Stability analysis of BHs by the S-deformation method

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Introduction

Linear perturbations of BHs to study

$$\begin{cases} \text{gravitational wave} \\ \text{slowly rotating BH} \\ \text{stability} \quad \text{etc} \end{cases} \qquad g_{\mu\nu} = g_{\mu\nu}^{(0\mathrm{th})} + \epsilon h_{\mu\nu}$$

Linear gravitational perturbation on a highly symmetric BH usually reduces to

$$\left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} - V(x) \right] \tilde{\Phi} = 0$$

$$\left[-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}-V(x)\right]\tilde{\Phi}=0$$

$$\tilde{\Phi}(t,x)=e^{-i\omega t}\Phi(x)$$

$$\left[-\frac{d^2}{dx^2}+V\right]\Phi=\omega^2\Phi$$

unstable mode → $\omega^2 < 0$ mode

(negative energy bound state)

To prove (mode) stability, we need to show the non-existence of $\omega^2 < 0$ mode

$$\left[-\frac{d^2}{dx^2}+V\right]\Phi=\omega^2\Phi$$

$$\Longrightarrow\left[\bar{\Phi}\frac{d\Phi}{dx}\right]_{-\infty}^{\infty}+\int dx\left[\left|\frac{d\Phi}{dx}\right|^2+V|\Phi|^2\right]=\omega^2\int dx|\Phi|^2$$

$V \geq 0$ implies non-existence of $\omega^2 < 0$ mode

Sometimes, V contains negative regions

S-deformation [Kodama and Ishibashi 2003]

$$-\frac{d}{dx}\left[\bar{\Phi}\frac{d\Phi}{dx}+S|\Phi|^2\right]+\left|\frac{d\Phi}{dx}+S\Phi\right|^2+\left(V+\frac{dS}{dx}-S^2\right)|\Phi|^2=\omega^2|\Phi|^2$$

For continuous S

$$-\left[\bar{\Phi}\frac{d\Phi}{dx}+S|\Phi|^2\right]_{-\infty}^{\infty}+\int dx\left[\left|\frac{d\Phi}{dx}+S\Phi\right|^2+\left(V+\frac{dS}{dx}-S^2\right)|\Phi|^2\right]=\omega^2\int dx\,|\Phi|^2$$

We can say $\omega^2 \geq 0$ if $V+\frac{dS}{dx}-S^2 \geq 0$

In general, it is hard to find an appropriate S analytically

In that case, numerical approach (e.g. solving PDE) was used so far

Today's talk

We propose a simple method for finding an appropriate S-deformation

Also, extend this method to coupled systems

Contents

- Introduction
- S-deformation method
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Very easy method

[Kimura 2017]
[Kimura & Tanaka2018]

Just solve $V + \frac{dS}{dx} - S^2 = 0$ numerically

At this stage, the existence of regular S is just a sufficient condition for the stability.

In fact, we can (almost) always find a regular solution if the spacetime is stable

Comment

If we consider

$$\boldsymbol{V + \frac{dS}{dx} - S^2 = W (\geq 0)}$$

$$\boldsymbol{\Longleftrightarrow \quad V - W + \frac{dS}{dx} - S^2 = 0}$$

This corresponds to a deeper potential.

More difficult (or dangerous).

Solving the Eq with $\boldsymbol{W = 0}$ is the most efficient.

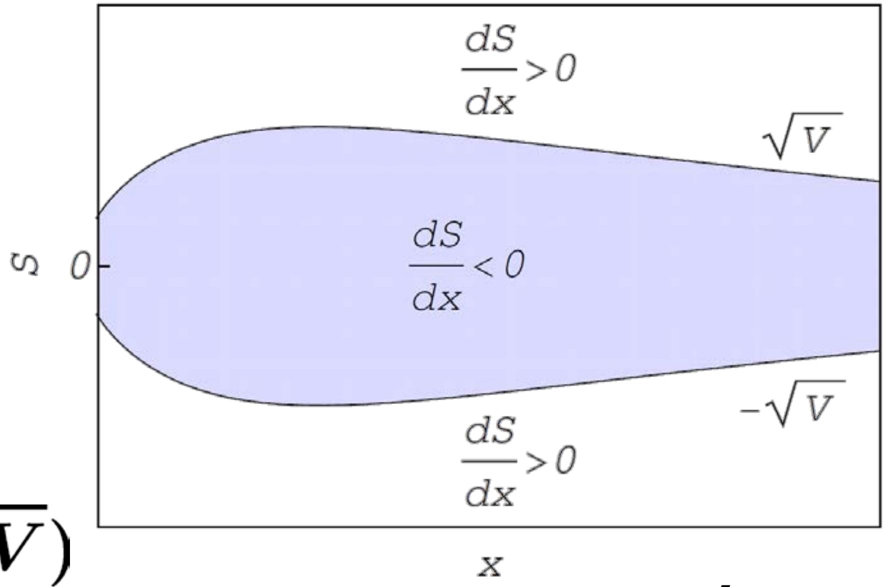
Positive potential (manifestly stable case)

Proposition. *If the potential is positive and bounded above in* $-\infty < x < \infty$*, there exists regular S*

