# Causal structure & Shock formation in The most general scalar-tensor theories

最も一般的なスカラー・テンソル理論における因果構造と衝撃波形成

Norihiro Tanahashi [Osaka U]

with Seiju Ohashi [KEK]

in prep.

# Causal structure and Shock formation in The most general scalar-tensor theories

Norihiro Tanahashi (Osaka U) w/ Seiju Ohashi (KEK), in prep.

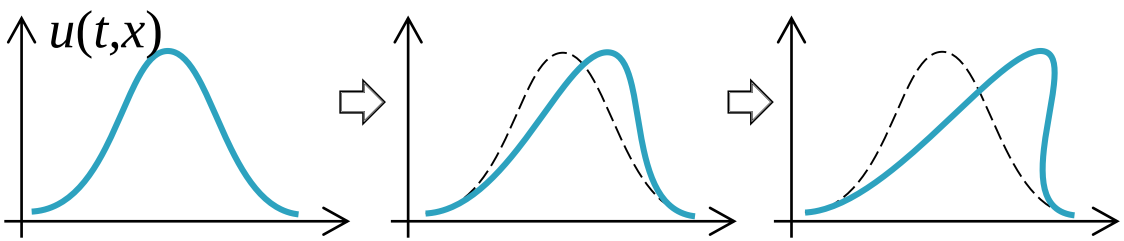
## Scalar & gravitational wave in modified gravity theories

- General relativity: GW propagates at the light speed $c$
- Modified gravity : (GW sound speed) $\neq c$ , environment dependent

**Q1**: Does it affect the definition of black hole horizon?

## Waveform distortion & Shock formation

Ex.) Burgers' eq. $$\partial_t u + u\,\partial_x u = 0$$




**Q2**: Does this occurs for scalar & gravitational waves in modified gravity?

- **Study these phenomena in (bi-)Horndeski theory.**

# Causal structure and Shock formation in The most general scalar-tensor theories

Norihiro Tanahashi (Osaka U) w/ Seiju Ohashi (KEK), in prep.

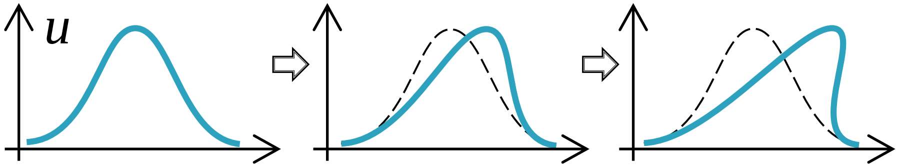
## Q1: Black hole horizon in bi-Horndeski theory

- GR: Horizon of the metric is the event horizon for any waves
- Bi-Horndeski: Metric horizon may not be an event horizon

- We studied stationary black holes, and found that a metric horizon becomes an event horizon only when the scalar fields are constant in the stationary direction.

## Q2: Shock formation for scalar & gravitational waves in Horndeski theory

$$\partial_t u + u\,\partial_x u = 0 \quad \Rightarrow$$




- We developed a formalism to study shock formation in Horndeski theory.
- We found that, in Horndeski theory,
  - shock formation occurs for scalar field wave
  - shock formation does not occur for gravitational wave

# Contents

# (Bi-)Horndeski theory

- Horndeski theory [Horndeski 1974]
  - One scalar field $\phi$ & gravity in 4-dim. spacetime
  - The most general covariant theory with 2nd-order EoM

$$\mathcal{L} = K(\phi, X) - G_3(\phi, X)\Box\phi + G_4(\phi, X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right]$$
$$+ G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{G_{5X}}{6}\left[(\Box\phi)^3 - 3\Box\phi(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right]$$
$$\left(X \equiv -\frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi\right)$$

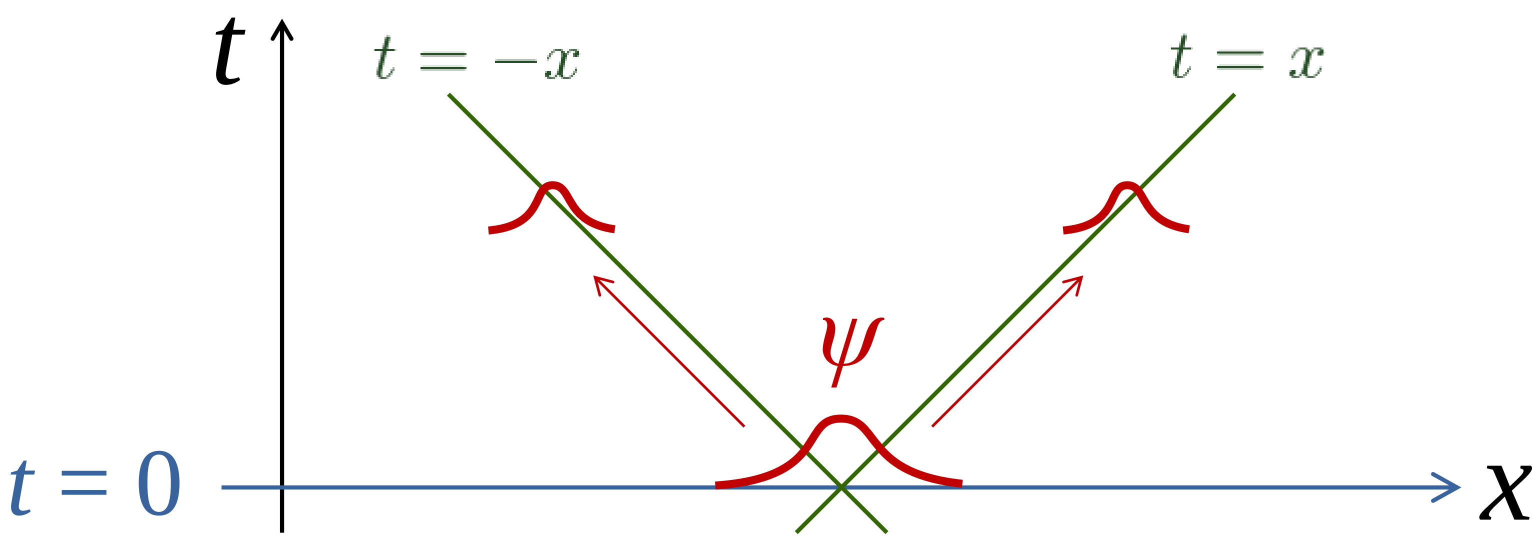
- Bi-Horndeski theory [Ohashi, NT, Kobayashi, Yamaguchi 2015]
  - Two scalar field $\phi_1, \phi_2$ & gravity in 4-dim. spacetime
  - The most general covariant theory with 2nd-order EoM
  - EoM has been constructed, not Lagrangian yet

