Noncommutative Solitons and Integrable Systems

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MH, ``Commuting Flows and Conservation Laws for NC Lax Hierarches,'' [hep-th/0311206]

cf. MH, ``Solitons on NC spaces''

to appear in 数研講究録 [牧場]

1. Introduction

- NC spaces: $[x^i, x^j] = i\theta^{ij}$
- NC gauge theories have been studied intensively.
- Distinguished features:
  - Resolution of singularities
    - → New physical objects
  - -- Easy treatment
    - → Various applications

NC solitons play central roles.

NC extension of soliton theories is worth studying.

Soliton equations in diverse dimensions

4	Anti-Self-Dual Yang-Mills eq. $F_{\mu\nu} = -\tilde{F}_{\mu\nu}$
3	Bogomol'nyi eq.
2 (+1)	KP eq. BCS eq. AKNS system ...
1 (+1)	KdV eq. mKdV eq Boussinesq eq. NLS eq. Burgers eq. sine-Gordon eq. Sawada-Kotera eq. ...

Ward's conjecture:

Almost all integrable equations are reductions of the ASDYM eqs.

ASDYM eq.

↓ Reductions

KP eq. BCS eq.
KdV eq. Boussinesq eq.
NLS eq. mKdV eq.
sine-Gordon eq.
Burgers eq. …
(Almost all !)

e.g. [Mason-Woodhouse]

NC Ward's conjecture:

Almost all NC integrable equations are reductions of the NC ASDYM eqs.

MH&K.Toda,PLA316('03)77[hepth/0211148]

NC ASDYM eq.

Successful

↓ Reductions ↓

NC KP eq. NC BCS eq.
NC KdV eq. NC Boussinesq eq.
NC NLS eq. NC mKdV eq.
NC sine-Gordon eq.
NC Burgers eq. …
(Almost all !?)

Successful?

Sato's theory may answer

2. NC Gauge Theory

- NC gauge theories are equivalent to ordinary gauge theories in background of magnetic fields.
- They are obtained from ordinary commutative gauge theories by replacing products of fields with star-products.
- The star product:

$$f(x)*g(x) := f(x)\exp\left(\frac{i}{2}\theta^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right)g(x)$$

$$= f(x)g(x) + \frac{i}{2}\theta^{ij}\partial_i f(x)\partial_j g(x) + O(\theta^2)$$

A deformed product

$$f*(g*h) = (f*g)*h$$ Associative

$$[x^i, x^j]_* := x^i * x^j - x^j * x^i = i\theta^{ij}$$ NC !

(Ex.) 4-dim. (Euclidean) G=U(N) Yang-Mills theory

- Action

$$S = \int d^4x\, Tr\, F_{\mu\nu} F^{\mu\nu}$$
$$= \int d^4x\, Tr \left(F_{\mu\nu} F_{\mu\nu} + \tilde{F}_{\mu\nu} \tilde{F}_{\mu\nu} \right)$$
$$= \int d^4x\, Tr \left[\left(F_{\mu\nu} \mp \tilde{F}_{\mu\nu} \right)^2 \pm 2 F_{\mu\nu} \tilde{F}_{\mu\nu} \right]$$

- Eq. Of Motion:

$$[D^\nu, [D_\nu, D_\mu]] = 0$$

- BPS eq. (=(A)SD eq.)

$$F_{\mu\nu} = \pm \tilde{F}_{\mu\nu}$$ → instantons

$$\updownarrow$$

$$F_{z_1 \bar{z}_1} + F_{z_2 \bar{z}_2} = 0, \quad F_{z_1 z_2} = 0$$

(Ex.) 4-dim. NC (Euclidean) G=U(N) Yang-Mills theory (All products are star products)