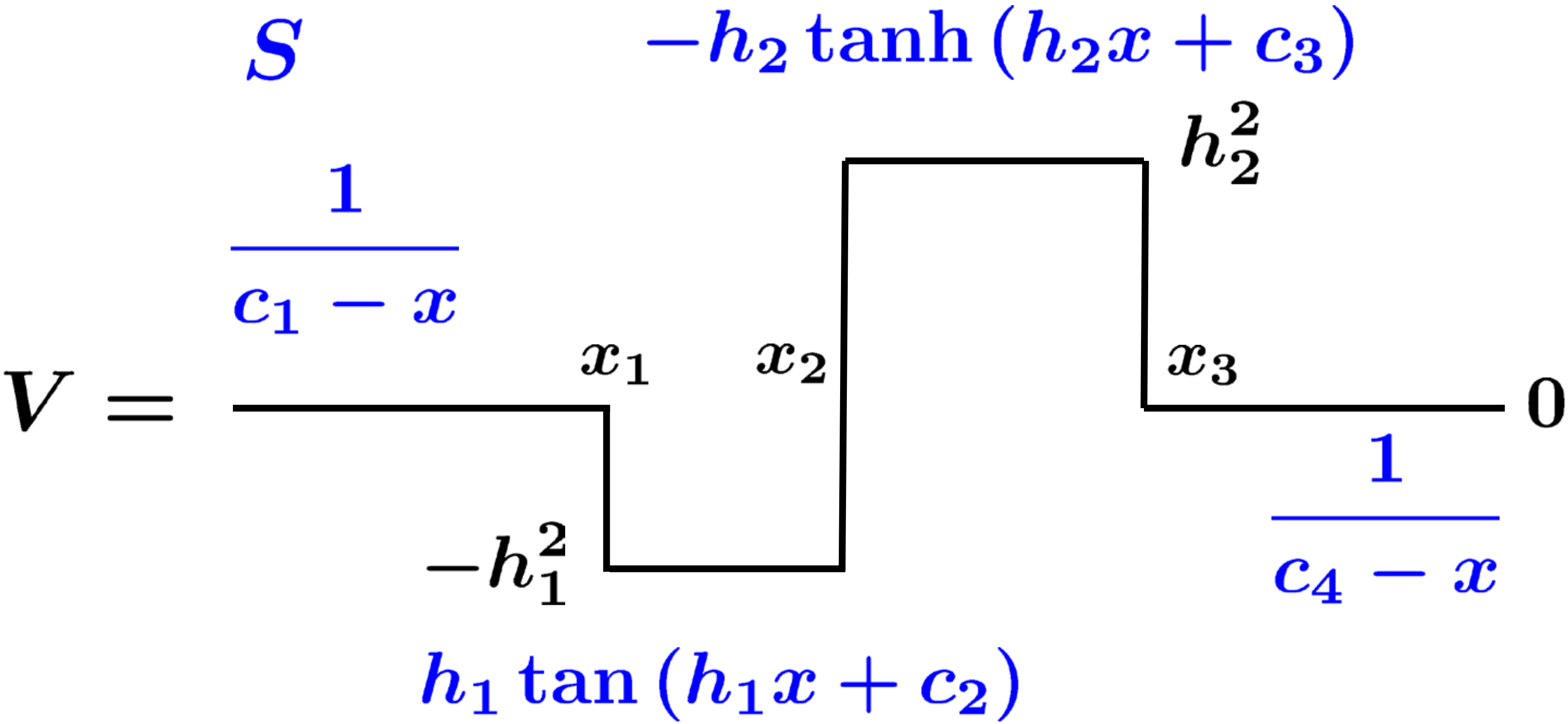
Sketch of proof:

We only need to exclude the possibility that S is divergent at some point

$$dS/dx = (S - \sqrt{V})(S + \sqrt{V})$$

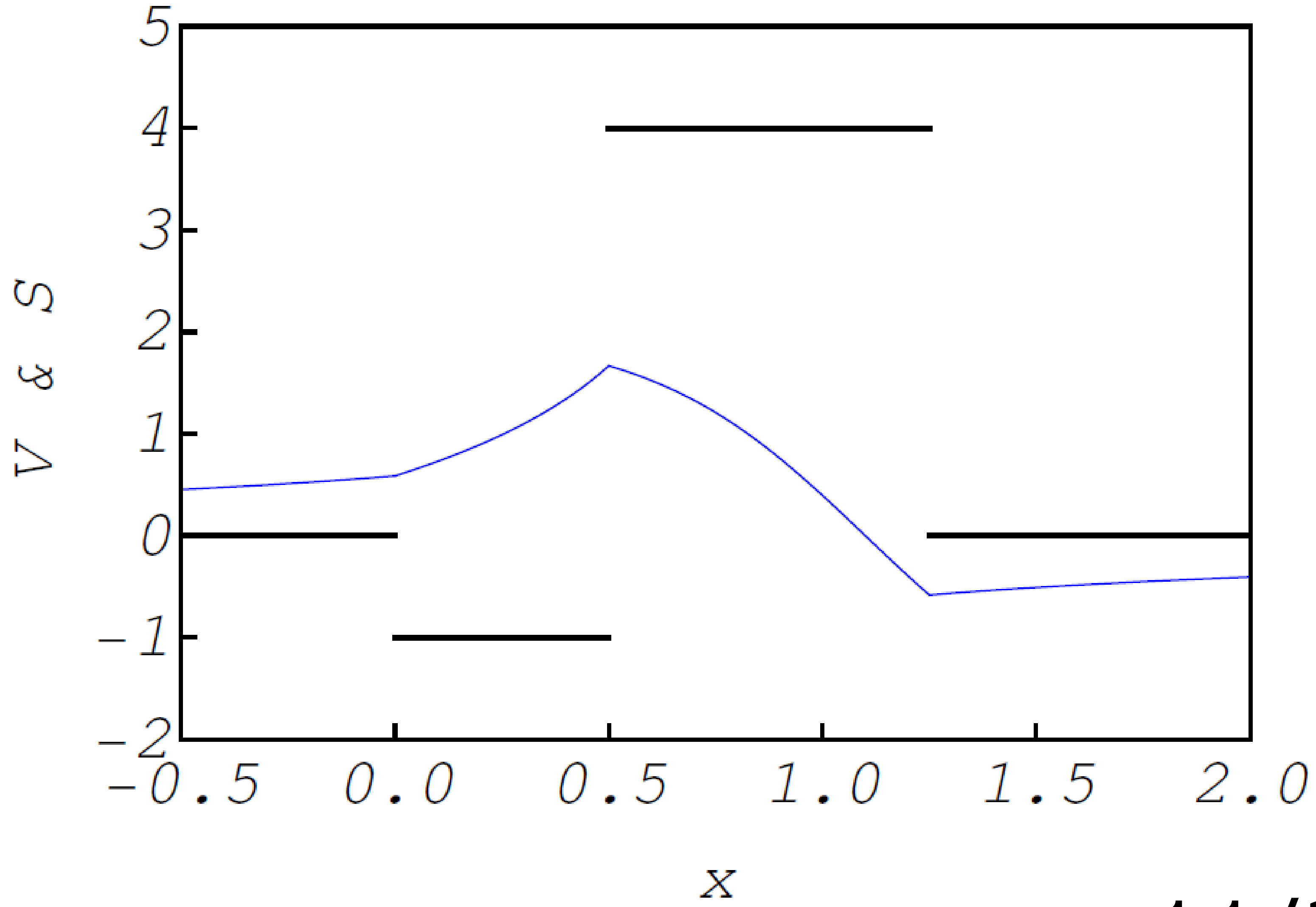


Toy model

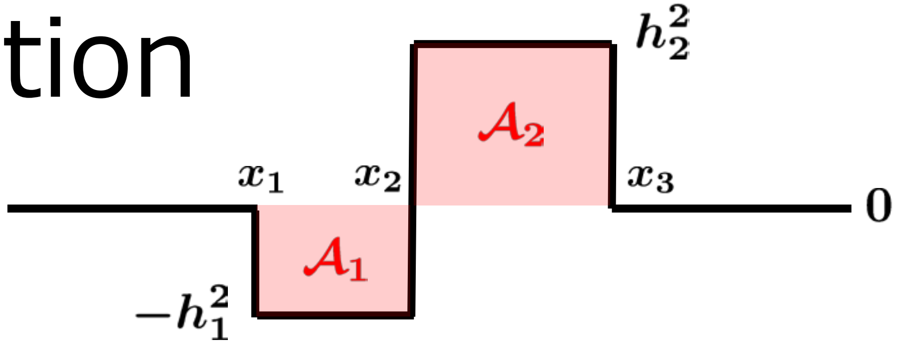


- continuity at $x = x_1, x_2, x_3$
- $S|_{x \to \pm\infty} \sim \mp$
- $S|_{x_1} > 0, S|_{x_2} > 0, S|_{x_3} < 0$

typical case



Existence condition



$$\Gamma := \frac{\mathcal{A}_2}{\mathcal{A}_1}$$

Condition for existence of regular S

$$\Gamma > \Gamma_{\mathrm{cr}}$$

$$\frac{\sqrt{x_3 - x_2}}{\sqrt{x_2 - x_1}} \tan\left(h_1(x_2 - x_1)\right) = \sqrt{\Gamma_{\mathrm{cr}}} \tanh\left(h_1 \sqrt{\Gamma_{\mathrm{cr}}} \sqrt{x_2 - x_1} \sqrt{x_3 - x_2}\right)$$

Condition for non-existence of bound state

$$\Gamma > \Gamma_{\mathrm{cr}}$$

Regular S exists if and only if spacetime is stable (i.e., no $\omega^2 < 0$ mode case)

Relation with Schrödinger Eq.

$$V + \frac{dS}{dx} - S^2 = 0$$ is the Riccati equation

$$\frac{1}{\phi}\frac{d\phi}{dx} := -S \quad \to \quad -\frac{d^2\phi}{dx^2} + V\phi = 0$$

Schrödinger Eq. with zero energy

A solution which does not have any zero corresponds to a regular S

Nodal theorem

A theorem in the Sturm−Liouville theory

$$\left[-\frac{d^2}{dx^2}+V\right]\Phi = E\Phi$$

If we solve the Schrödinger Eq. with the boundary condition $\Phi = 0, d\Phi/dx = 1$ at a sufficiently large distance, the number of zeros coincides with the number of the negative energy bound states.

There should exist a regular S for stable spacetime

Under some assumption, we can show that S constructed from a sol. with decaying boundary condition is regular if the spacetime is stable.

Proposition. *There exists a regular S-deformation for stable spacetimes*

general regular S

Φ_L : decaying at $x \to -\infty$

Φ_R : decaying at $x \to \infty$

$$\left[-\frac{d^2}{dx^2} + V\right] \Phi = 0$$

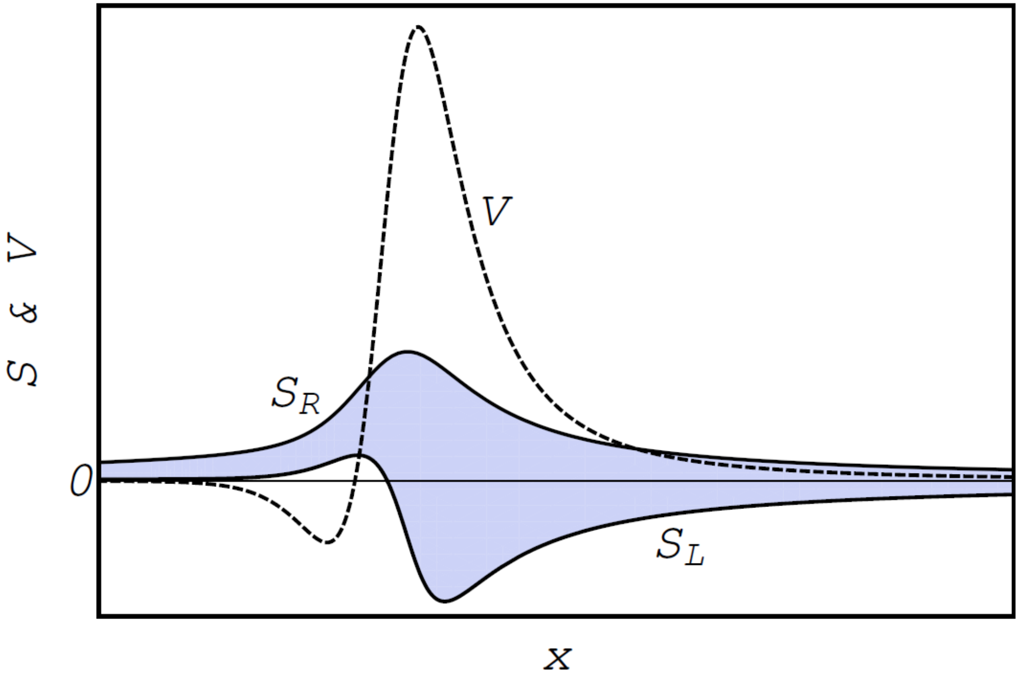
$$S_L := -\frac{1}{\Phi_L}\frac{d\Phi_L}{dx} \qquad S_R := -\frac{1}{\Phi_R}\frac{d\Phi_R}{dx}$$

