Oscillators, Resonators, and the Instability of AdS

Benson Way (UBC)

Matthew Choptuik, Oscar Dias, Jorge Santos, B.W. arXiv:1706.06101, 1803.02830

Basic question:

What happens to small energy excitations in (global) AdS?

Nonlinear Stability

Do all finite, but small perturbations remain small?

DeSitter: Yes, 33 pages [G. Friedrich]

Minkowski: Yes, 526 pages [D. Christodoulou, S. Klainerman]

(Global) Anti-deSitter Space: ??

Nonlinear Stability

Proof of stability depends upon decay of linear fields.

DeSitter: Fast Decay, extend local results to global

Minkowski: Borderline Decay, nonlinearities matter

(Global) Anti-deSitter: No Decay (reflecting boundary)

Conjectured Instability

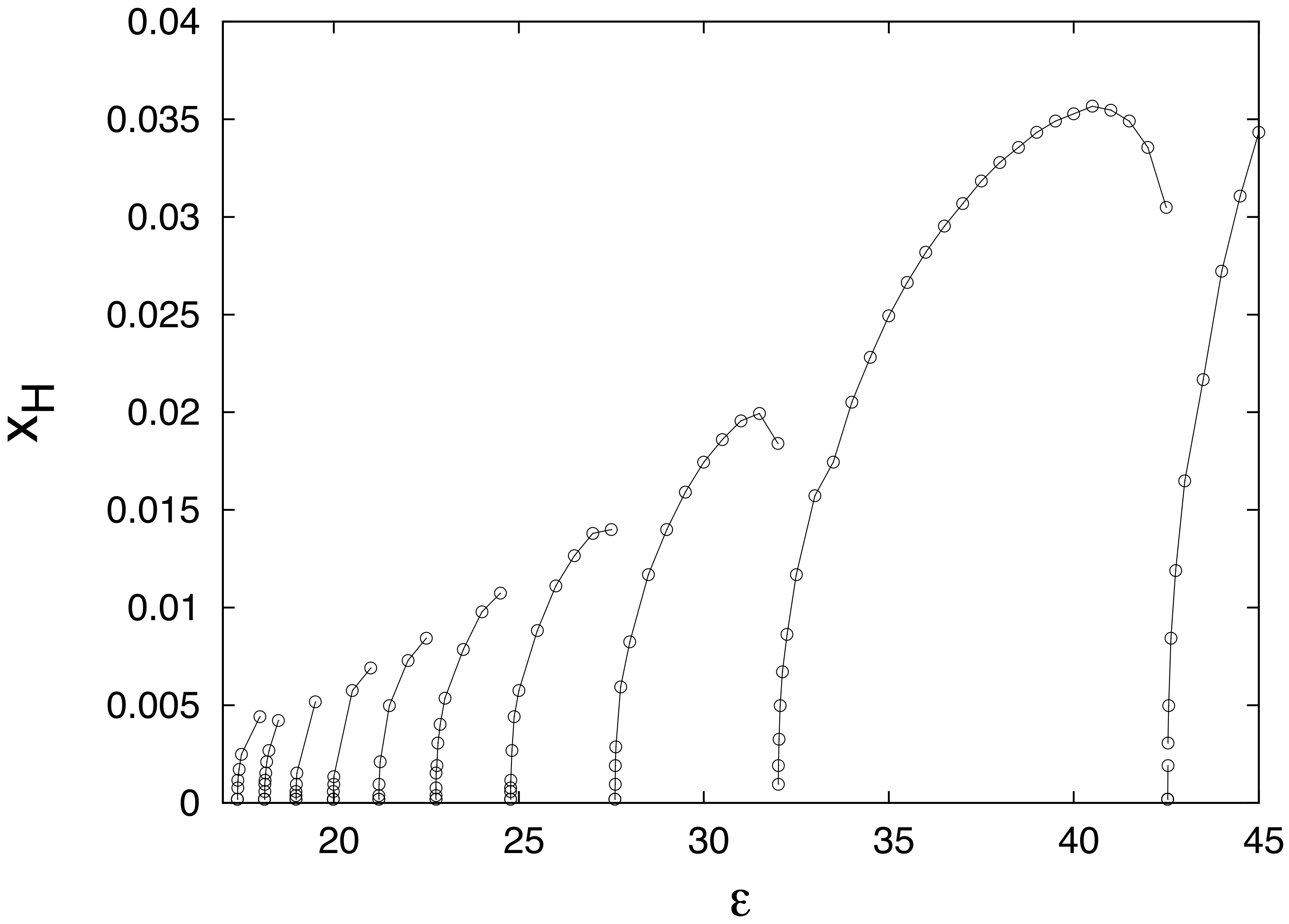
Reflecting boundary prevents dissipation, so nonlinear excitations may become large. [M. Dafermos and G. Holzegel]

If the Einstein equation is sufficiently ergogodic, generic perturbations will eventually explore black hole configurations. [M.T. Anderson]

Part I: Spherical Symmetry

Instability of AdS

First evidence for instability with a scalar field.[P. Bizon and A. Rostworowski, arXiv:1104.3702]



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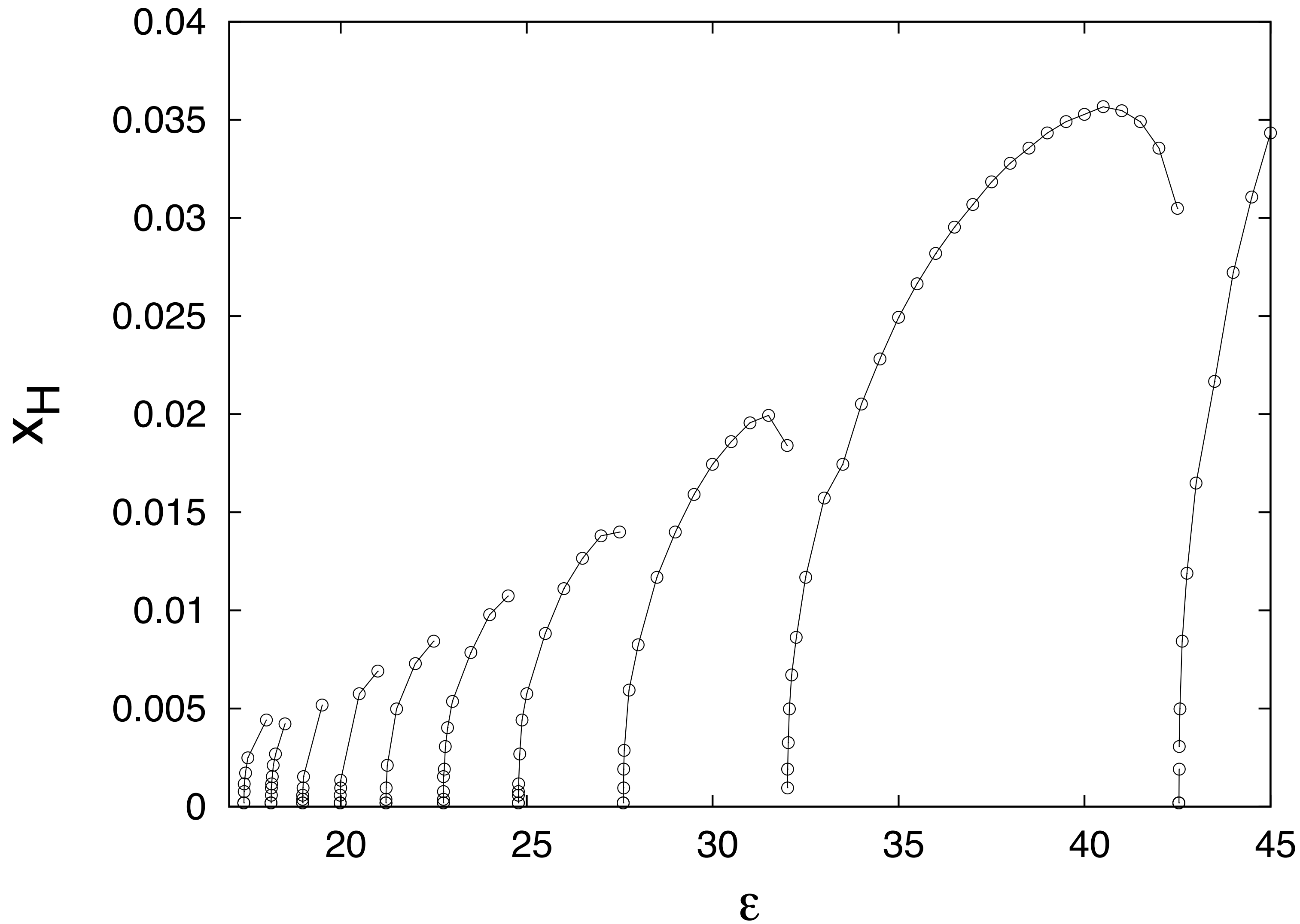

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$$(L = 1)$$

Proof of instability in Einstein-massless-Vaslov system.

[G. Moschidis]

Stability of AdS

But not all initial data form black holes!
(checked numerically for $t \gg 1/E$.)[V. Balasubramanian, A. Buchel, S.R.Green, S. L. Liebling, and L. Lehner; M. Maliborski and A. Rostworowski.]

We call the (open) set of non-collapsing initial data the "islands of stability".

Stability of AdS

Why do some data collapse and not others?

Can we characterise collapsing vs non-collapsing data?
I.e. determine if initial data will collapse without doing the full evolution.

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Vague partial answer: Non-collapsing data resemble normal modes.

Perturbation Theory

$$\varphi = \epsilon\,\varphi^{(1)} + \epsilon^3 \varphi^{(3)} + \dots\,, \qquad g = g^{(0)}_{\mathrm{AdS}} + \epsilon^2 g^{(2)} + \dots$$

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Choice of a_k determines next order.