- Action

$$S = \int d^4x \, Tr \, F_{\mu\nu} * F^{\mu\nu}$$
$$= \int d^4x \, Tr \left(F_{\mu\nu} * F_{\mu\nu} + \tilde{F}_{\mu\nu} * \tilde{F}_{\mu\nu} \right)$$
$$= \int d^4x \, Tr \left[\left(F_{\mu\nu} \mp \tilde{F}_{\mu\nu} \right)_*^2 \pm 2 F_{\mu\nu} * \tilde{F}_{\mu\nu} \right]$$

- Eq. Of Motion:

$$[D^\nu, [D_\nu, D_\mu]_*]_* = 0$$

- BPS eq. (=(A)SD eq.)

$$F_{\mu\nu} = \pm \tilde{F}_{\mu\nu}$$ → NC instantons

$$\updownarrow$$

$$F_{z_1 \bar{z}_1} + F_{z_2 \bar{z}_2} = 0, \quad F_{z_1 z_2} = 0$$

ADHM construction of instantons

ADHM eq. (G=```U(k)”): k times k matrix eq.

ADHM data $B_{1,2} : k \times k,$

$I : k \times N, \quad J : N \times k$

1:1

Instatntons $A_\mu : N \times N$

ASD eq. (G=U(N), C_2=-k): N times N PDE

ADHM construction of BPST instantons (N=2,k=1)

ADHM eq. (G=``U(1)’’)

$$B_{1,2} = \alpha_{1,2}, \qquad I = (\rho, 0),\ J = \begin{pmatrix} 0 \\ \rho \end{pmatrix}$$

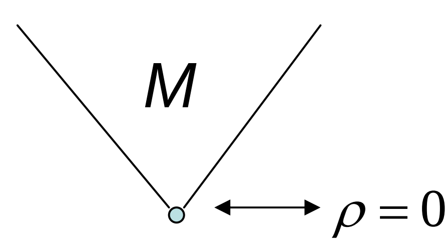
position   size

$$A_\mu = \frac{i(x-b)^\nu \eta^{(-)}_{\mu\nu}}{(x-b)^2 + \rho^2},\ F_{\mu\nu} = \frac{2i\rho^2}{((x-b)^2+\rho^2)^2}\eta^{(-)}_{\mu\nu}$$

ASD eq. (G=U(2), C_2=-1)

$\downarrow\ \rho \to 0$

singular



singularity

ADHM construction of NC BPST instantons (N=2,k=1)

ADHM eq. (G=``U(1)'')

$$B_{1,2} = \alpha_{1,2}, \qquad I = (\sqrt{\rho^2 + \zeta}, 0),\ J = \begin{pmatrix} 0 \\ \rho \end{pmatrix}$$

position ↕ $B_{1,2}$

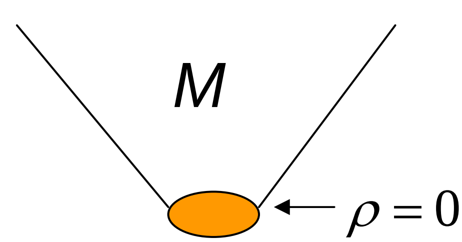
Size → slightly fat? ↕ I

$A_\mu, F_{\mu\nu}$: something smooth

↓ $\rho \to 0$

Regular!
(U(1) instanton!)

ASD eq. (G=U(2), C_2=-1)



Resolution of singularity

3. NC Sato's Theory

- Sato's Theory : one of the most beautiful theory of solitons
  - Based on the exsitence of hierarchies and tau-functions
- Sato's theory reveals essential aspects of solitons:
  - Construction of exact solutions
  - Structure of solution spaces
  - Infinite conserved quantities
  - Hidden infinite-dim. symmetry

Is it possible to extend it

to NC spaces ? → YES!