General regular S is given by

$$S = -\frac{1}{\Phi}\frac{d\Phi}{dx}$$ with $$\Phi = c_L \Phi_L + c_R \Phi_R$$

$$(c_L c_R \geq 0, c_L^2 + c_R^2 \neq 0)$$

This satisfies $S_L \leq S \leq S_R$



Eq. for S

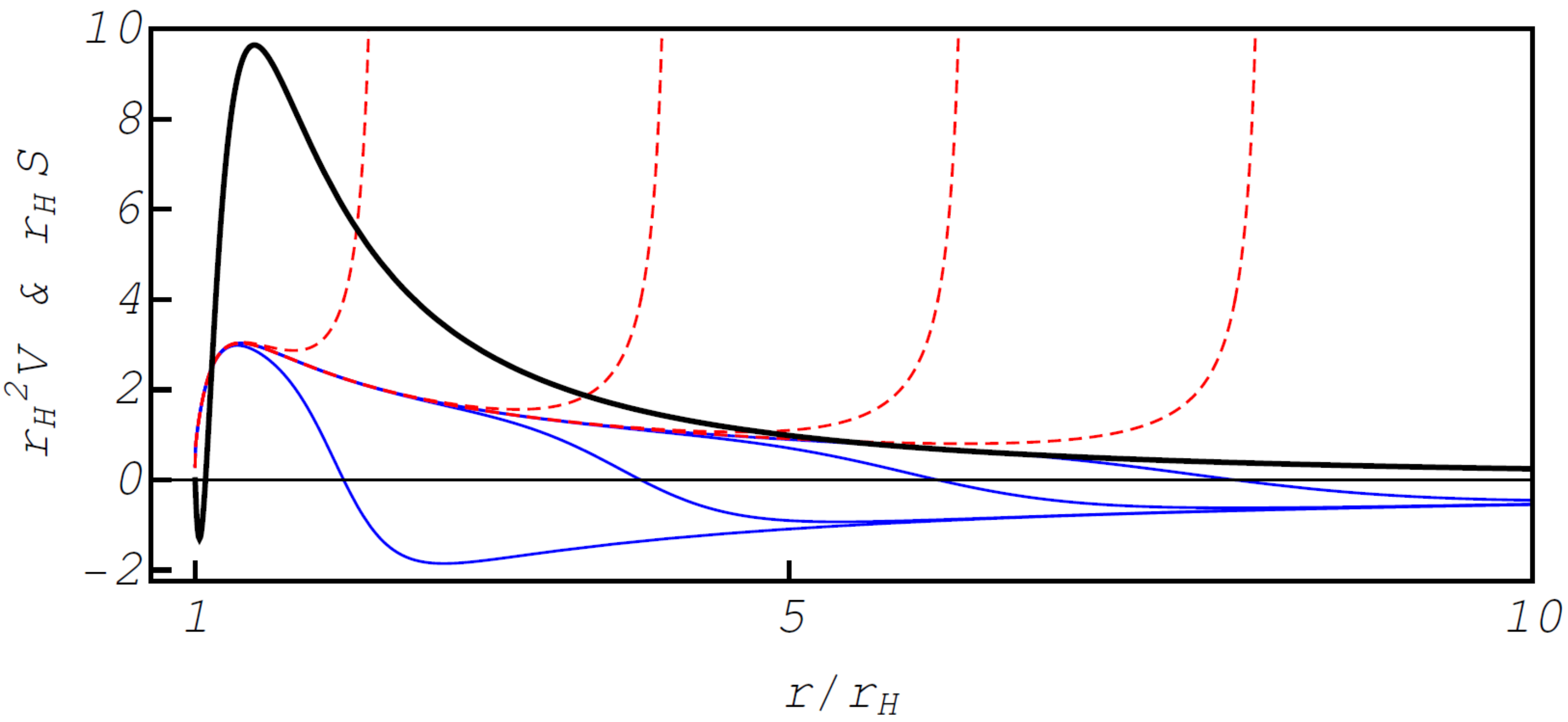
$$V + \frac{dS}{dx} - S^2 = 0$$

Shaded region corresponds to boundary conditions for regular S

If V > 0 in asymptotic region,
$S_L < 0 < S_R$ there

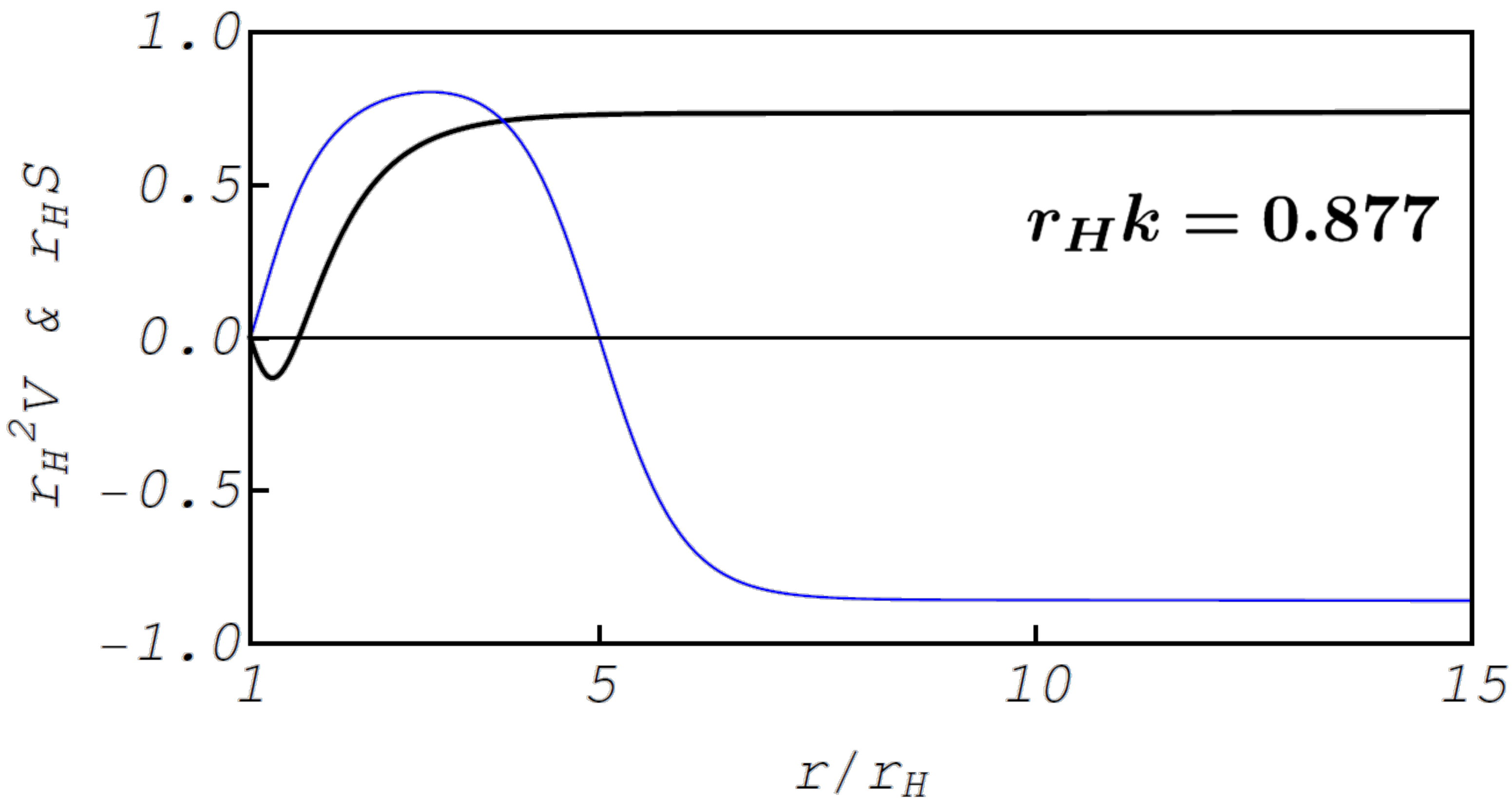
$S = 0$ at large x is an appropriate BC

10 Dim Schwarzschild BH



We can find regular S without fine-tuning

Black string



If $r_H k < r_H k_{\rm cr} \simeq 0.876$ there exists an unstable mode [Gregory and Laflamme, 1993]

Extension to multiple degrees of freedom

If there exist two or more physical degrees of freedom, and they are coupled, master Eqs sometimes become

$$\left[-\frac{d^2}{dx^2}+\boldsymbol{V}\right]\boldsymbol{\Phi}=\omega^2\boldsymbol{\Phi}$$

$\boldsymbol{V}$: $n\times n$ Hermitian matrix

$\boldsymbol{\Phi}$: n components vector

We assume the coupling term $\mathcal{L}\sim\boldsymbol{\Phi}^\dagger\boldsymbol{V}\boldsymbol{\Phi}$