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$$\varphi^{(3)} \sim t\sin(\omega_j t) \qquad t_{\mathrm{pert}} \sim 1/\epsilon^2 \sim 1/E.$$

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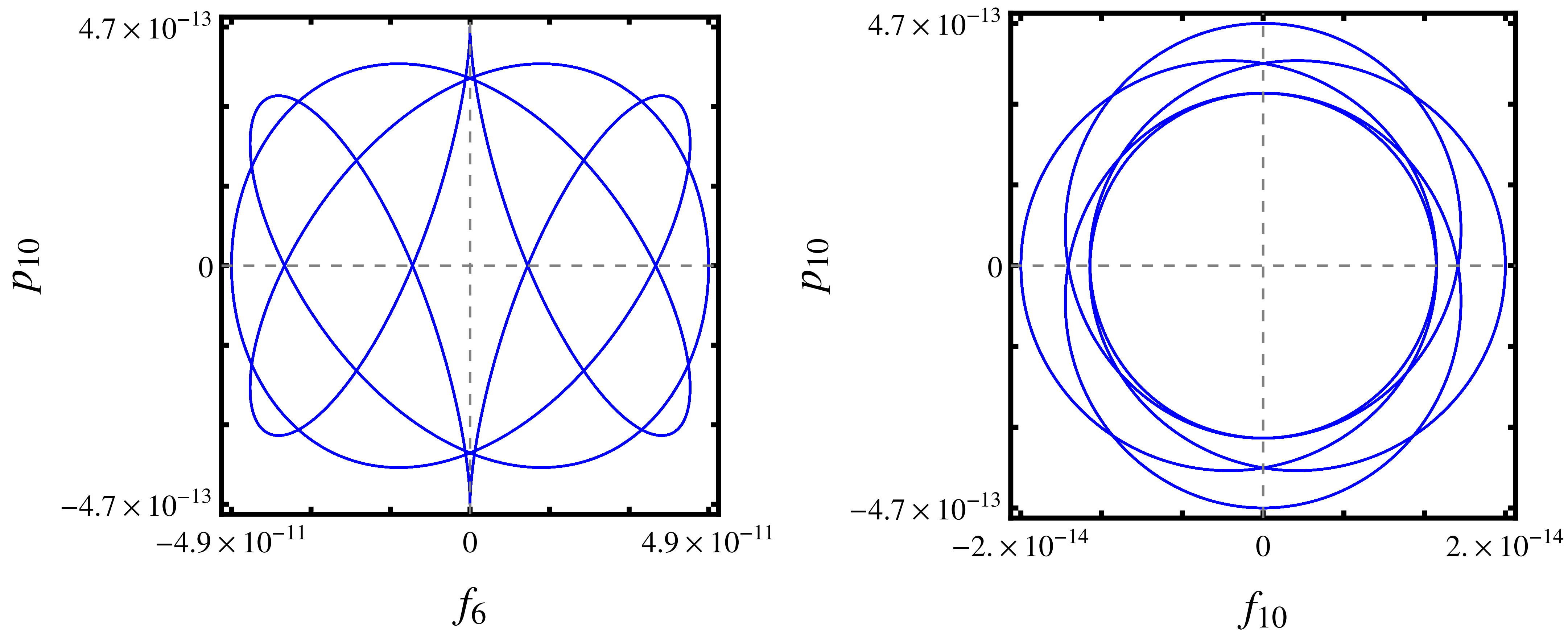
But secular terms are absent if all but one a_k vanishes. Perturbation theory can continue to all orders.

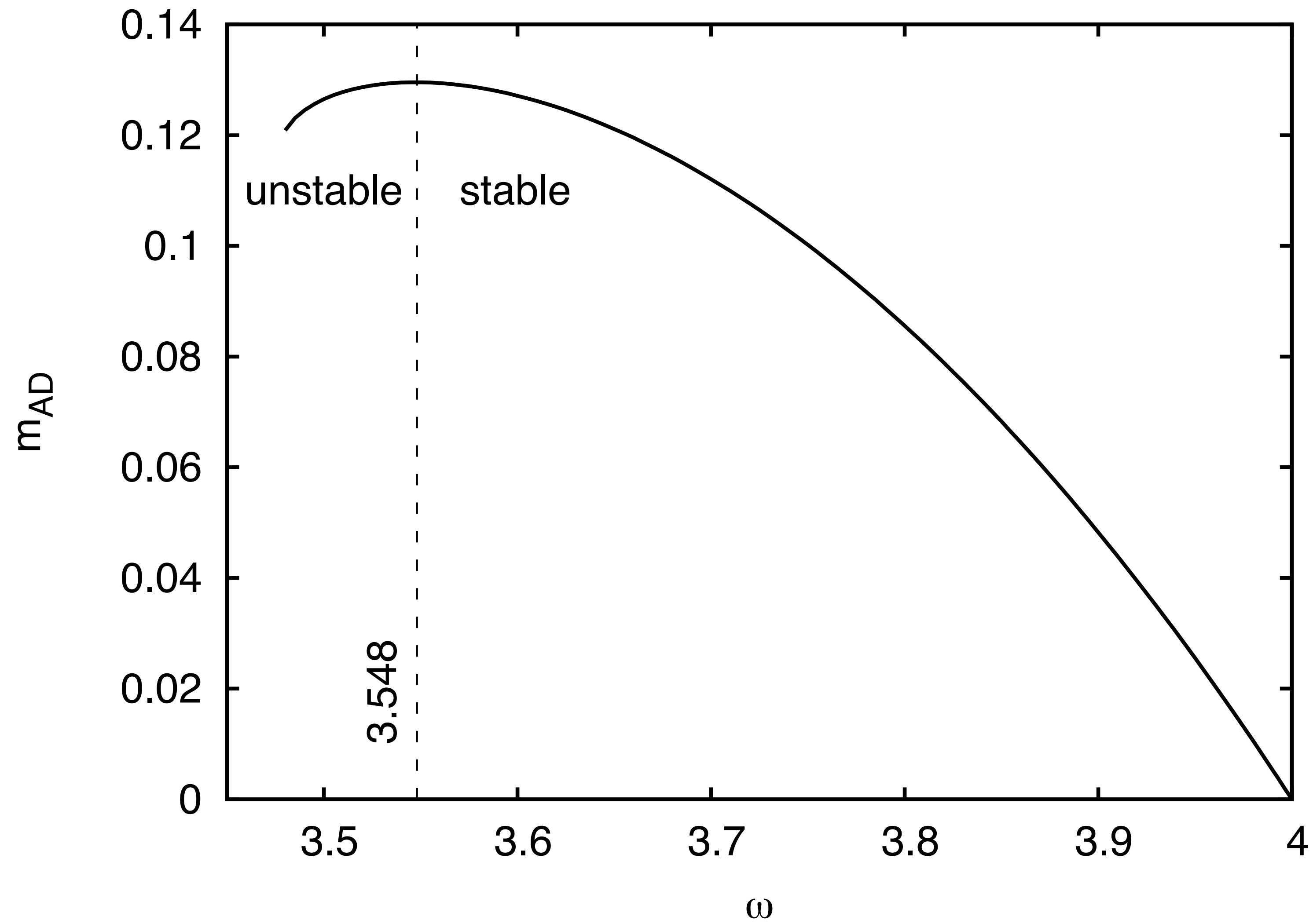
Oscillons

Can construct a non-perturbative, harmonically oscillating solution: an oscillon.



[M. Maliborski and A. Rostworowski; arXiv:1303.3186]

Oscillons



[G. Fodor, P. Forgacs, and P. Grandclement; arXiv:1503.07746]

Islands of Stability

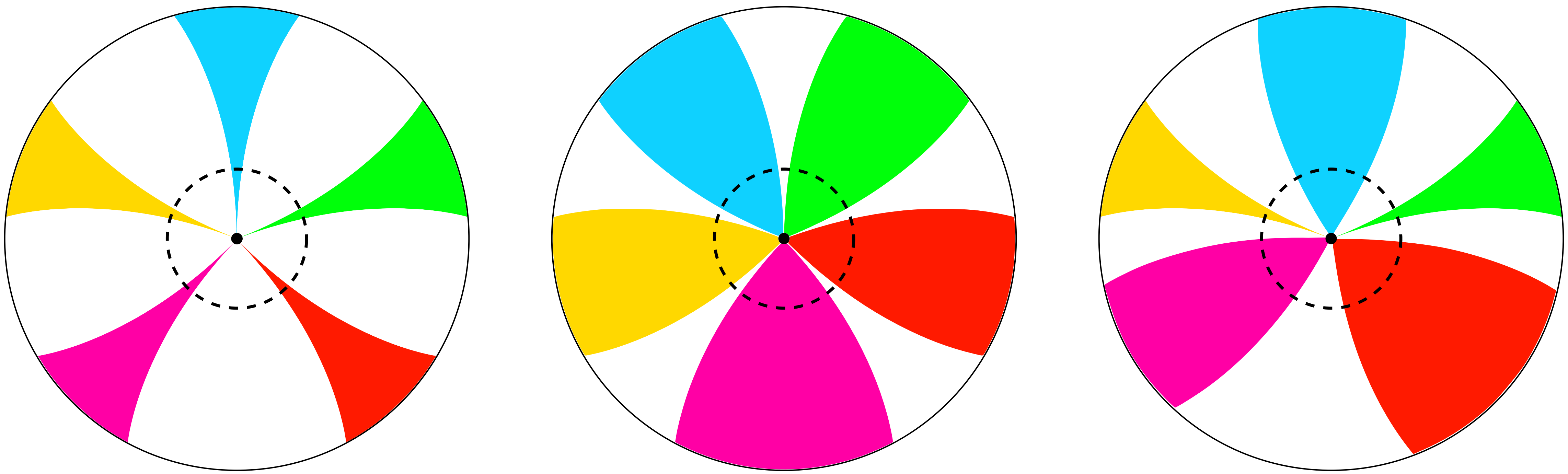
Oscillons are essentially non-linear normal modes.

Similar solutions exists for complex scalars (boson stars) and gravity (geons). Call these ‘oscillators.’

All non-collapsing solutions that were found are ‘close’ to an oscillator (i.e. single-mode dominated)!

Islands of Stability

What do we mean by 'close'? How big are these islands, and what is their shape?



[Figure from G. Martinon; arXiv:1708.05600]

New Perturbation Theory

Oscillators and data near them are stable, yet they are multi-mode data. [Pert. theory has secular terms.]

Secular resonances imply breakdown of perturbation theory, not necessarily collapse!

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Need some way to handle resonances.

One approach: resum perturbation theory. [V. Balasubramanian, A. Buchel, S.R.Green, S. L. Liebling, and L. Lehner; B. Craps, O. Evnin, and J. Vanhoof; N. Deppe; F. V. Dimitrakopoulos, B. Freivogel, M. Lippert, J. F. Pedraza, I.-S. Yang]

Two Timescale Formalism (TTF)

Introduce time dependence in modes:

$$a_k \to a_k(\epsilon^2 t) = a_k(\tau)$$

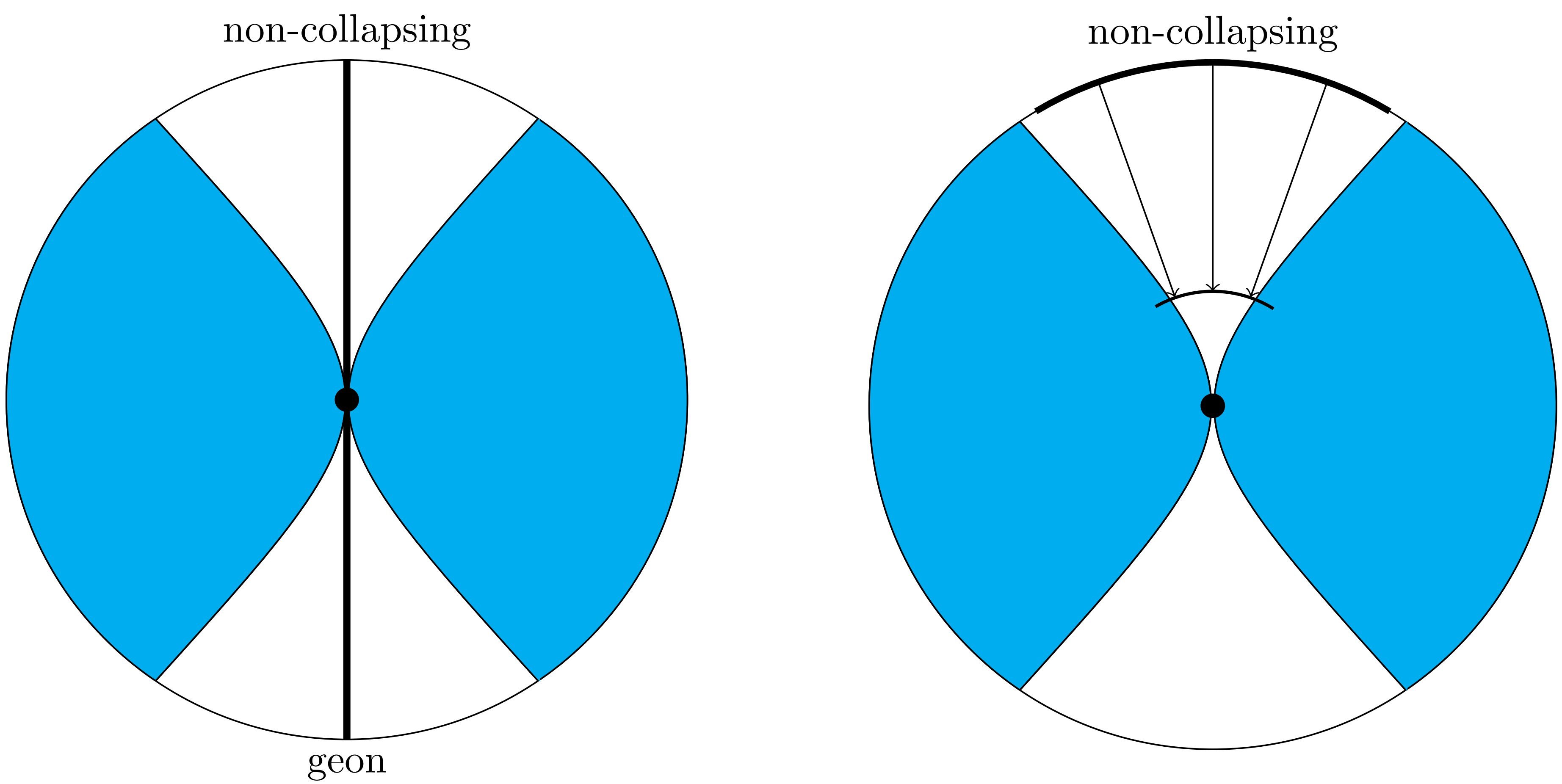
At third order, get new set of dynamical equations valid to $t \sim 1/\epsilon^2$: the TTF equations.