NC (KP) Hierarchy:

$$\frac{\partial L}{\partial x^m} = [B_m, L]_*$$

$$\partial_m u_2 \partial_x^{-1} + \quad F_{m2}(u)\partial_x^{-1} +$$

$$\partial_m u_3 \partial_x^{-2} + \quad F_{m3}(u)\partial_x^{-2} +$$

$$\partial_m u_4 \partial_x^{-3} + \cdots \quad F_{m4}(u)\partial_x^{-3} + \cdots$$

Huge amount of ```NC evolution equations''

$$L := \partial_x + u_2 \partial_x^{-1} + u_3 \partial_x^{-2} + u_4 \partial_x^{-3} + \cdots$$

$$B_m := (\underbrace{L * \cdots * L}_{m\ times})_{\geq 0}$$

$$B_1 = \partial_x$$

$$B_2 = \partial_x^2 + 2u_2$$

$$B_3 = \partial_x^3 + 3u_2\partial_x + 3(u_3 + u_2')$$

$$u_k = u_k(x^1, x^2, x^3, \cdots) \qquad [x^i, x^j] = i\theta^{ij}$$

Noncommutativity is introduced here

Negative powers of differential operators

$$\partial_x^n \circ f := \sum_{j=0}^{\infty} \binom{n}{j} (\partial_x^j f) \partial_x^{n-j}$$

$$\binom{n}{j} = \frac{n(n-1)(n-2)\cdots(n-(j-1))}{j(j-1)(j-2)\cdots 1}$$

$$\partial_x^3 \circ f = f\partial_x^3 + 3f'\partial_x^2 + 3f''\partial_x^1 + f'''$$

$$\partial_x^2 \circ f = f\partial_x^2 + 2f'\partial_x + f''$$

$$\partial_x^{-1} \circ f = f\partial_x^{-1} - f'\partial_x^{-2} + f''\partial_x^{-3} - \cdots$$

$$\partial_x^{-2} \circ f = f\partial_x^{-2} - 2f'\partial_x^{-3} + 3f''\partial_x^{-4} - \cdots$$

Star product

$$f(x) * g(x) := f(x) \exp\left(\frac{i}{2}\theta^{ij}\overleftarrow{\partial}_i \overrightarrow{\partial}_j\right) g(x)$$

which makes theories "noncommutative":

$$[x^i, x^j]_* := x^i * x^j - x^j * x^i = i\theta^{ij}$$

Closer look at NC (KP) hierarchy

For m=2

$$\partial_x^{-1})\quad \partial_2 u_2 = 2\underline{u_3'} + u_2''$$

$$\partial_x^{-2})\quad \partial_2 u_3 = 2\underline{u_4'} + u_3'' + 2u_2 * u_2' + 2[u_2, u_3]_*$$

$$\partial_x^{-3})\quad \partial_2 u_4 = 2\underline{u_5'} + u_4'' + 4u_3 * u_2' - 2u_2 * u_2'' + 2[u_2, u_4]_*$$

$$\vdots$$

Infinite kind of fields are represented in terms of one kind of field $u_2 \equiv u$

MH&K.Toda, [hep-th/0309265]

For m=3

$$\partial_x^{-1})\quad \partial_3 u_2 = u_2''' + 3u_3'' + 3u_4'' + 3u_2' * u_2 + 3u_2 * u_2'$$

$$\vdots$$

$$\downarrow$$

$$u_t = \frac{1}{4}u_{xxx} + \frac{3}{4}(u * u)_x + \frac{3}{4}\partial_x^{-1}u_{yy} + \frac{3}{4}[u, \partial_x^{-1}u_{yy}]_*$$

NC KP equation

$$u_x := \frac{\partial u}{\partial x} \qquad u = u(x^1, x^2, x^3, \cdots)$$

$$\partial_x^{-1} := \int^x dx' \quad etc. \qquad x^1 \leftrightarrow x,\; x^2 \leftrightarrow y,\; x^3 \leftrightarrow t$$

(KP hie.) → (various hies.)