# Wave propagation surface = Characteristic surface

- Massless scalar in flat space

$$0 = g^{\mu\nu}\nabla_\mu\nabla_\nu\psi = \left(-\partial_t^2 + \partial_x^2\right)\psi$$

$$\Rightarrow \quad \psi = f_1(t-x) + f_2(t+x)$$



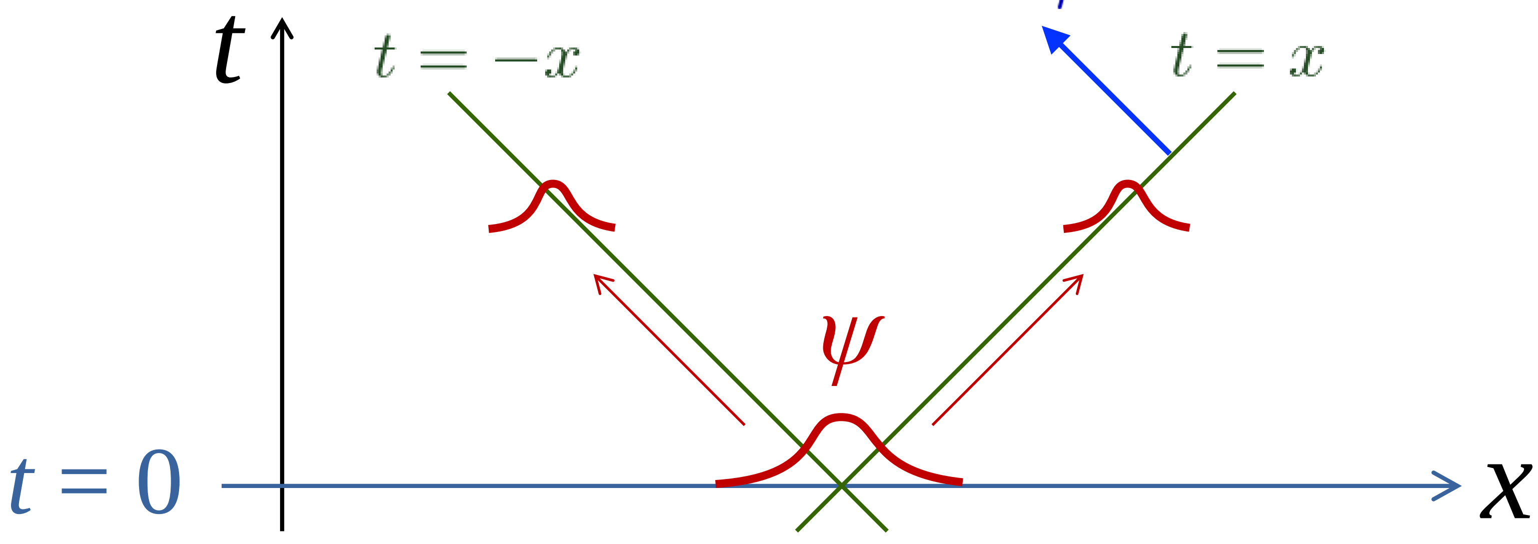
# Wave propagation surface = Characteristic surface

- Massless scalar in flat space $\left(\psi = e^{in_\mu x^\mu}\tilde{\psi}_k\right)$

$$0 = g^{\mu\nu}\nabla_\mu\nabla_\nu\psi = -g^{\mu\nu}n_\mu n_\nu e^{in_\mu x^\mu}\tilde{\psi}_k$$

$$\Rightarrow \quad \psi = f_1(t-x) + f_2(t+x) \;\Leftrightarrow\; g^{\mu\nu}n_\mu n_\nu = 0$$

$n_\mu$ : normal vector



# Wave propagation surface = Characteristic surface

- EoM $E$ of dynamical variable $v$ :

$$0 = E\left(v, \partial v, \partial^2 v\right) = \frac{\partial E}{\partial\left(\partial_t^2 v\right)} \partial_t^2 v + \cdots$$
$$= \frac{\partial E}{\partial\left(\partial_\mu \partial_\nu v\right)} n_\mu n_\nu v + \cdots$$

- Characteristic equation

$$\frac{\partial E_a}{\partial\left(\partial_\mu \partial_\nu v_b\right)} n_\mu n_\nu = 0$$

- A surface whose normal $n_\mu$ satisfies this equation is a Characteristic surface.

Physically, a characteristic surface is a wave propagation surface.