The TTF equations have additional conserved quantities and a scaling symmetry:

$$a_k(\tau) \to \epsilon a_k(\tau/\epsilon^2)$$

New Perturbation Theory

Assuming late-time validity of TTF, scaling symmetry implies non-cuspy islands.



[F. Dimitrakopoulos, I.S. Yang; arXiv:1507.02684]

No Perturbation Theory

What about timescales much longer than $t \sim 1/\epsilon^2$?

Is it possible to chart the islands of stability?

Why are oscillators stable?

Try non-perturbative approach.

Multioscillators

Oscillators are extensions of AdS normal modes.

$$\varphi \sim a\cos(\omega_{\mathrm{AdS}} t) P(x)$$

$$\varphi_{\mathrm{osc}} = \sum a_{nk} \cos(n\omega_{\mathrm{osc}} t) P_k(x)$$

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What about normal modes of oscillators?

$$\varphi \sim \varphi_{\mathrm{osc}} + e^{i\omega t} \underbrace{\delta\varphi_{\mathrm{osc}}(t,x)}_{\text{freq. } \omega_{\mathrm{osc}}}$$

No resonances since φ_{osc} deforms AdS.

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$$\varphi = \sum a_{n_1 n_2 k} \cos(n_1 \omega_1 t) \cos(n_2 \omega_2 t) P_k(x)$$

A double-oscillator!

Multioscillators

Why stop there?

$$\varphi = \sum a_{n_1 \ldots n_N k} \cos(n_1 \omega_1 t) \ldots \cos(n_N \omega_N t) P_k(x)$$

An infinite-parameter family of non-collapsing solutions!

Might account for apparent stability of single-oscillators. Many, perhaps all, perturbations of single-oscillators will land on a multi-oscillator.

Constructing Multioscillators

In exponential form,

$$\varphi(t,x) = \sum_{k_1 \ldots k_N} A_{k_1 \ldots k_N}(x) e^{ik_1\omega_1 t + \ldots + ik_N \omega_N t}$$

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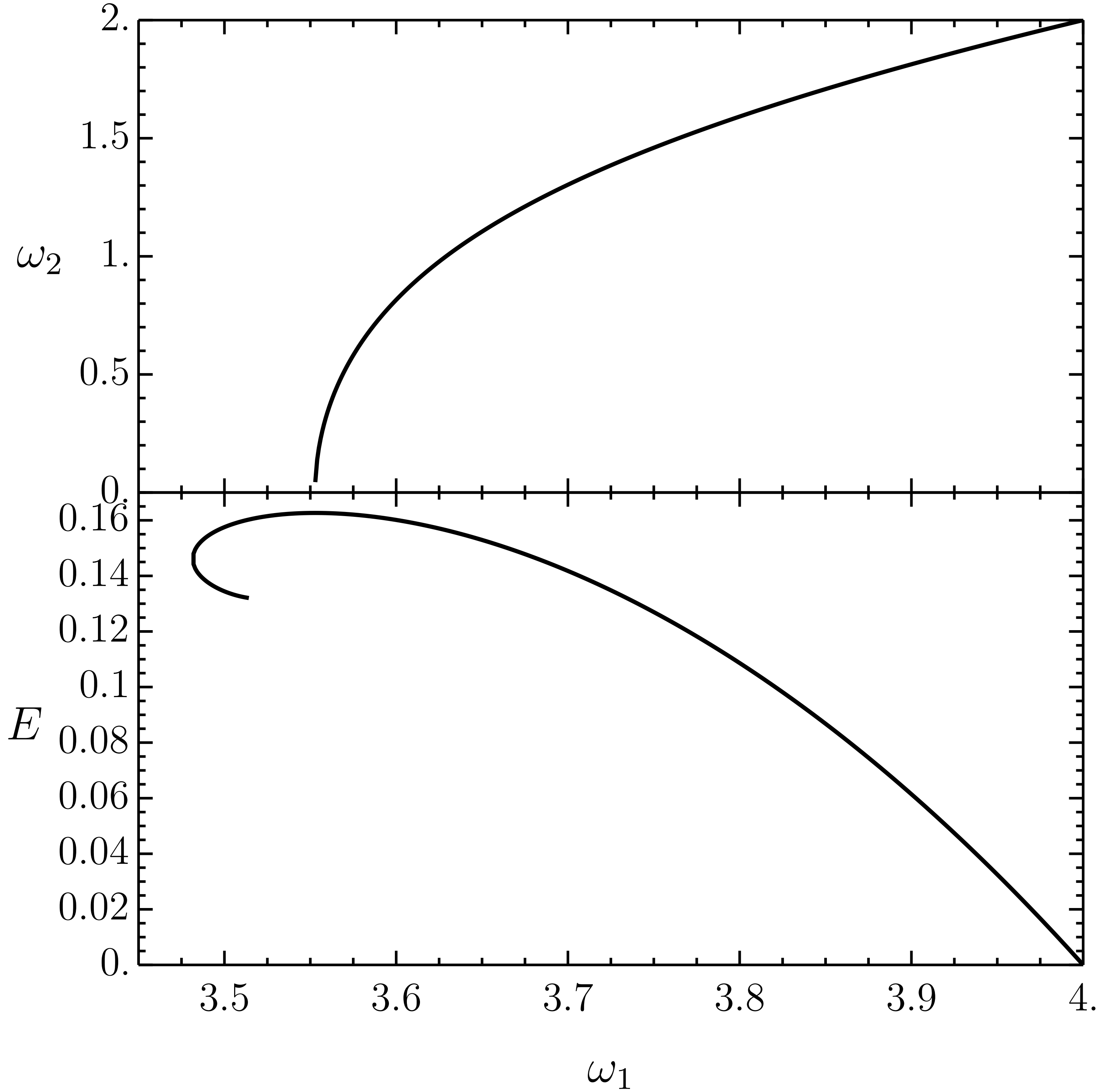
Solve N+1 dimensional PDE, where time coordinates have periodic boundary conditions.

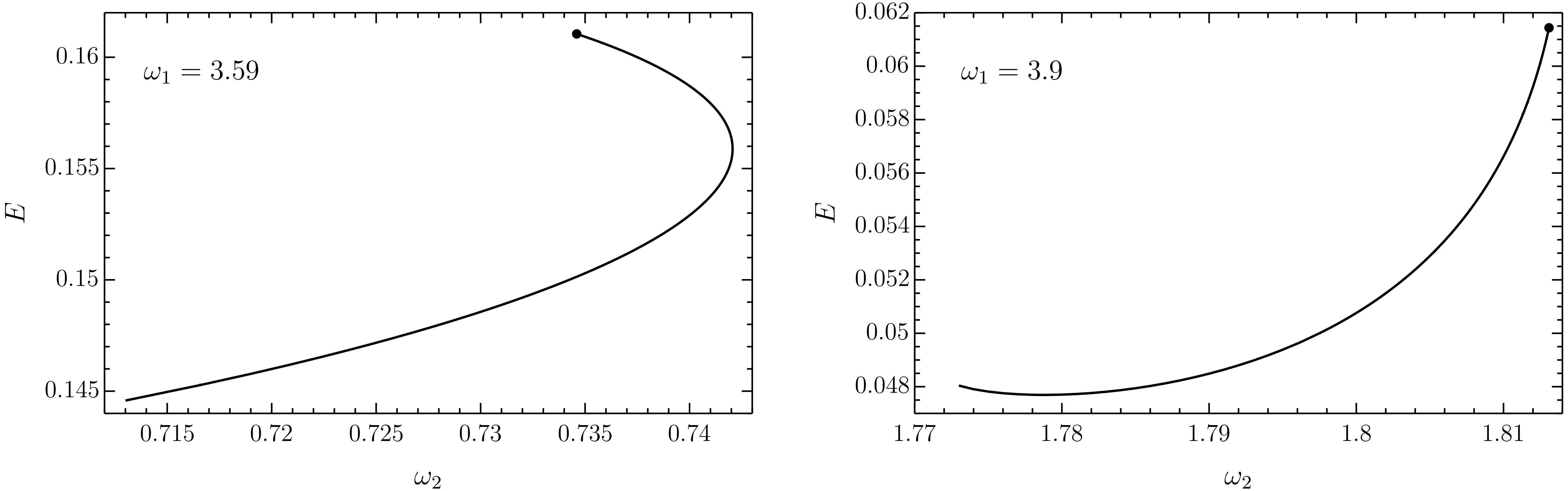
Boson Star Normal Modes

(Complex scalar field)



Two-Frequency Oscillators

(Complex scalar field)