- KdV hierarchy

Reduction condition

$$L^2 = B_2 (=: \partial_x^2 + u)$$: 2-reduction

gives rise to NC KdV hierarchy

which includes NC KdV eq.:

$$u_t = \frac{1}{4} u_{xxx} + \frac{3}{4} (u * u)_x$$

Note

$$\frac{\partial u}{\partial x_{2l}} = 0$$: dimensional reduction

l -reduction yields other NC hierarchies which include NC Boussinesq, coupled KdV Sawada-Kotera, mKdV hierarchies and so on.

NC Burgers hierarchy

MH&K.Toda,JPA36('03)11981[hepth/0301213]

- NC (1+1)-dim. Burgers equation:

$$\dot{u} = u'' + 2u * u'$$

: Non-linear&

Infinite order diff. eq. w.r.t. time !

↓ NC Cole-Hopf transformation

$$u = \tau^{-1} * \tau'$$

$$\dot{\tau} = \tau''$$

: Linear &

first order diff. eq. w.r.t. time

Integrable !

NC Burgers eq. can be derived from G=U(1) NC ASDYM eq. (One example of NC Ward conjecture)

4. Conserved Quantities

- We have obtained wide class of NC hierarchies and NC (soliton) equations.

- Are they integrable or special

  from viewpoints of soliton theories? → YES !

  Now we show the existence of infinite number of conserved quatitities which suggests a hidden infinite-dimensional symmetry.

Conservation Laws

- Conservation laws:

$$\partial_t \sigma = \partial_i J^i \qquad \sigma$$: Conserved density

Then $Q := \int_{space} dx\,\sigma$ is a conserved quantity.

$$\because \partial_t Q = \int_{space} dx \partial_t \sigma = \int_{\substack{spatial \\ \inf inity}} dS_i J^i = 0$$

Conservation laws for hierarchies

Follwing G.Wilson's approach, we have:

$$\partial_m res_{-1} L^n = \partial_x J + \underline{[A, B]_*} = \partial_x J + \underline{\theta^{ij} \partial_j \Xi_i}$$

troublesome

We have succeeded in the explicit evaluation !

Noncommutativity should be introduced
in space-time directions only.

$res_{-r} L^n$: coefficient of ∂_x^{-r} in L^n

Main Results

Conserved densities for NC hierarchies (n=1,2,...)

$$\sigma = res_{-1}L^n + \theta^{mi}\sum_{k=1}^{m}\sum_{l=n+1}^{n+1+m-k}(-1)^{m+n-k-l}\binom{m-k}{l-n-1}\times a_{n-l}\lozenge\partial_i b_{m-k}^{(m+n-k-l+1)}$$

$$t \equiv x^m$$

$$L^n = \partial_x^n + \sum_{l=1}^{\infty} a_{n-l}\partial_x^{n-l}$$

$$B_m = \partial_x^m + \sum_{k=1}^{\infty} b_{m-k}\partial_x^{m-k}$$

$\lozenge$: Strachan's product

$$f(x)\lozenge g(x) := f(x)\left(\sum_{s=0}^{\infty}\frac{(-1)^s}{(2s+1)!}\left(\frac{1}{2}\theta^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right)^{2s}\right)g(x)$$

Example: NC KP and KdV hierarchies $([t,x]=i\theta)$

$$\sigma = res_{-1}L^n - 3\theta((res_{-1}L^n)\lozenge u_3' + (res_{-2}L^n)\lozenge u_2')$$

5. Conclusion and Discussion

- We proved the existence of infinite conserved quantities for wide class of NC hierarchies, which is one of the definitions of integrabilities of field equations, and gave the infinite conserved densities explicitly.

- Our results strongly suggest that infinite-dim. symmetry would be hidden in NC (soliton) equations.
  → theories of tau-functions
  (Hirota's bilinearization)
  → the completion of NC Sato's theory

- The interpretation of space-time noncommutativity should be clarified.

The interplay between mathematics and physics is important and fruitful.
(NLYH is wonderful!)

- Mirror symmetry
- Seiberg-Witten's theory (Nekrasov's conjecture)
- The next is …

Noncommutative Solitons !?