For any Hermitian S,

$$-\left[\boldsymbol{\Phi}^\dagger \frac{d\boldsymbol{\Phi}}{dx} + \boldsymbol{\Phi}^\dagger \boldsymbol{S}\boldsymbol{\Phi}\right]_{-\infty}^{\infty} + \int dx \left[\left|\frac{d\boldsymbol{\Phi}}{dx} + \boldsymbol{S}\boldsymbol{\Phi}\right|^2 + \boldsymbol{\Phi}^\dagger \underbrace{\left(\boldsymbol{V} + \frac{d\boldsymbol{S}}{dx} - \boldsymbol{S}^2\right)}_{=\tilde{\boldsymbol{V}}} \boldsymbol{\Phi}\right] = \omega^2 \int dx \, |\boldsymbol{\Phi}|^2$$

If $\tilde{\boldsymbol{V}}$ is positive definite, spacetime is stable

We can still find a regular S by solving

$$\boldsymbol{V} + \frac{d\boldsymbol{S}}{dx} - \boldsymbol{S}^2 = \boldsymbol{0}$$

Comment

If V is bounded, S is bounded iff Tr S is bounded

We only need to plot Tr S

Schwarzschild BH in dCS

[Molina, Pani, Cardoso, Gualtieri 2010]

$$-\frac{d^2}{dx^2}\Phi_1 + V_{11}\Phi_1 + V_{12}\Phi_2 = \omega^2\Phi_1 \qquad f = 1 - \frac{2M}{r}$$

$$-\frac{d^2}{dx^2}\Phi_2 + V_{12}\Phi_1 + V_{22}\Phi_2 = \omega^2\Phi_2 \qquad f\,d/dr = d/dx$$

$$V_{11} = f\left[\frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3}\right]$$