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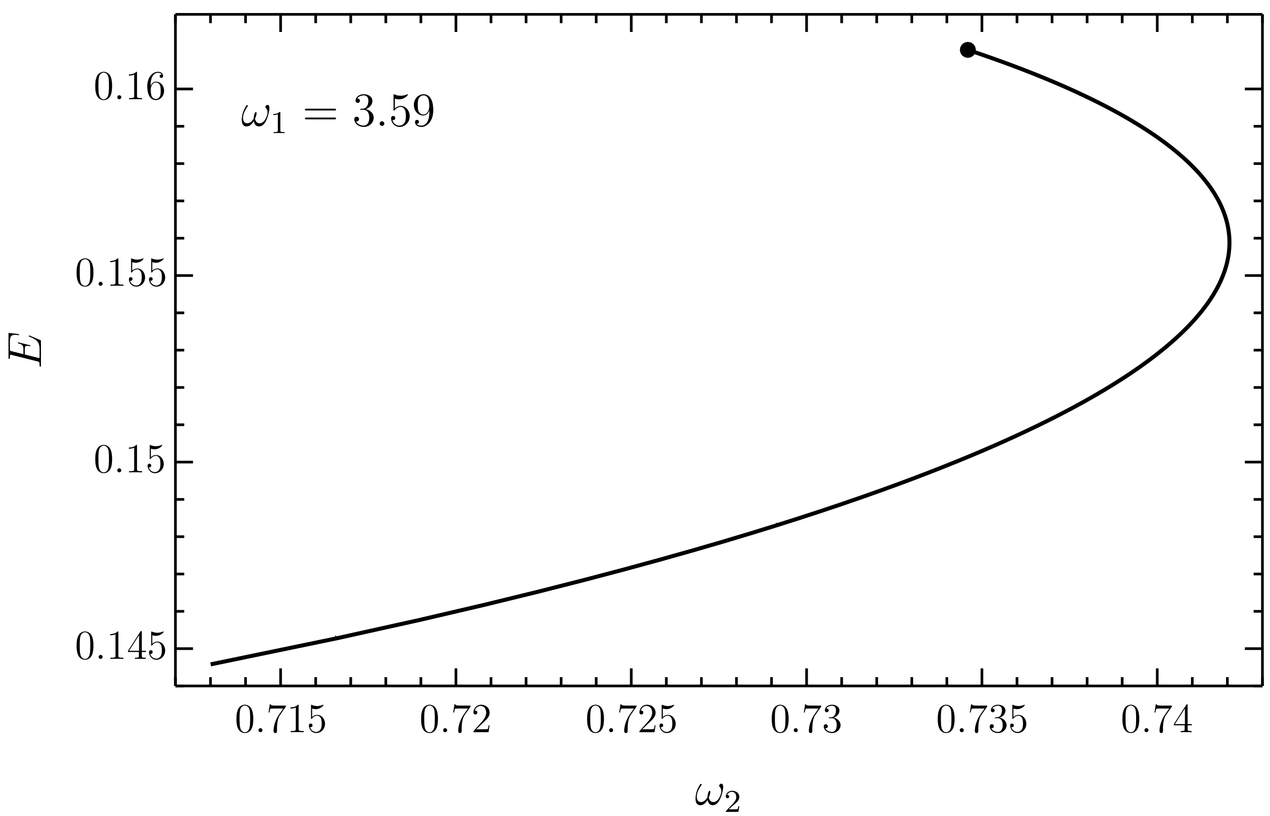
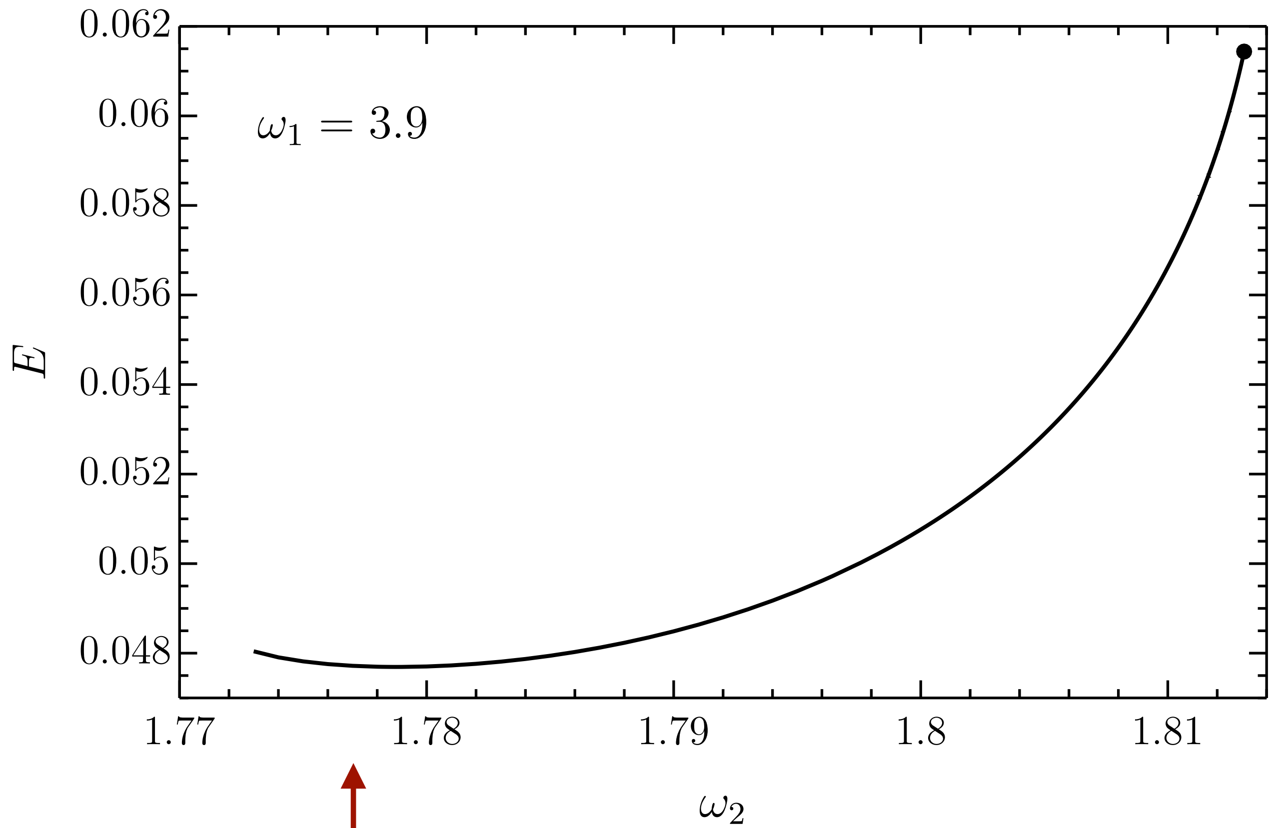
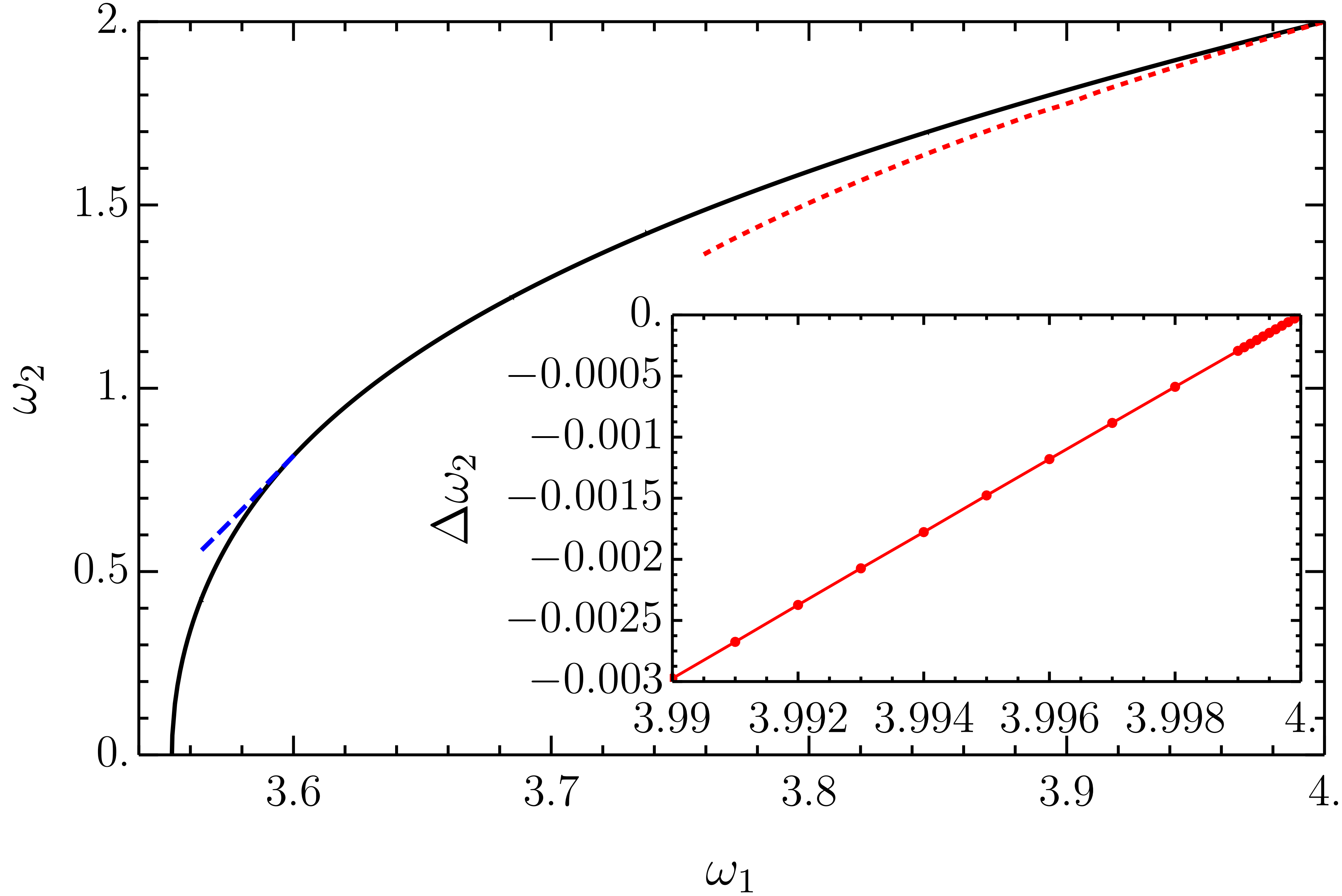




Change in stability?

No Cusps

(Complex scalar field)