# Wave propagation surface = Characteristic surface

- EoM of scalar-tensor theory:

$$g_{ab}\ \text{EoM}:\quad 0 = E_{ab} = \frac{\partial E_{ab}}{\partial\left(\partial_t^2 g_{cd}\right)}\partial_t^2 g_{cd} + \frac{\partial E_{ab}}{\partial\left(\partial_t^2 \phi_J\right)}\partial_t^2 \phi_J + \cdots$$

$$\phi_I\ \text{EoM}:\quad 0 = E_I = \frac{\partial E_I}{\partial\left(\partial_t^2 g_{cd}\right)}\partial_t^2 g_{cd} + \frac{\partial E_I}{\partial\left(\partial_t^2 \phi_J\right)}\partial_t^2 \phi_J + \cdots$$

- Characteristic equation

$$\det P = 0 \quad \text{where} \quad P \equiv n_\mu n_\nu \begin{pmatrix} \frac{\partial E_{ab}}{\partial(\partial_\mu \partial_\nu g_{cd})} & \frac{\partial E_{ab}}{\partial(\partial_\mu \partial_\nu \phi_J)} \\ \frac{\partial E_I}{\partial(\partial_\mu \partial_\nu g_{cd})} & \frac{\partial E_I}{\partial(\partial_\mu \partial_\nu \phi_J)} \end{pmatrix}$$

- A surface whose normal $n_\mu$ satisfies $\det P = 0$ is a Characteristic surface

- $\det P = 0 \quad \Leftrightarrow \quad P$ has eigenvectors $v$ with eigenvalues $= 0$
  $= \quad$ Propagating modes ( $P \cdot v = 0$ )

# Contents

# Causal structure and Shock formation in The most general scalar-tensor theories

Norihiro Tanahashi (Osaka U) w/ Seiju Ohashi (KEK), in progress

- Scalar & gravitational wave in modified gravity theories
  - General relativity: GW propagates at the light speed $c$
  - Modified gravity : (GW sound speed) $\neq c$ , environment dependent

**Q1**: Does it affect the definition of black hole horizon?

- Waveform distortion & Shock formation

  Ex.) Burgers' eq. $\partial_t u + u\,\partial_x u = 0$

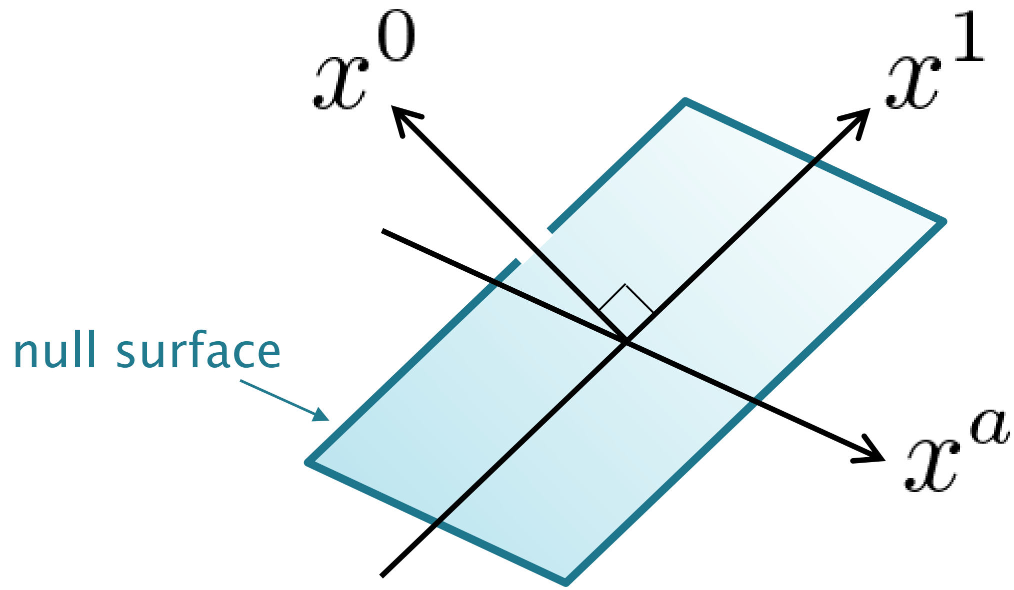



**Q2**: Does this occurs for scalar & gravitational waves in modified gravity?

- **Study these phenomena in (bi-)Horndeski theory.**

# Causal structure in Bi-Horndeski theory

- Is a null surface a wave propagation surface?




$x^0$ = const. surface is null

$$\Updownarrow$$

$$g^{00} = g^{0a} = 0$$
$$g_{11} = g_{1a} = 0$$

Ex.) General relativity + a canonical scalar field

$$P \cdot v = \begin{pmatrix} 0 & 0 & g^{ab}\left(g^{01}\right)^2 & 0 \\ 0 & -g^{cb}\left(g^{10}\right)^2 & 0 & 0 \\ g^{cd}\left(g^{01}\right)^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} g_{11} \\ g_{1b} \\ g_{ab} \\ \phi \end{pmatrix} = 0 \Longrightarrow$$

| | |
|---|---|
| GW modes | $\times 2$ |
| Scalar mode | $\times 1$ |

propagates at light speed

# Causal structure in Bi-Horndeski theory

- Is a null surface a wave propagation surface?
  - ✓ In Bi-Horndeski theory, a null surface is NOT characteristic in general.
  - ◆ How about a BH horizon?
  - $\Leftrightarrow$ Does a Killing horizon become a characteristic surface?

