such that

$$\psi^i = g^i(\phi^j, \partial_\mu\phi^j, \cdots, \partial_{(\ell)}\phi^j)$$

- Transformed action

$$S = \int d^D x\, L'(\psi^i, \partial_\mu\psi^i, \cdots, \partial_{(n+m)}\psi^i)$$

EL equations for ψ^i are $(2n+2m)$th-order differential equations:

$$\mathcal{E}_i^{(\psi)} \equiv \sum_{p=0}^{n+m} (-1)^p\, \partial_{(p)} \frac{\partial L'[\psi]}{\partial(\partial_{(p)}\psi^i)} = 0.$$

Theorem

If the transformation is invertible, then the solution space for the old-frame EL equations $\mathcal{E}_i^{(\phi)} = 0$ and the solution space for the new-frame EL equations $\mathcal{E}_i^{(\psi)} = 0$ have a common number of DOFs.

Proof of the theorem

Step 1: <u>Linearization of the transformation law</u>

We linearize $\phi^i = f^i(\psi^j, \partial_\mu\psi^j, \cdots, \partial_{(m)}\psi^j)$ to obtain

$$\delta\phi^i = \hat{P}^i_j\,\delta\psi^j, \qquad \hat{P}^i_j \equiv \sum_{s=0}^{m} \frac{\partial f^i}{\partial(\partial_{(s)}\psi^j)}\partial_{(s)}.$$

The inverse operator matrix $\hat{Q}^i_j$ satisfies

$$\hat{P}^i_j \hat{Q}^j_k = \hat{Q}^i_j \hat{P}^j_k = \delta^i_k.$$

For an invertible transformation, $\hat{Q}^i_j$ does not contain integral operators.

Step 2: <u>Relation between the old- and new-frame EL equations</u>

Variation of the action

$$\delta S = \delta \int d^D x\, L[\phi^i] = \int d^D x\, \mathcal{E}_i^{(\phi)}\delta\phi^i$$

Substitute $\delta\phi^i = \hat{P}^i_j\delta\psi^i$:

$$\delta S = \int d^D x\, \mathcal{E}_i^{(\phi)}(\hat{P}^i_j\delta\psi^j).$$

Then, integration by parts yields

$$\delta S = \int d^D x\, \left(\hat{P}^{\dagger i}{}_j \mathcal{E}_i^{(\phi)}\right)\delta\psi^j,$$

where $\hat{P}^{\dagger i}{}_j$ is the adjoint operator of $\hat{P}^i_j$. Therefore,

$$\hat{P}^{\dagger i}{}_j \mathcal{E}_i^{(\phi)} = \mathcal{E}_j^{(\psi)},$$

meaning that the solution space for ϕ^i is mapped to a subspace of the solution space for ψ^i:

$$(\text{the solution space for } \phi^i) \subset (\text{the solution space for } \psi^i).$$

Step 3: <u>Opposite direction</u>

Since both $\hat{P}^i_j$ and $\hat{Q}^i_j$ are derivative-operator-valued matrices,

$$\hat{P}^{\dagger i}{}_j \hat{Q}^{\dagger j}{}_k = \hat{Q}^{\dagger i}{}_j \hat{P}^{\dagger j}{}_k = \delta^i_k.$$

Thus, the inverse matrix of $\hat{P}^{\dagger i}{}_j$ is given by $\hat{Q}^{\dagger i}{}_j$, which allows us to write

$$\mathcal{E}_i^{(\phi)} = \hat{Q}^{\dagger j}{}_i \mathcal{E}_j^{(\psi)}.$$

Then we have

$$(\text{the solution space for } \phi^i) \supset (\text{the solution space for } \psi^i).$$

Therefore, the two solution spaces have the same number of DOFs. This completes the proof of the Theorem. ■

Outlook – noninvertible transformation

- A noninvertible transformation could change (either increase or decrease) the number of DOFs.
  e.g. <u>mimetic gravity model</u>
  - obtained by a noninvertible disformal transformation from the Einstein-Hilbert action:

2 DOFs $\qquad S_{\rm EH} = \dfrac{M_{\rm Pl}^2}{2}\displaystyle\int d^4x\sqrt{-\tilde{g}}\tilde{R}$

+1 DOF $\qquad \tilde{g}_{\mu\nu} = X g_{\mu\nu}$: noninvertible

3 DOFs $\qquad S_{\rm MG} = \dfrac{M_{\rm Pl}^2}{2}\displaystyle\int d^4x\sqrt{-g}\left(XR + \frac{3}{2X}\nabla^\mu X\nabla_\mu X\right)$

- How is the number of DOFs changed by a noninvertible transformation?
- If some DOFs are added by a noninvertible transformation, are the additional DOFs healthy or ghost?
- If a theory has ghost DOFs, is it possible to kill them by some noninvertible transformation?