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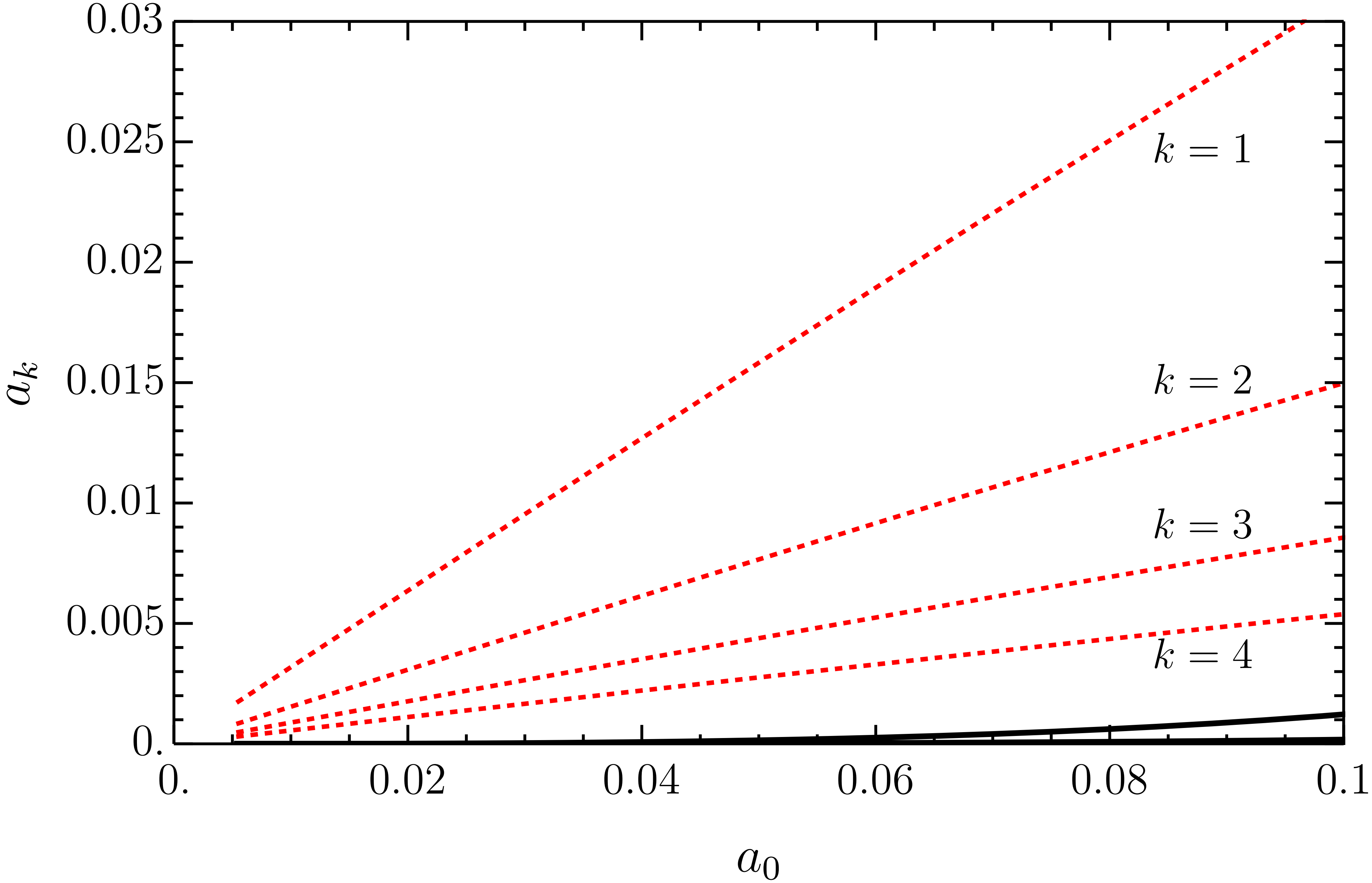


Summary of Results

- Propose the existence of an infinite-parameter family of multi-oscillator solutions near single-oscillators.
- Constructed double-oscillators.
- Two-frequency oscillators contain turning points.
- Phase diagram does not cusp near AdS.

Further Questions

- Are some multi-oscillators linearly unstable? If so, what is their endpoint?
- Are all non-collapsing solutions in the multi-oscillator family?

Evidence for Multi-oscillator Islands

Nonlinear evolution with moderate energies to moderately long times find either:

- Regular collapse with $t \sim 1/E$.
- Irregular collapse not on this timescale.
- Stable non-collapse with quasi-periodic behaviour.

Part II: Angular Momentum

Adding Angular Momentum

With $J \neq 0$,

- Generic.
- Angular momentum barrier may affect collapse.
- Endpoint of instability is not just Schwarzschild-AdS. (Spoilers: not just Kerr-AdS either.)

Phase Diagram

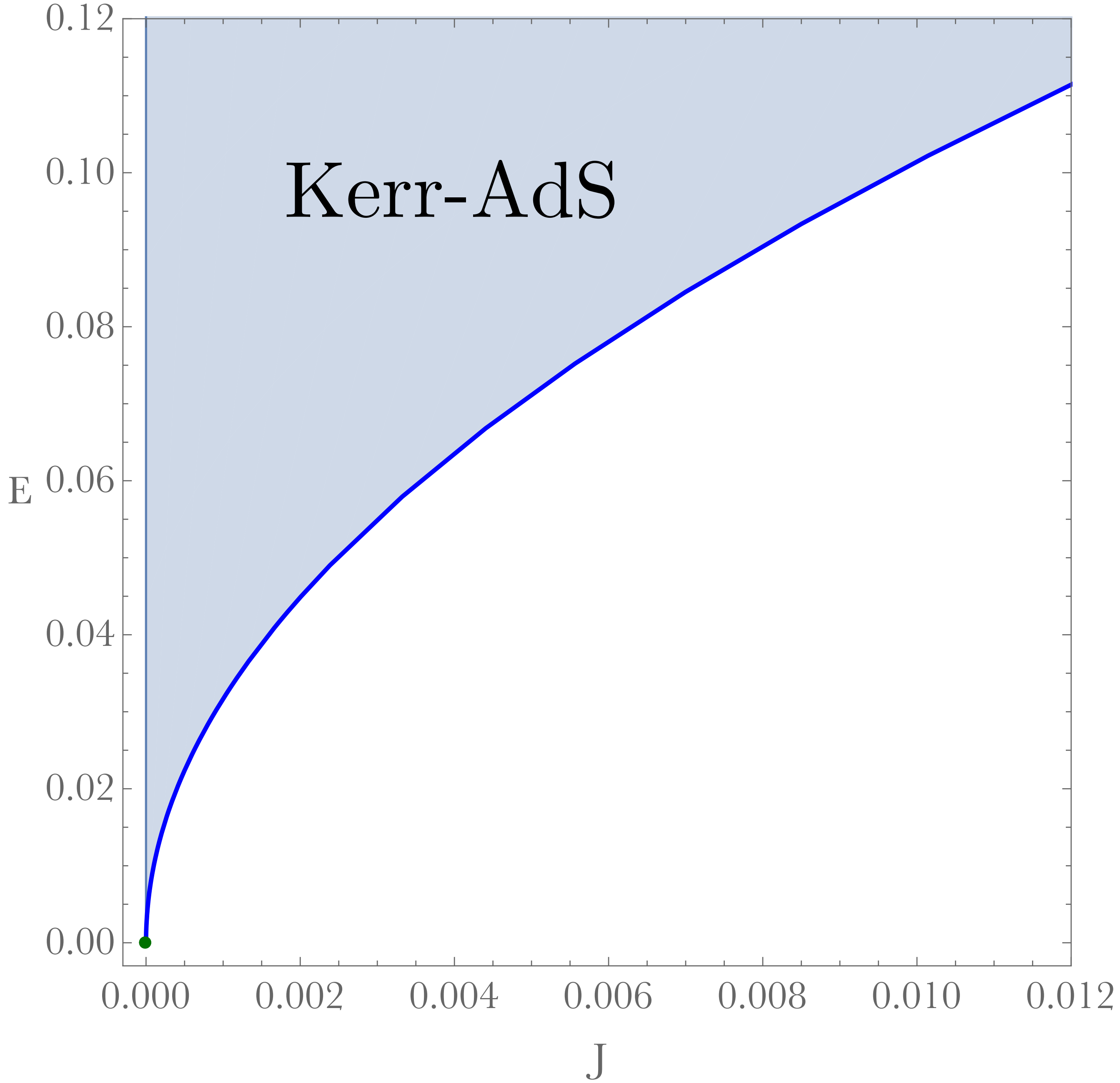
There are typically three types of 'stationary' solutions in global AdS:

- 'Bald' black holes.
- (Multi)-oscillators.
- Hairy black holes.

[V. Cardoso O.C. Dias, G.S. Hartnett, G. Horowitz, L. Lehner, D. Marolf, J.E. Santos, B.W.]

Phase Diagram

For pure gravity in AdS_4 ,



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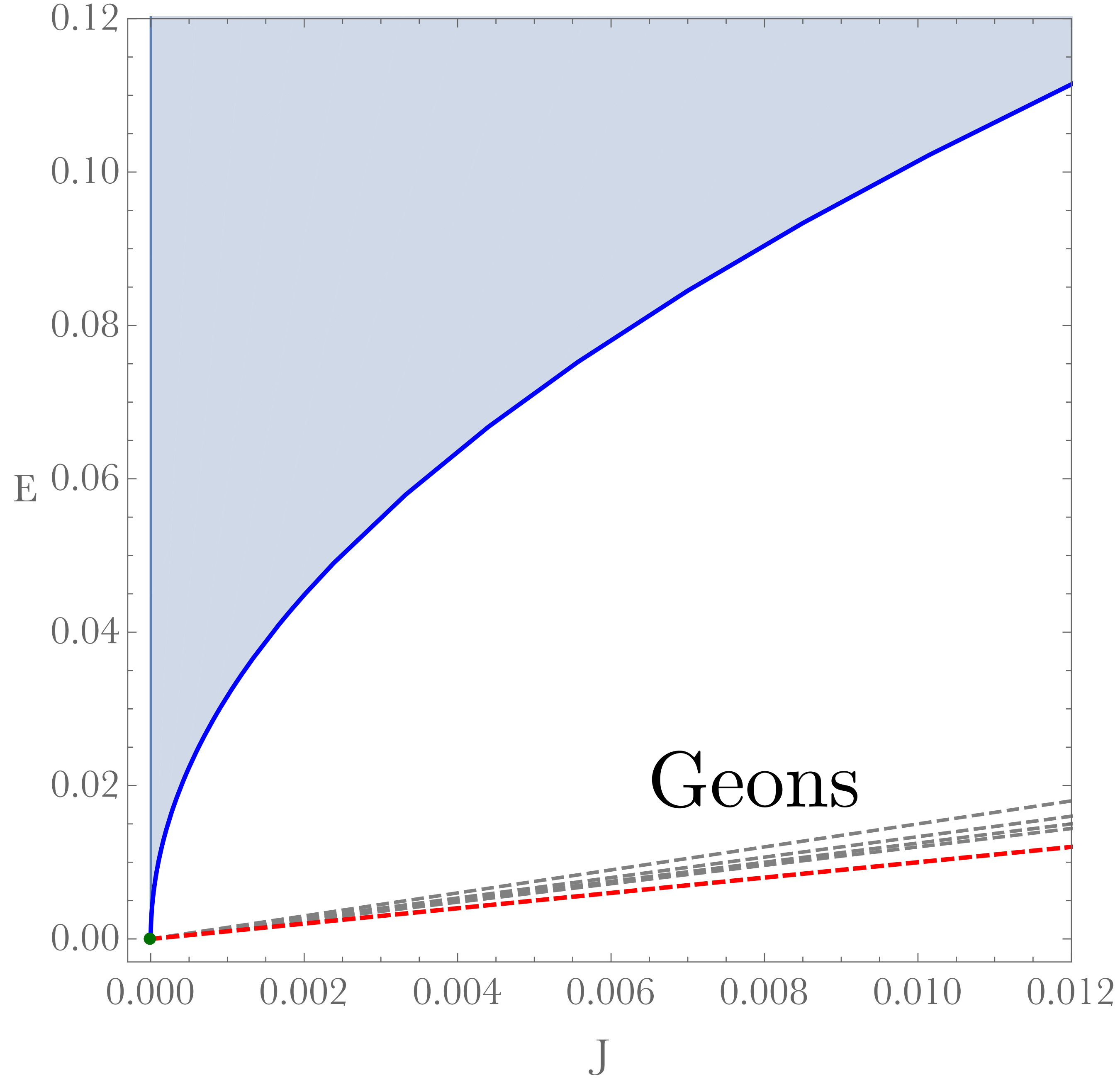