$$\partial_{x^1} g_{ij} = \partial_{x^1}^2 g_{ij} = \partial_{x^1}\partial_{x^k} g_{ij} = 0$$

$$P \cdot v = \begin{pmatrix} 0 & 0 & \mathcal{A}_{11,ab} & \mathcal{B}_{11}^I \\ 0 & 2\mathcal{A}_{1c,1b} & \mathcal{A}_{1c,ab} & \mathcal{B}_{1c}^I \\ \mathcal{A}_{cd,11} & 2\mathcal{A}_{cd,1b} & \mathcal{A}_{cd,ab} & \mathcal{B}_{cd}^I \\ \mathcal{C}_{11}^J & 2\mathcal{C}_{1c}^J & \mathcal{C}_{cd}^J & \mathcal{D}^{JI} \end{pmatrix} \begin{pmatrix} g_{11} \\ g_{1b} \\ g_{ab} \\ \phi_I \end{pmatrix} = 0 \Longrightarrow$$

No propagation modes

$\therefore$ K.H. is NOT characteristic in general.

# Causal structure in Bi-Horndeski theory

- Is a null surface a wave propagation surface?

✓ In Bi-Horndeski theory, a null surface is NOT characteristic in general.

◆ How about a BH horizon?

⇔ Does a Killing horizon become a characteristic surface?

$$\partial_{x^1} g_{ij} = \partial_{x^1}^2 g_{ij} = \partial_{x^1}\partial_{x^k} g_{ij} = 0$$

◆ If also $\partial_{x^1}\phi_I = \partial_{x^1}^2\phi_I = \partial_{x^1}\partial_{x^k}\phi_I = 0$ are satisfied,

$$P \cdot v = \begin{pmatrix} 0 & 0 & \mathcal{A}_{11,ab} & \mathcal{B}^I_{11} \\ 0 & 2\mathcal{A}_{1c,1b} & 0 & 0 \\ \mathcal{A}_{cd,11} & 0 & 0 & 0 \\ \mathcal{C}^J_{11} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} g_{11} \\ g_{1b} \\ g_{ab} \\ \phi_I \end{pmatrix} = 0 \implies$$

| | |
|---|---|
| GW modes | × 2 |
| Scalar mode | × 2 |

⇒ Killing horizon becomes an event horizon for GW & scalar wave.

# Causal structure and Shock formation in The most general scalar-tensor theories

Norihiro Tanahashi (Osaka U) w/ Seiju Ohashi (KEK), in progress

## Q1: Black hole horizon in bi-Horndeski theory

- GR: Horizon of the metric is the event horizon for any waves
- Bi-Horndeski: Metric horizon may not be an event horizon

- We studied stationary black holes, and found that a metric horizon becomes an event horizon only when the scalar fields are constant in the stationary direction.

## Q2: Shock formation for scalar & gravitational waves in Horndeski theory

$$\partial_t u + u\,\partial_x u = 0 \quad \Rightarrow$$



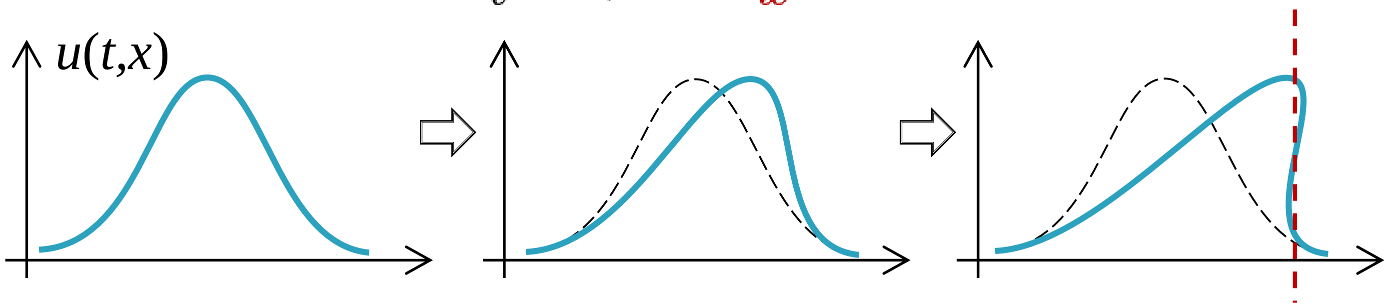

- We developed a formalism to study shock formation in Horndeski theory.
- We found that, in Horndeski theory,
  - shock formation occurs for scalar field wave
  - shock formation does not occur for gravitational wave

