

Non-Commutative Solitons and Integrable Equations

Masashi HAMANAKA

(Tokyo U. present) → (Nagoya U. from Feb. 2004)

MH, “Commuting Flows and Conservation
Laws for NC Lax Hierarches,” [[hep-th/0311206](#)]

cf. MH, “NC Solitons and D-branes,”

Ph.D thesis (2003) [[hep-th/0303256](#)]

1. Introduction

- Non-Commutative (NC) spaces are defined by noncommutativity of the coordinates:

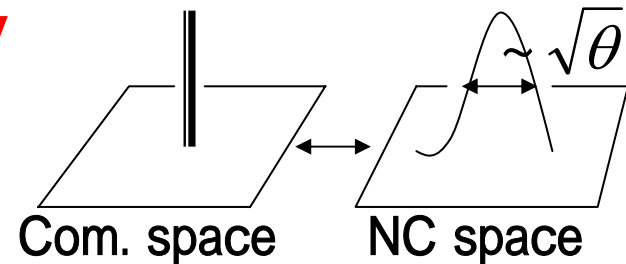
$$[x^i, x^j] = i\theta^{ij} \quad \theta^{ij} : \text{NC parameter}$$

This looks like CCR in QM: $[q, p] = i\hbar$

(\rightarrow "space-space uncertainty relation" \rightarrow)

Resolution of singularity

(\rightarrow New physical objects)



e.g. resolution of small instanton singularity

(\rightarrow U(1) instantons)

NC gauge theories



Com. gauge theories
in background of
magnetic fields

(real physics)

- Gauge theories are realized on D-branes which are solitons in string theories
- In this context, NC solitons are (lower-dim.) D-branes

Analysis of NC solitons



Analysis of D-branes

(easy)

Various successful applications

e.g. confirmation of Sen's conjecture on decay of D-branes

NC extension of soliton theories
are worth studying !

Soliton equations in diverse dimensions

4	Anti-Self-Dual Yang-Mills eq. (instantons) $F_{\mu\nu} = -\tilde{F}_{\mu\nu}$	NC extension (Successful)
3	Bogomol'nyi eq. (monopoles)	NC extension (Successful)
2 (+1)	KP eq. BCS eq. DS eq. ...	NC extension (This talk)
1 (+1)	KdV eq. Boussinesq eq. NLS eq. Burgers eq. sine-Gordon eq. Sawada-Kotera eq	NC extension (This talk)



Dim. of space

Ward's observation:

Almost all integrable equations are reductions of the ASDYM eqs.

R.Ward, Phil.Trans.Roy.Soc.Lond.A315(85)451

ASDYM eq.

↓ Reductions

KP eq. BCS eq.
KdV eq. Boussinesq eq.
NLS eq. mKdV eq.
sine-Gordon eq. Burgers eq. ...

(Almost all !)

e.g. [Mason&Woodhouse]

NC Ward's observation: Almost all
NC integrable equations are
reductions of the **NC** ASDYM eqs.

MH&K.Toda, PLA316(03)77[hep-th/0211148]

NC ASDYM eq.

Successful

↓ Reductions

NC KP eq. **NC** BCS eq.

NC KdV eq. **NC** Boussinesq eq.

NC NLS eq. **NC** mKdV eq.

NC sine-Gordon eq. **NC** Burgers eq. ...

(Almost all !?)

Successful?

Sato's theory may answer

Plan of this talk

1. Introduction
2. NC Gauge Theory
3. NC Sato's Theory
4. Conservation Laws
5. Conclusion and Discussion

2. NC Gauge Theory

Here we discuss NC gauge theory of instantons.
 (Ex.) 4-dim. (Euclidean) $G=U(N)$ Yang-Mills theory

- Action

$$\begin{aligned}
 S &= -\frac{1}{2} \int d^4x \text{Tr} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} \int d^4x \text{Tr} (F_{\mu\nu} F_{\mu\nu} + \tilde{F}_{\mu\nu} \tilde{F}_{\mu\nu}) \\
 &= -\frac{1}{4} \int d^4x \text{Tr} \left[\underbrace{(F_{\mu\nu} \mp \tilde{F}_{\mu\nu})^2}_{=0 \Leftrightarrow \text{BPS}} \pm \underbrace{2F_{\mu\nu} \tilde{F}_{\mu\nu}}_{\Leftrightarrow C_2} \right]
 \end{aligned}$$