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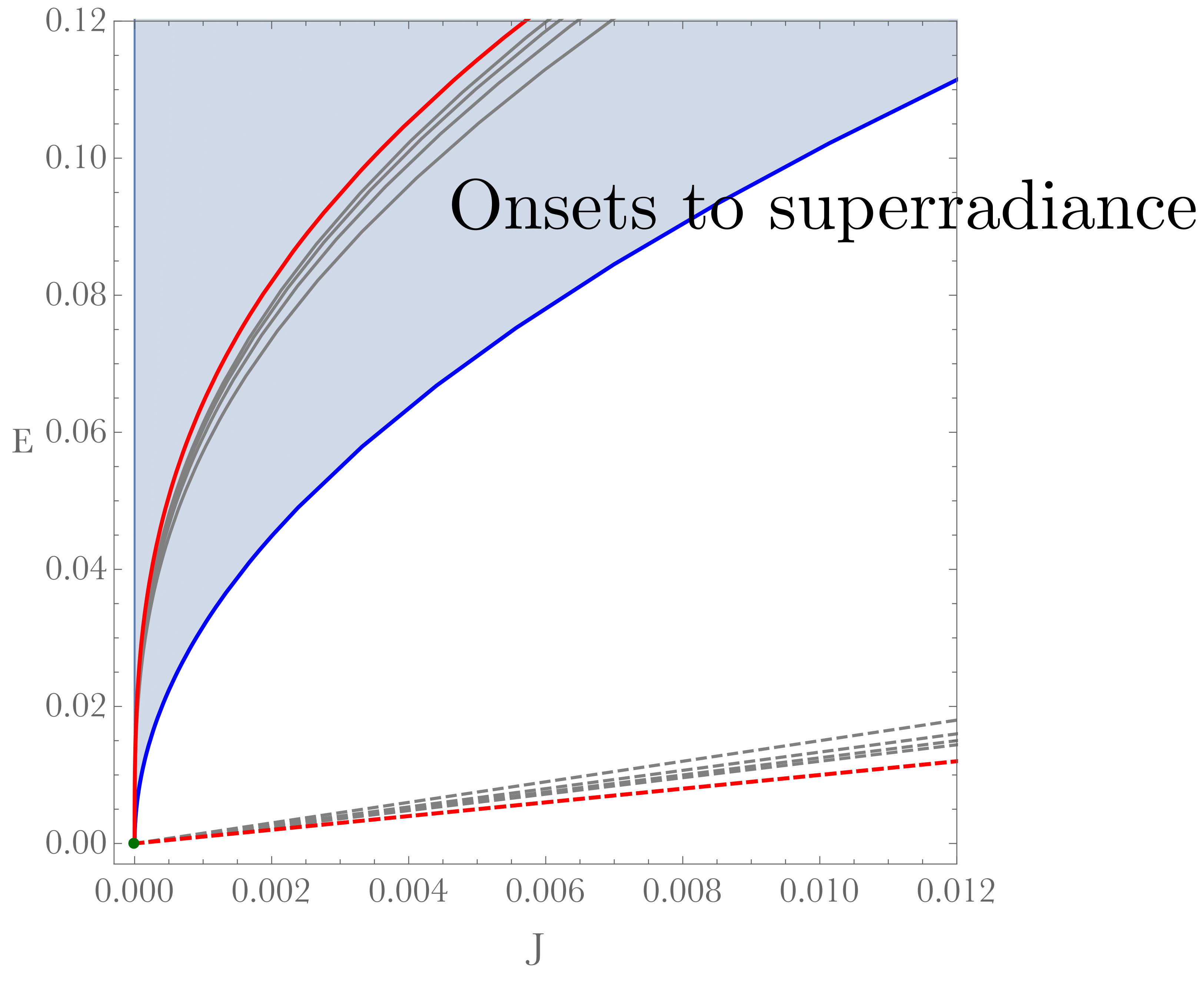

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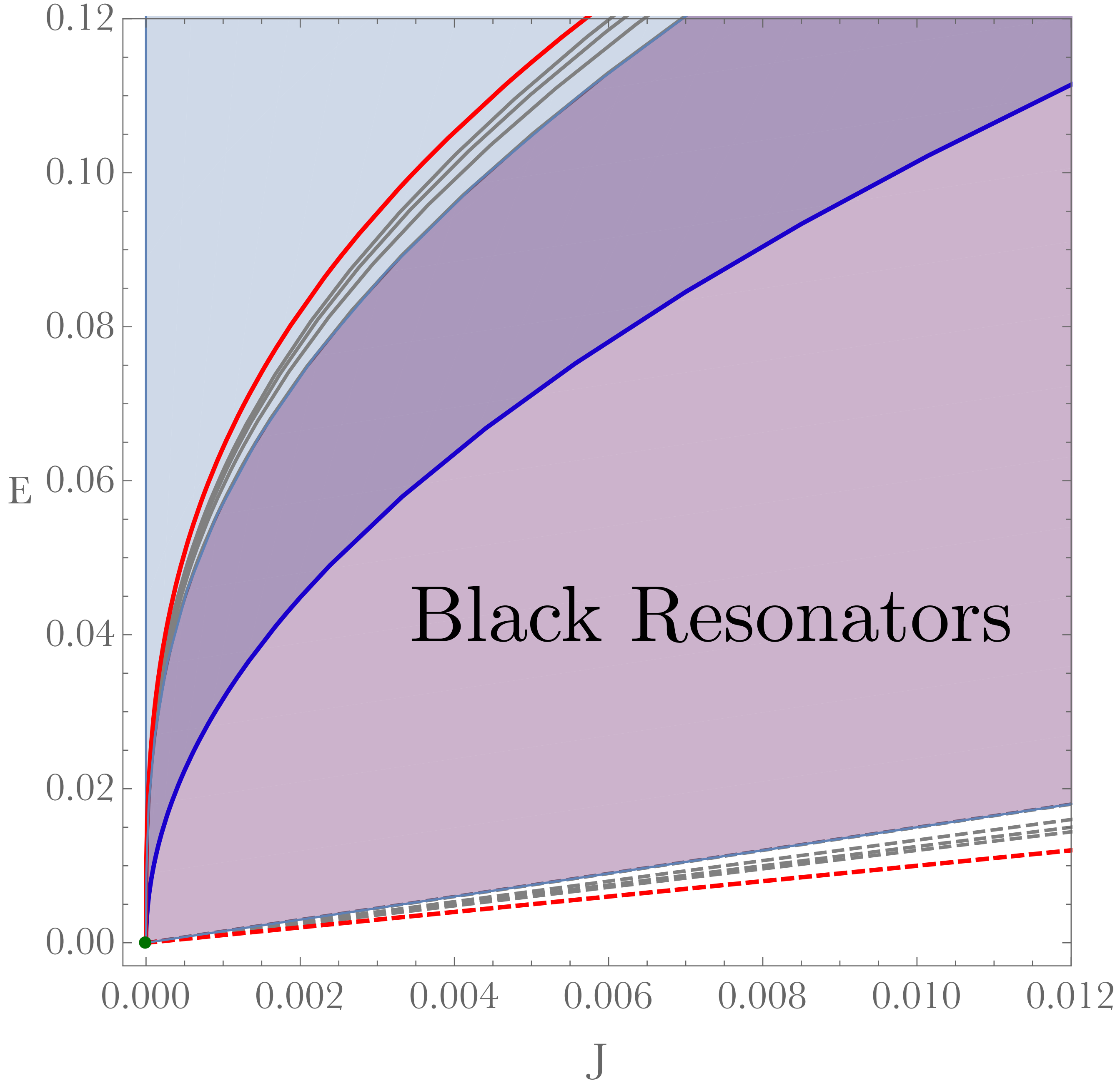


Questions

With $J \neq 0$:

- What is the typical timescale for collapse?
- What is the final state after gravitational collapse?

With pure gravity, need 3+1 numerical computation!

Making Life Easier

Need to exploit lots of symmetry to make this problem easier.

- Go to 5 dimensions. Set $J \equiv J_1 = J_2$.
- Add a complex doublet scalar field that can carry angular momentum and also respect these symmetries.

Making Life Easier

Ansatz:

$$ds^2 = \frac{1}{(1-\rho^2)^2}\Big\{ -\alpha^2\left[1-\rho^2(2-\rho^2)\frac{\beta^2}{a}\right]\mathrm{d}t^2 + \frac{4\alpha\beta}{a}\rho\,\mathrm{d}t\mathrm{d}\rho + \frac{\mathrm{d}\rho^2}{a(2-\rho^2)} +$$

$$+\rho^2(2-\rho^2)\left[\frac{1}{b^2}\Big(\mathrm{d}\psi + \cos^2(\tfrac{\theta}{2})\mathrm{d}\phi - \Omega\mathrm{d}t\Big)^2 + \frac{b}{4}\Big(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\Big)\right]\Big\}\,,$$

$$\Pi = (\Pi_{\mathfrak{R}} + i\,\Pi_{\mathfrak{I}})\left[\begin{array}{c} e^{i\psi}\sin(\frac{\theta}{2}) \\ e^{i(\psi+\phi)}\cos(\frac{\theta}{2}) \end{array}\right]\,.$$

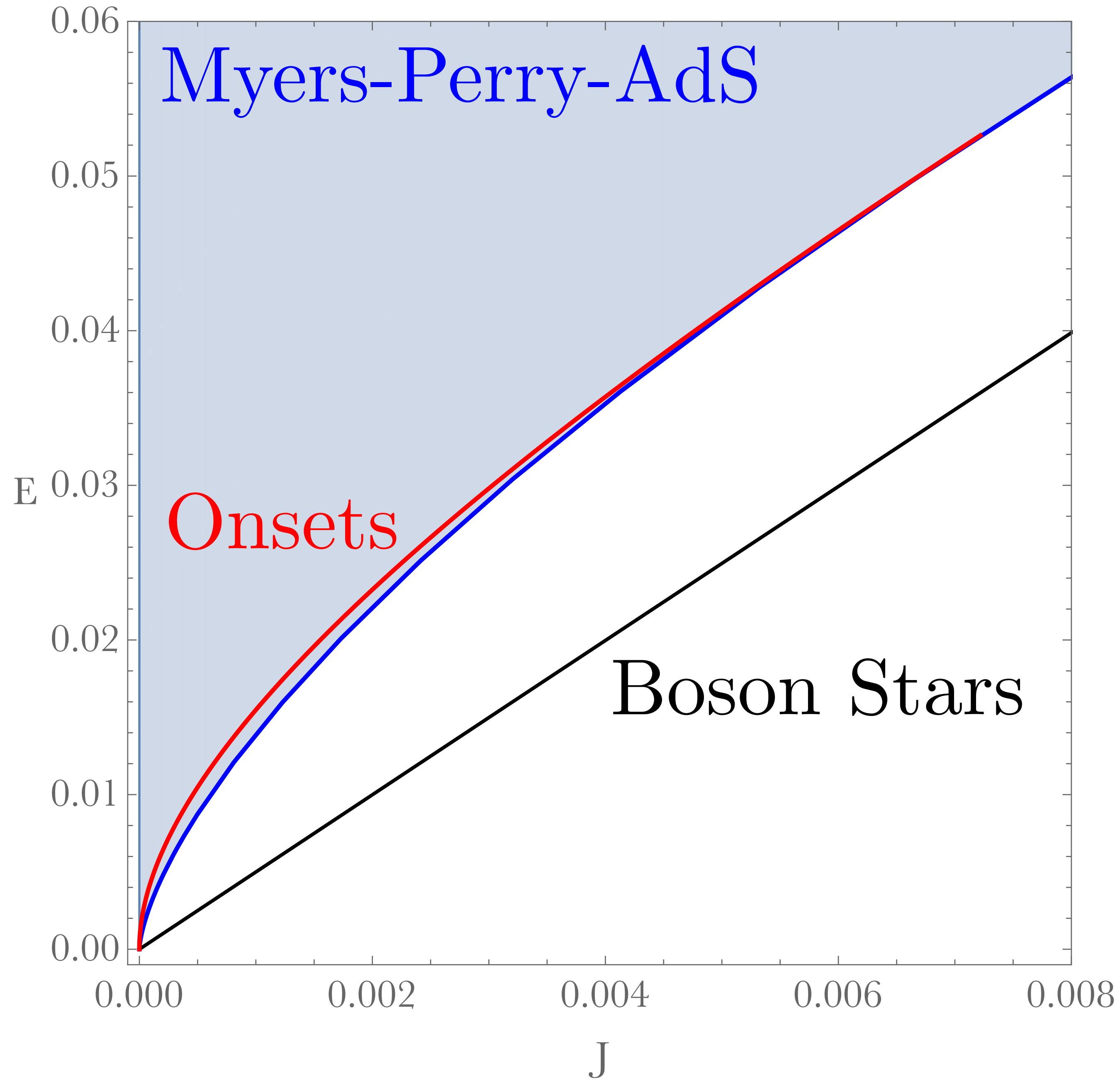
Ansatz with 1+1 $\{\rho, t\}$ evolution!