# Contents

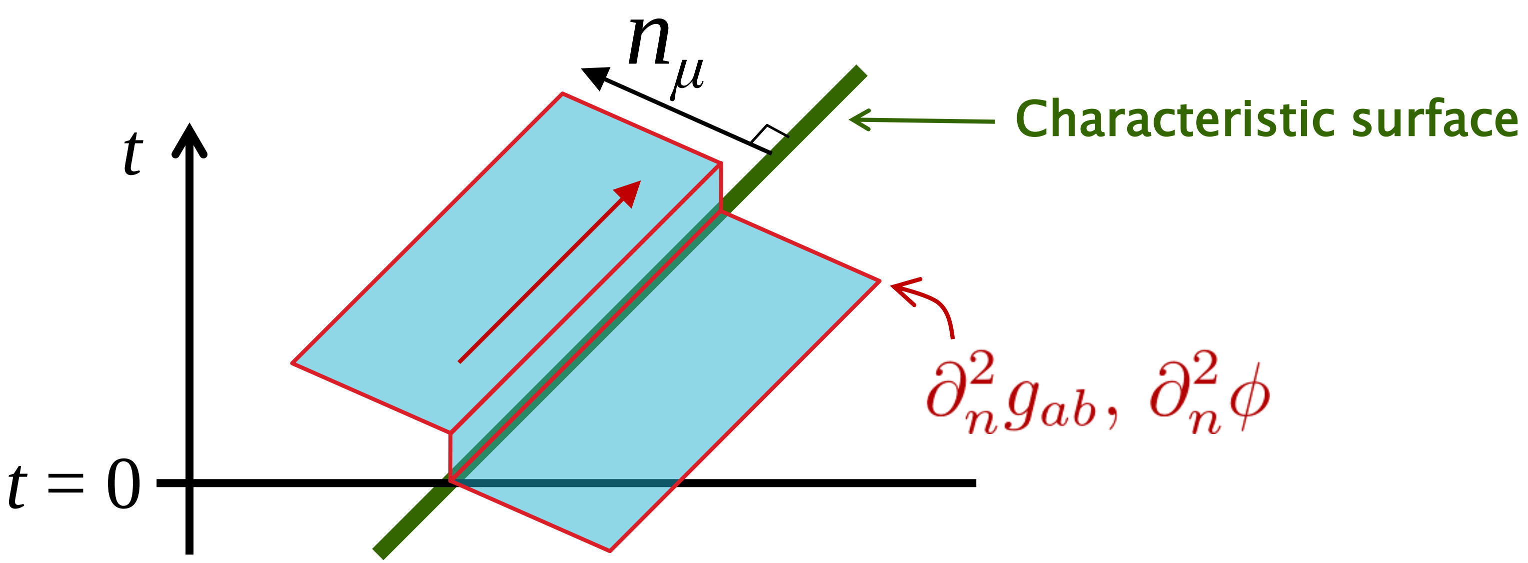
# Shock formation in Horndeski theory

- Shock formation = Divergence in gradient of waveform

Ex.) Burgers' eq. $\partial_t u + u\,\partial_x u = 0$




- For simplicity, we look at wave with discontinuity in second derivative:

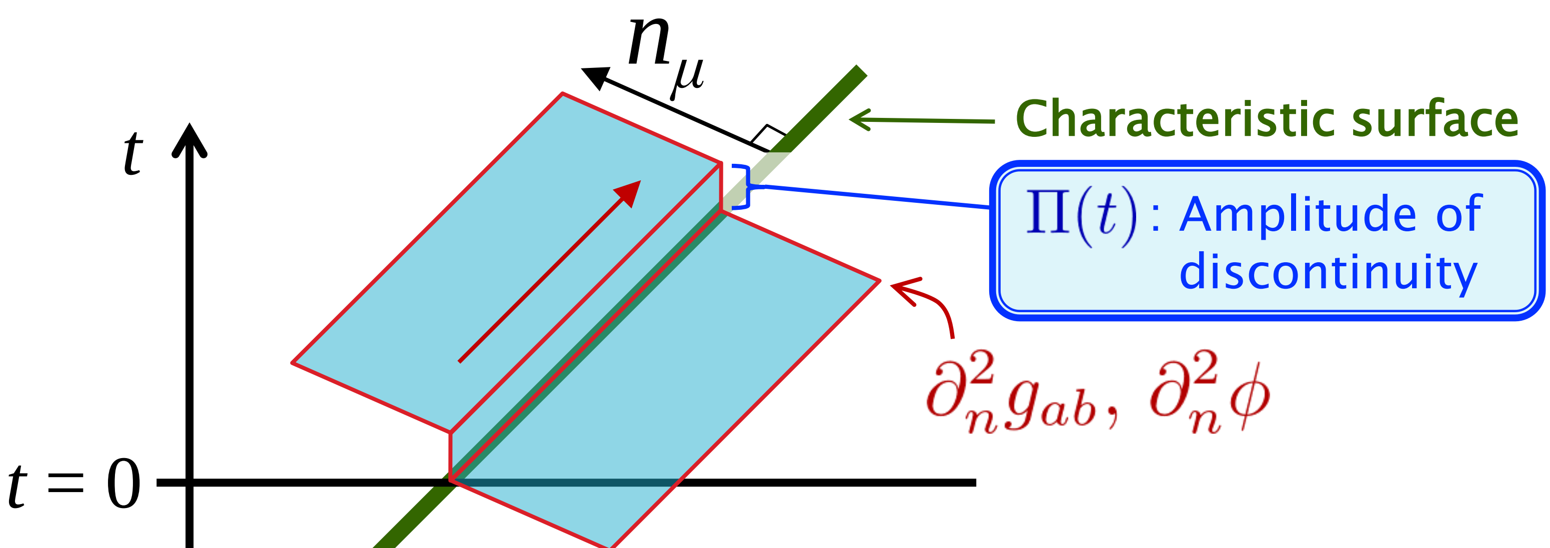


# Shock formation in Horndeski theory

- Discontinuity ≈ High–frequency wave
- Look at amplitude of discontinuity $\Pi(t) = \left[\partial_n^2 g_{ab}\right]\,,\ \left[\partial_n^2 \phi\right]$
  - Evolution equation of $\Pi(t)$:

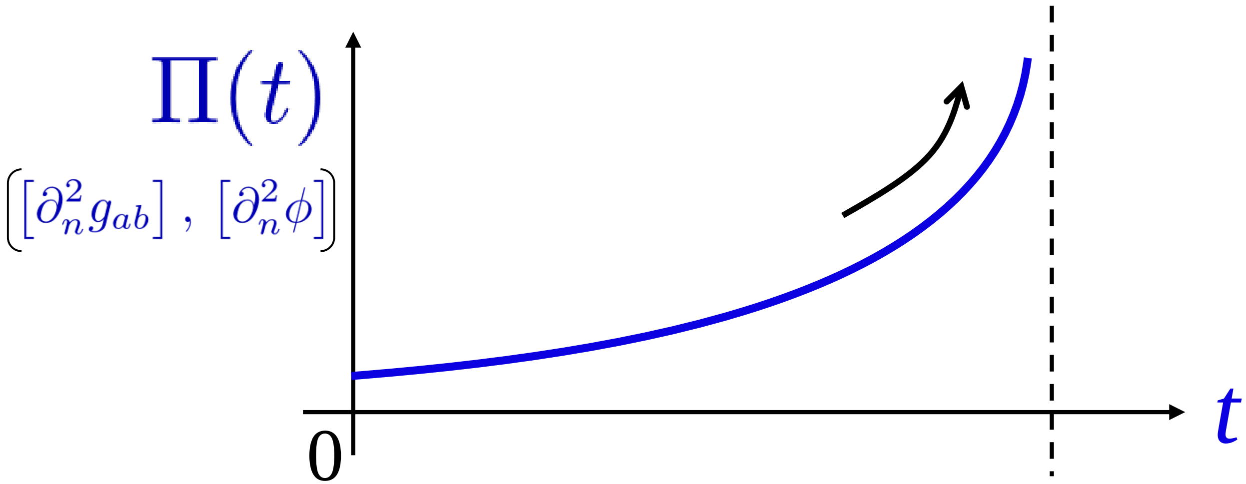
$$\dot{\Pi} + M\,\Pi + N\,\Pi^2 = 0$$

- For simplicity, we look at wave with discontinuity in second derivative:

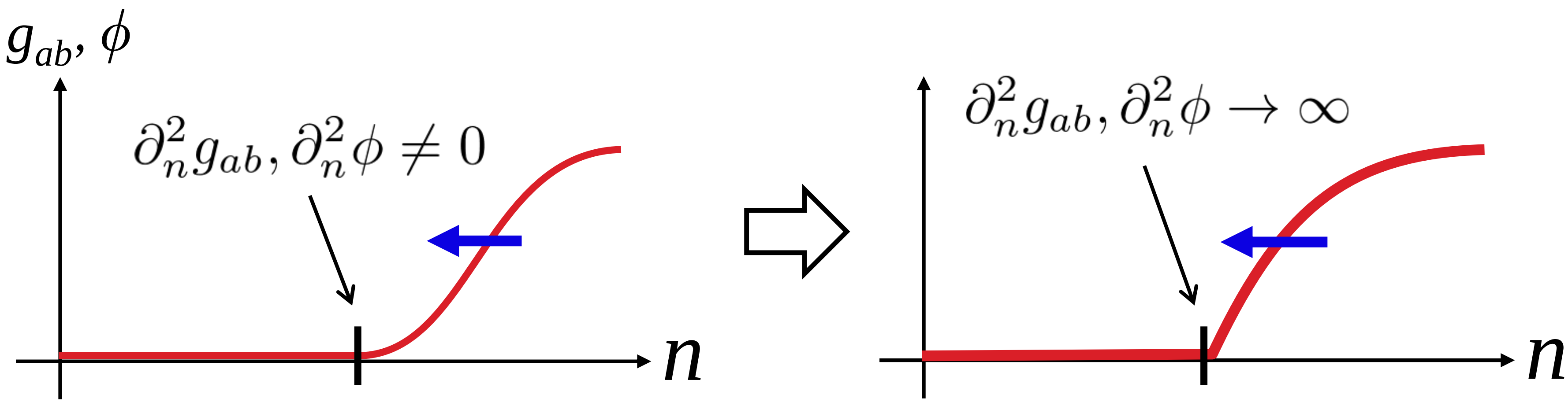


- Amplitude of discontinuity: $\Pi(t) = \left[\partial_n^2 g_{ab}\right],\ \left[\partial_n^2 \phi\right]$

- Our "shock formation" = 2nd derivative $\Pi(t) \to \infty$ at finite $t$




- In terms of $g_{ab}$ and $\phi$, it corresponds to "sharpening":



1. EoM: $\begin{cases} E_{ab} = \dfrac{\partial E_{ab}}{\partial(\partial_n^2 g_{cd})}\partial_n^2 g_{cd} + \dfrac{\partial E_{ab}}{\partial(\partial_n^2 \phi_J)}\partial_n^2 \phi_J + \cdots = 0 \\ E_I = \dfrac{\partial E_I}{\partial(\partial_n^2 g_{cd})}\partial_n^2 g_{cd} + \dfrac{\partial E_I}{\partial(\partial_n^2 \phi_J)}\partial_n^2 \phi_J + \cdots = 0 \end{cases}$

$$P \equiv \begin{pmatrix} \dfrac{\partial E_{ab}}{\partial g_{cd,nn}} & \dfrac{\partial E_{ab}}{\partial \phi_{J,n}} \\ \dfrac{\partial E_I}{\partial g_{cd,nn}} & \dfrac{\partial E_I}{\partial \phi_{J,n}} \end{pmatrix}$$

Write them collectively as

$$E_a = P_a{}^b\, \partial_n^2 v_b + \cdots = 0$$

2. Take discontinuous part

$$[E_a] = P_a{}^b\, [\partial_n^2 v_b] = 0 \quad \Rightarrow \quad [\partial_n^2 v_b] = \Pi(x^i) r_b$$

(discontinuous part)