$$V_{12} = f\frac{24M\sqrt{\pi(\ell+2)(\ell+1)\ell(\ell-1)}}{\sqrt{\beta}r^5}$$

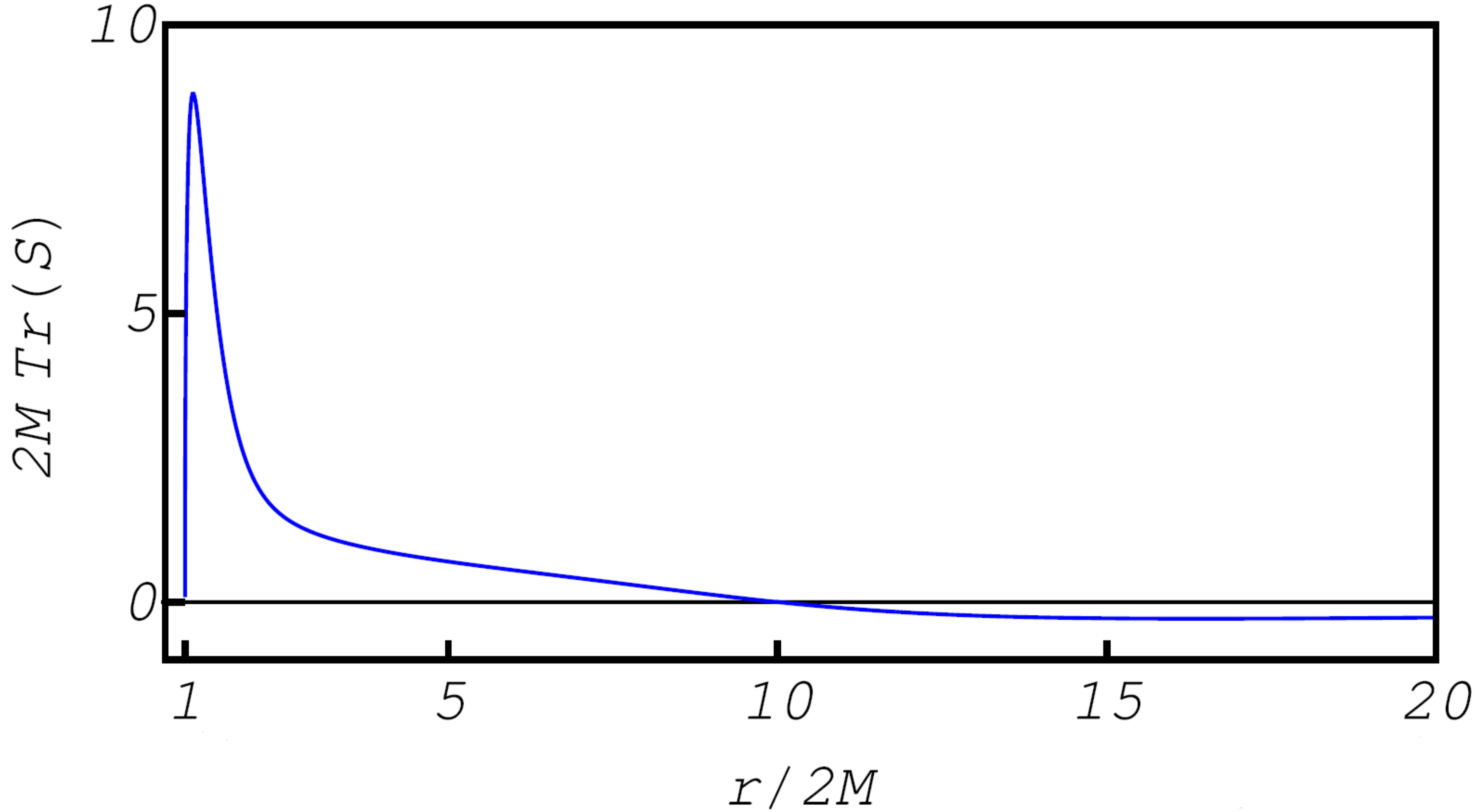
$$V_{22} = f\left[\frac{\ell(\ell+1)}{r^2}\left(1 + \frac{576\pi M^2}{\beta r^6}\right) + \frac{2M}{r^3}\right]$$

Schwarzschild BH in dCS

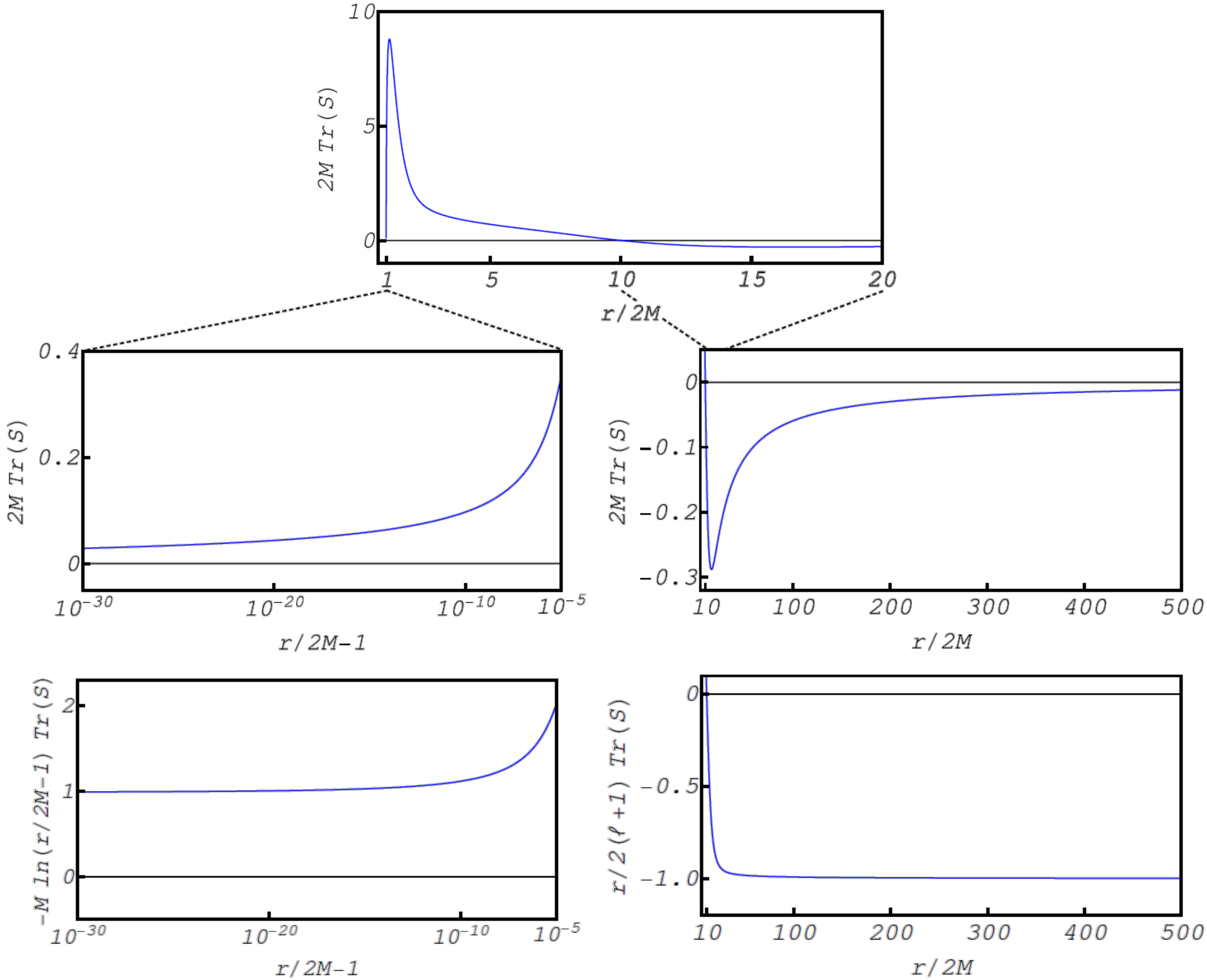
We solve $\boldsymbol{V + \frac{dS}{dx} - S^2 = 0}$ numerically