One limitation: restricted to azimuthal wavenumbers with $m = 1$.

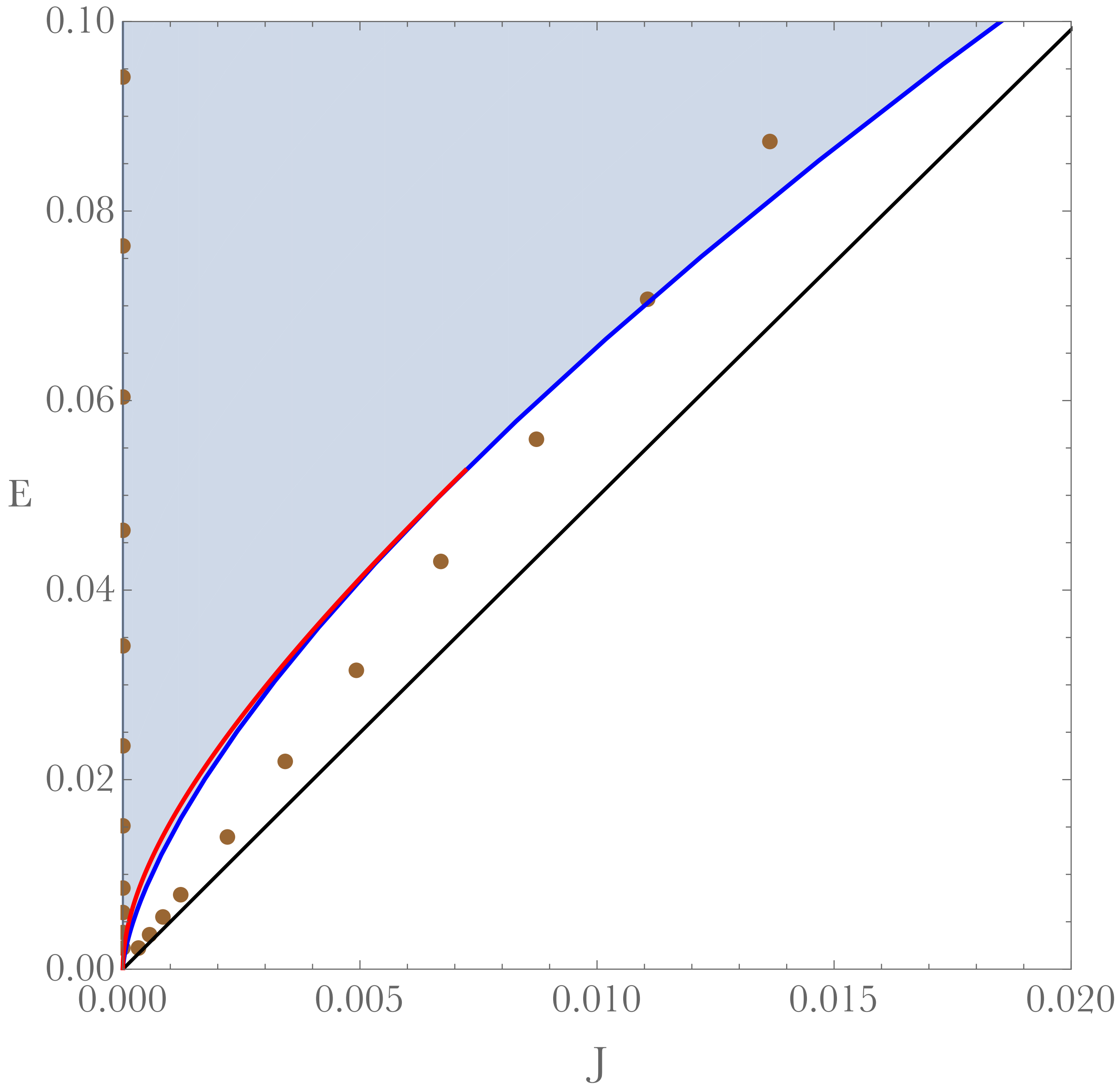
Stationary Solutions

- ‘Bald’ black holes: Myers-Perry-AdS
- (Single) Oscillators: Boson Stars
- Hairy black holes (with complex scalar hair).

Phase Diagram



Collapse Timescale



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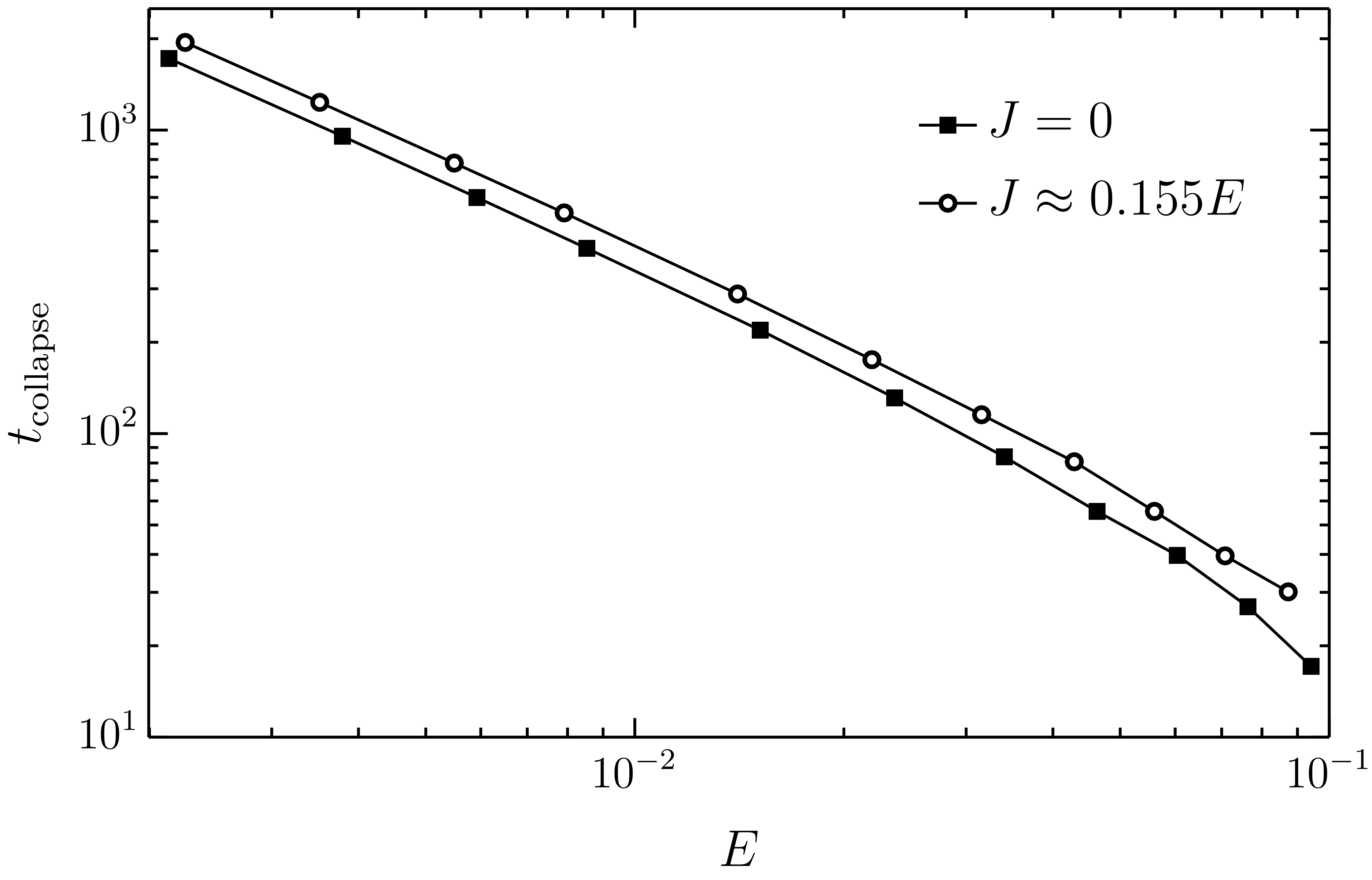
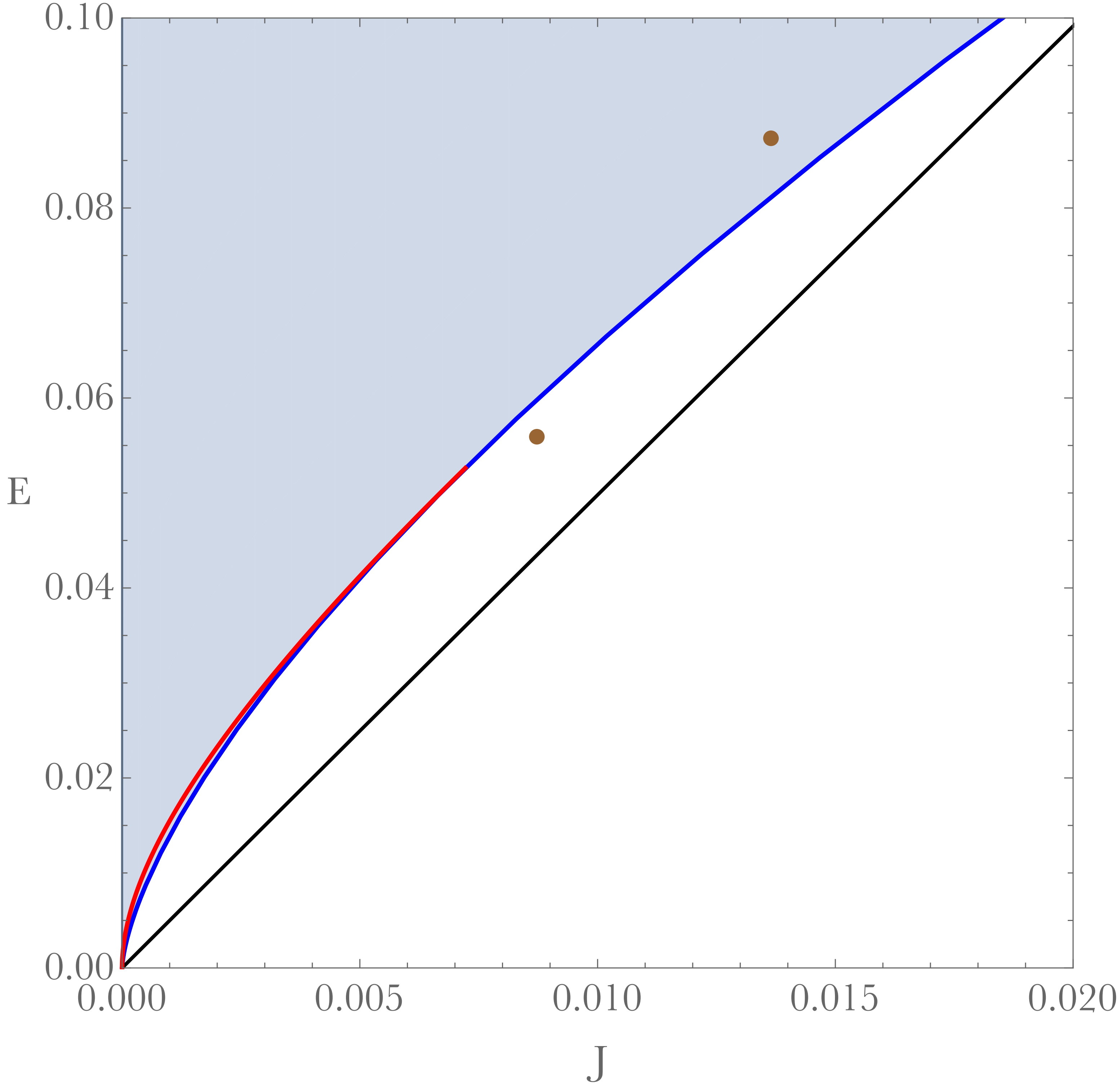