For $r_b$ s.t. $P \cdot r = 0$

3. Transport equation of amplitude $\Pi(x^i)$

$$[\,\partial_n E_a\,] = 0 \quad \Rightarrow \quad \dot{\Pi} + M\,\Pi + N\,\Pi^2 = 0$$

where

$$N = \frac{\partial P^{ab}}{\partial\,(\partial_n v_c)} r_a r_b r_c$$

◆ What happens when $N \neq 0$ ?

$$\dot{\Pi} + M\,\Pi + N\,\Pi^2 = 0$$

$$\Phi(t) = \int_0^t M(t')dt'$$

$$\Rightarrow \qquad \Pi(t) = \frac{\Pi(0)e^{-\Phi(t)}}{1 + \Pi(0)\int_0^t N(t')e^{-\Phi(t')}dt'}$$

- GR : $N = 0$ $\Rightarrow$ $\Pi(s)$ stays finite
- Modified grav.: $N \neq 0$ $\Rightarrow$ Denominator may vanish due to $N \neq 0$
  $\Rightarrow$ Amplitude $\Pi(t)$ diverges
  $\Rightarrow$ Shock formation

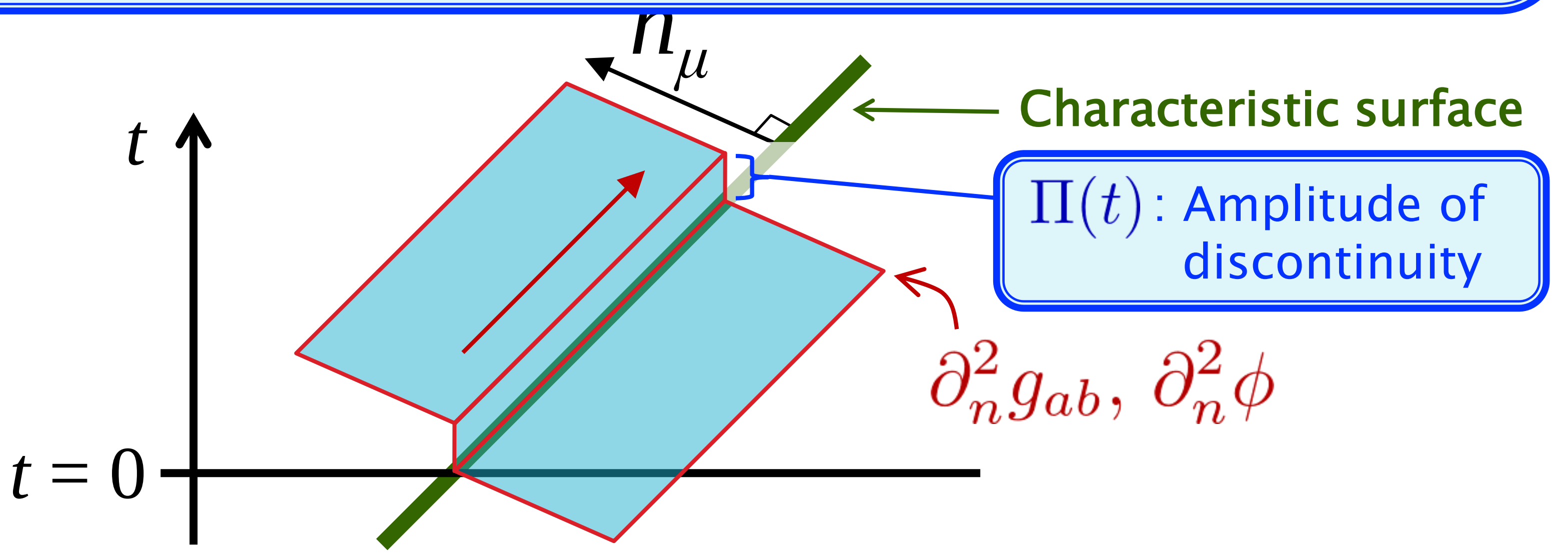
# Shock formation in Horndeski theory

- Discontinuity propagates on the characteristic surface.
- Look at amplitude of discontinuity $\Pi(t) = \left[\partial_n^2 g_{ab}\right]\,,\ \left[\partial_n^2 \phi\right]$
  - Evolution equation of $\Pi(t)$:

$$\dot{\Pi} + M\,\Pi + N\,\Pi^2 = 0$$

➢ $N = 0$ : $\Pi(t)$ remains finite in time evolution. ⇐ GR & canonical scalar

➢ $N \neq 0$ : $\Pi(t)$ diverges within finite time ⇔ Shock formation



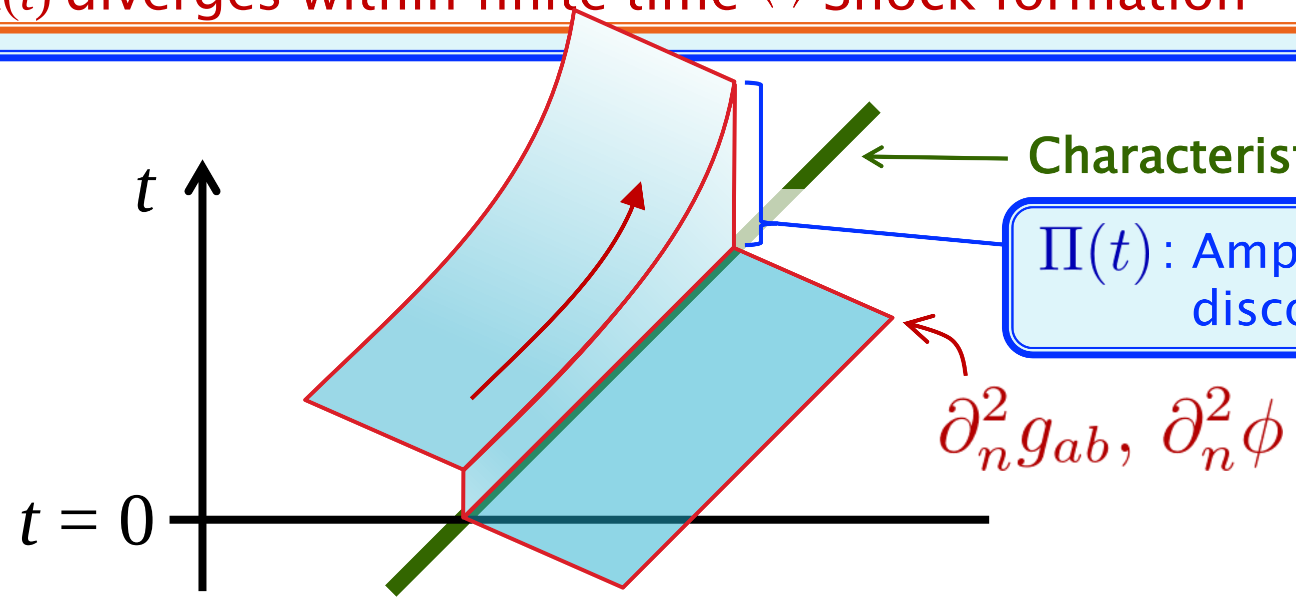
# Shock formation in Horndeski theory

- Discontinuity propagates on the characteristic surface.
- Look at amplitude of discontinuity $\Pi(t) = [\partial_n^2 g_{ab}]\ ,\ [\partial_n^2 \phi]$
  - Evolution equation of $\Pi(t)$:

$$\dot{\Pi} + M\,\Pi + N\,\Pi^2 = 0$$

Non-GR
Non-canonical scalar

- $N = 0$ : $\Pi(t)$ remains finite in time evolution.
- $N \neq 0$ : $\Pi(t)$ diverges within finite time ⇔ Shock formation

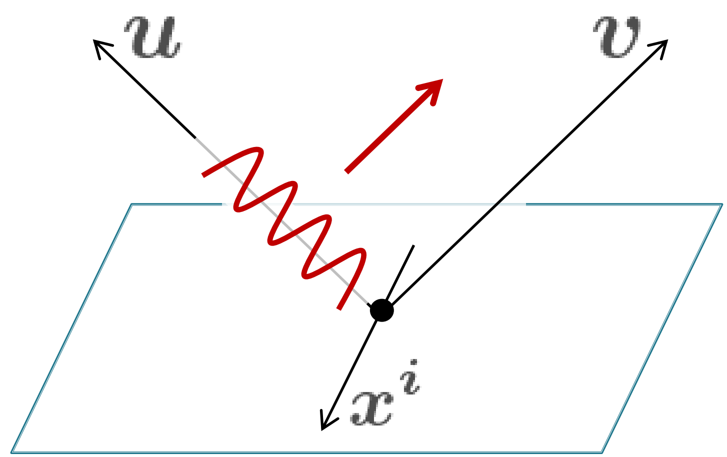


# Shock formation on Plane wave background

- Example: Perturbations on Plane wave solution

$$ds^2 = a_{ij}x^i x^j du^2 + 2du dv + \delta_{ij} dx^i dx^j, \quad \phi = \phi(u)$$

[Babichev 2012]




- ✓ Scalar & Gravitational perturbations propagate at different speeds
- ✓ Scalar : $N \neq 0$ → Shock formation
- ✓ GW : $N = 0$ → No shock formation

[Babichev 2016]
[Mukohyama, Namba, Watanabe 2016]
[de Rham, Motohashi 2016]

# Shock formation on Plane wave background

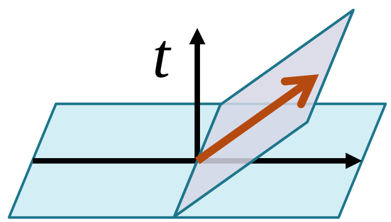
- Example: 2D maximally-symmetric dynamical spacetime

$$ds^2 = f(\tau,\chi)\left(-d\tau^2 + d\chi^2\right) + \rho(\tau,\chi)d\Omega^2, \quad \phi = \phi(\tau,\chi)$$

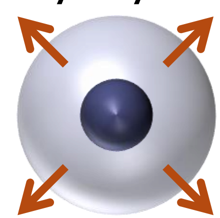
2-dim. flat or $S^2$ or $H^2$

and consider waves propagating in $(\tau, \chi)$ direction.

- Plane wave in FRW universe




- Spherical wave around spherically-sym. star/BH



✓ Scalar & Gravitational perturbations propagate at different speeds

✓ Scalar : $N \neq 0$ → Shock formation

[Babichev 2016]
[Mukohyama, Namba, Watanabe 2016]
[de Rham, Motohashi 2016]

✓ GW : $N = 0$ → No shock formation

# Summary

- Causal structure & Shock formation in (Bi-)Horndeski theory

  - Result 1: BH horizon in bi-Horndeski theory

    In Bi-Horndeski theory, a Killing horizon becomes an event horizon for $\Phi$ and GW if $\Phi$ is constant in the spacetime symmetry direction.

  - Result 2: Shock formation in Horndeski theory

    In shift-symmetric Horndeski theory, and for fluctuations on plane wave and 2D maximally symmetric background,

    - Shock formation occurs for scalar field wave
    - Shock formation does not occur for gravitational wave

- More complicated background in Horndeski
- Bi-Horndeski theory

Shock formation occurs even for gravitational wave?