with the boundary condition

S = 0 at a large distance



$$\ell = 2, \beta M^4 = 1/10$$



2M Tr(S)
r/2M
r/2M−1
−M ln(r/2M−1) Tr(S)
r/2(ℓ+1) Tr(S)

$r/2M-1$

1.0^{-30} 1.0^{-20} 1.0^{-10} 1.0^{-5}

$$S_{11} = \frac{-2x + c_+ + c_- - (c_+ - c_-)\cos\theta_0}{2(x-c_+)(x-c_-)},$$

$$S_{12} = \frac{-(x_+ - x_-)\sin\theta_0}{2(x-c_+)(x-c_-)},$$

$$S_{22} = \frac{-2x + c_+ + c_- + (c_+ - c_-)\cos\theta_0}{2(x-c_+)(x-c_-)}.$$

x

$$c_\pm = x - \frac{S_{11} + S_{22} \pm \sqrt{4S_{12}^2 + (S_{11} - S_{22})^2}}{2(S_{12}^2 - S_{11}S_{22})},$$

$$\cos\theta_0 = \frac{-S_{11} + S_{22}}{\sqrt{4S_{12}^2 + (S_{11} - S_{22})^2}}.$$

matched with the asymptotic sol

Remarks for general case

The nodal theorem for coupled systems suggest the existence of regular S
(we can explicitly show the existence of regular S for rapidly decaying potential)

$\boldsymbol{S_L} \leq \boldsymbol{S} \leq \boldsymbol{S_R}$ seems to hold

If V > 0 in asymptotic region,
$\boldsymbol{S} = \boldsymbol{0}$ at large x is a candidate for an appropriate BC

Merit of S-deformation method

- We do not need to care about boundary condition at infinity very much, we can solve equation from finite point
- It is clear that the existence of regular S is the sufficient condition for stability (proof of nodal theorem is very difficult)
- Any fine-tuning is not needed
- Easy to show the non-existence of zero mode (by showing two different S)

Summary

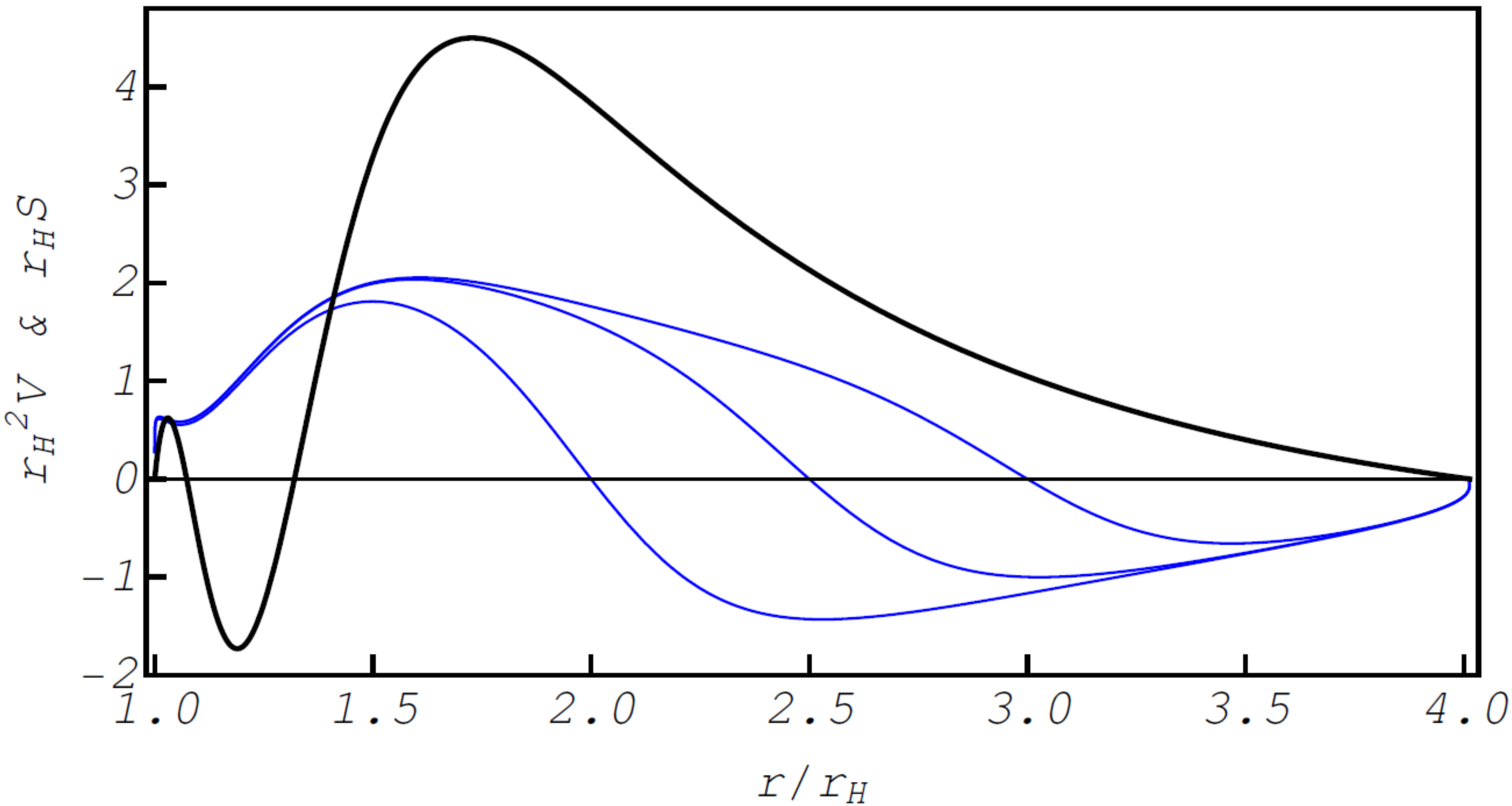
We proposed a simple method for finding S-deformation by solving $V + \frac{dS}{dx} - S^2 = 0$