Final State



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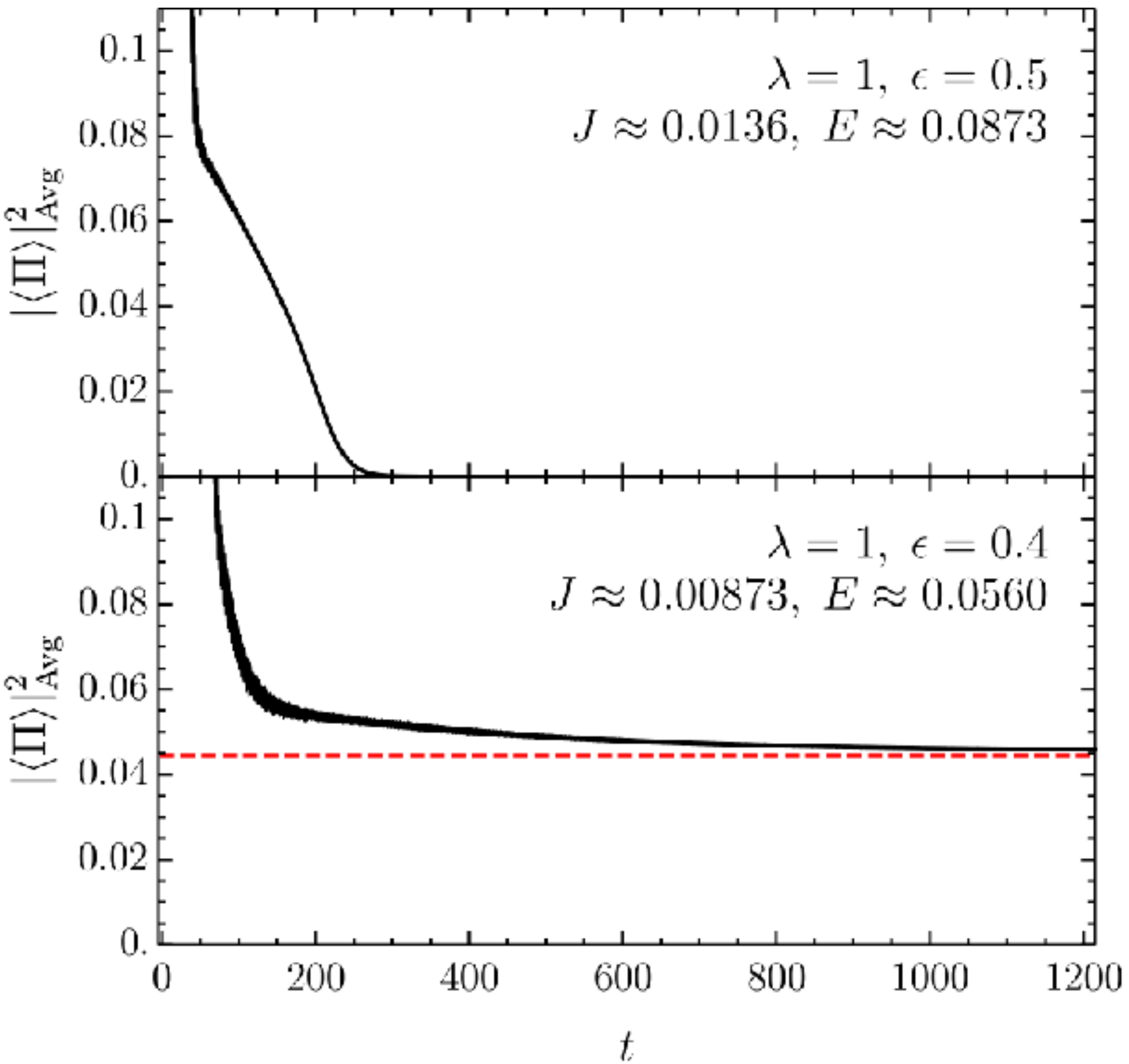

$\lambda = 1,\ \epsilon = 0.5$
$J \approx 0.0136,\ E \approx 0.0873$
$\lambda = 1,\ \epsilon = 0.4$
$J \approx 0.00873,\ E \approx 0.0560$
$|\langle \Pi \rangle|^2_{\mathrm{Avg}}$
$|\langle \Pi \rangle|^2_{\mathrm{Avg}}$
0.1
0.08
0.06
0.04
0.02
0.
0
200
400
600
800
1000
1200
t

Summary of Doublet Model Results

- In our runs, angular momentum delays collapse, but does not change the time scaling.
- Final state for small E and nonzero J is a hairy black hole.

Generic Final State

What is the generic final state of the AdS instability?

- Schwarzschild? Not generic.
- The hairy black hole? No, it's unstable. (Our calculation needed $m = 1$ symmetry.)
- Some other hairy black hole? No, these are also unstable. [S.R. Green, S. Hollands, A. Ishibashi, R.M. Wald]

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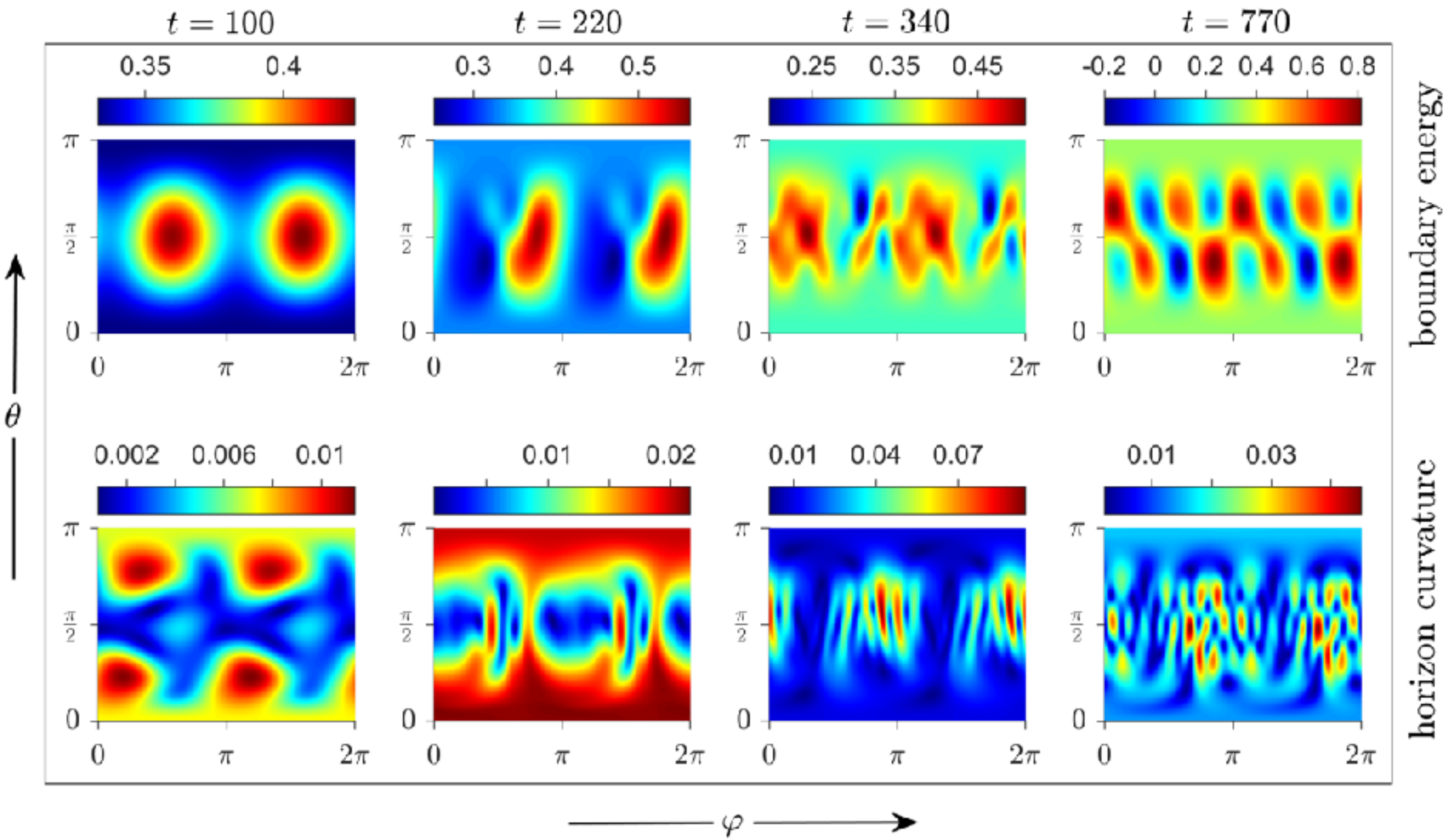
Our best guess: no stationary endpoint.

Perpetual evolution, maybe to ever smaller length scales.

Evolution of Superradiance



[P.M. Chesler, D.A. Lowe, arXiv: 1801.09711]

Some Speculation

Can the final 'endpoint' be constructed?

- Do multi-oscillating geons exist? [Same arguments of scalar field multi-oscillators apply, so it seems likely.]
- Are there black multi-resonators?
- If so, can they be stable?

Another possibility: evolution continues to ever smaller scales, essentially a violation of cosmic censorship!

Summary

- AdS is unstable to the formation of black holes, but not all initial data leads to collapse.
- Non-collapsing data intimately tied to oscillating solutions and normal modes.
- Space of noncollapsing data is infinite-dimensional and non-vanishing near AdS.
- Typical collapsing data forms horizons on a timescale of $t \sim 1/E$, including data with angular momenta.
- There may be no final endpoint because of superradiance instabilities.

Thank you