This is a good test for stability of BH

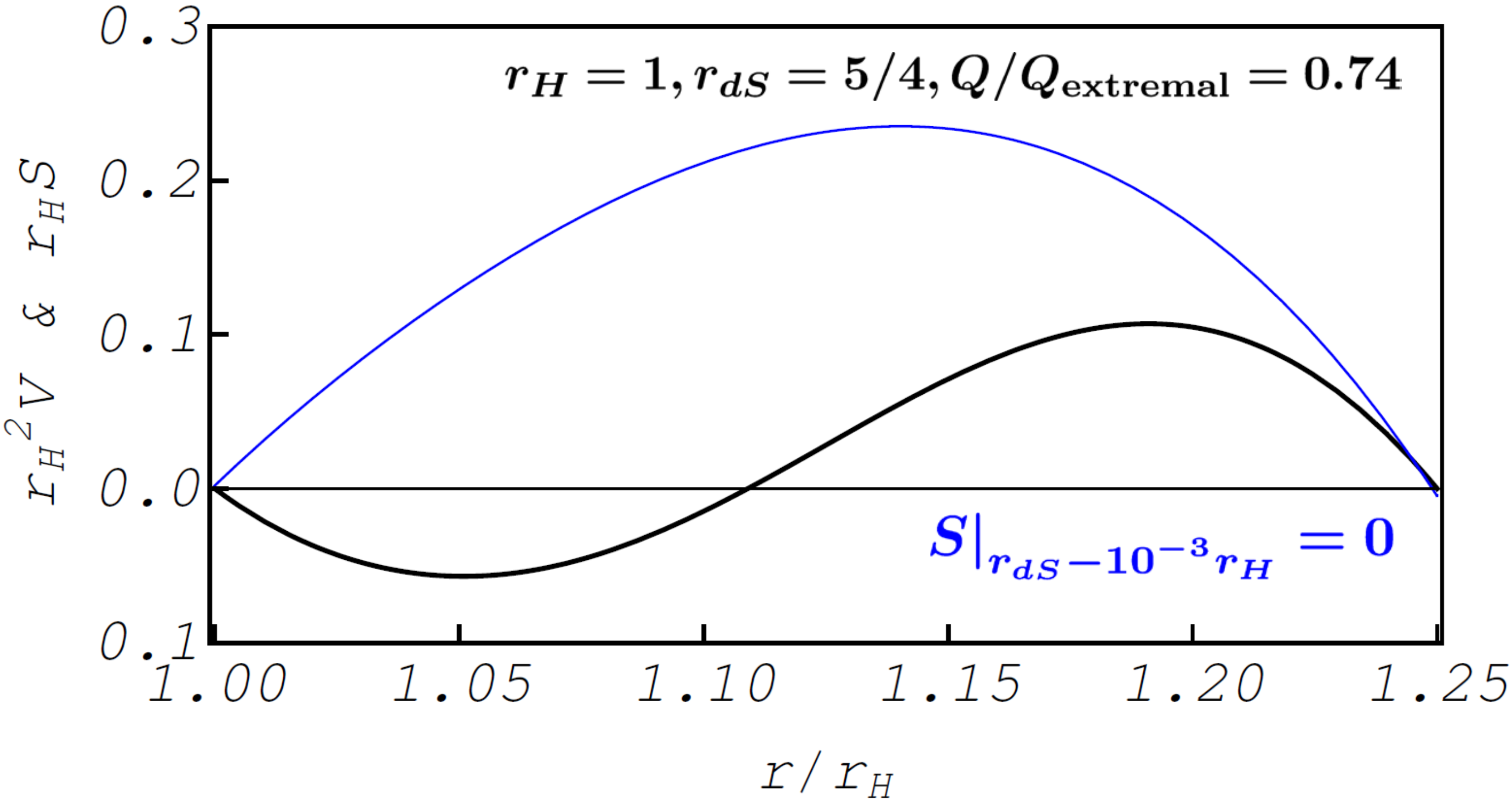
If stable, we can find regular S

We can guess the threshold of the parameter where unstable mode appears

10 Dim Schwarzschild-dS BH



10 Dim RN-dS BH



If $Q/Q_{\mathrm{extremal}} > q_{\mathrm{cr}} \simeq 0.75$, there exists an unstable mode [Konoplya and Zhidenko, 2008]

Supersymmetric quantum mechanics

From $V + \frac{dS}{dx} - S^2 = 0$

$$\left(-\frac{d}{dx} + S\right)\left(\frac{d}{dx} + S\right)\Phi = \omega^2\Phi$$

Supersymmetric quantum mechanics system