- Eq. Of Motion:

$$[D^\nu, [D_\nu, D_\mu]] = 0$$

$$(F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])$$

- BPS eq. (= (A)SDYM eq.)

$$F_{\mu\nu} = \pm \tilde{F}_{\mu\nu} \quad \rightarrow \text{instantons}$$

$$(\Leftrightarrow F_{z_1 \bar{z}_1} + F_{z_2 \bar{z}_2} = 0, \quad F_{z_1 z_2} = 0)$$

(Q) How we get NC version of the theories?

(A) They are obtained from ordinary commutative gauge theories by replacing products of fields

with star-products: $f(x)g(x) \rightarrow f(x) * g(x)$

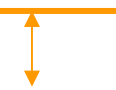
• The star product:

$$f(x) * g(x) := f(x) \exp\left(\frac{i}{2} \theta^{ij} \overleftarrow{\partial}_i \overrightarrow{\partial}_j\right) g(x) = f(x)g(x) + i \frac{\theta^{ij}}{2} \partial_i f(x) \partial_j g(x) + O(\theta^2)$$

$$f * (g * h) = (f * g) * h \quad \text{Associative}$$

$$[x^i, x^j]_* := x^i * x^j - x^j * x^i = i\theta^{ij} \quad \text{NC !}$$

A deformed product



Presence of background magnetic fields

In this way, we get NC-deformed theories with infinite derivatives in NC directions. (integrable???)

(Ex.) 4-dim. NC (Euclidean) $G=U(N)$

Yang-Mills theory

(All products are star products)

- Action

$$\begin{aligned}
 S &= -\frac{1}{2} \int d^4x \text{Tr} F_{\mu\nu} * F^{\mu\nu} = -\frac{1}{4} \int d^4x \text{Tr} (F_{\mu\nu} * F_{\mu\nu} + \tilde{F}_{\mu\nu} * \tilde{F}_{\mu\nu}) \\
 &= -\frac{1}{4} \int d^4x \text{Tr} \left[\underbrace{(F_{\mu\nu} \mp \tilde{F}_{\mu\nu})^2}_{=0 \Leftrightarrow \text{BPS}} \pm \underbrace{2F_{\mu\nu} * \tilde{F}_{\mu\nu}}_{\Leftrightarrow C_2} \right] \\
 &\quad (F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + \underbrace{[A_\mu, A_\nu]_*})
 \end{aligned}$$

- Eq. Of Motion:

$$[D^\nu, [D_\nu, D_\mu]_*]_* = 0$$

Don't omit even for $G=U(1)$

- BPS eq. (=NC (A)SDYM eq.)

($\because U(1) \cong U(\infty)$)

$$F_{\mu\nu} = \pm \tilde{F}_{\mu\nu} \quad \rightarrow \text{NC instantons}$$

$$(\Leftrightarrow F_{z_1 \bar{z}_1} + F_{z_2 \bar{z}_2} = 0, \quad F_{z_1 z_2} = 0)$$

ADHM construction of instantons

Atiyah-Drinfeld-Hitchin-Manin, PLA65(78)185

ADHM eq. ($G=U(k)$): k times k matrix eq.

D-brane s
interpretation

Douglas, Witten

$$[B_1, B_1^*] + [B_2, B_2^*] + I I^* - J^* J = 0$$

$$[B_1, B_2] + I J = 0$$

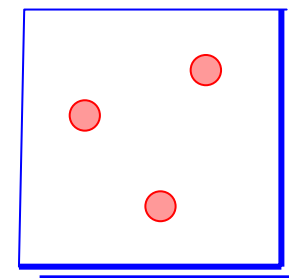
BPS

ADHM data $B_{1,2} : k \times k, I : k \times N, J : N \times k$

k D0-branes

1:1

Instantons $A_\mu : N \times N$



N D4-branes

ASD eq. ($G=U(N), C_2=-k$): N times N PDE

$$F_{z_1 \bar{z}_1} + F_{z_2 \bar{z}_2} = 0$$

$$F_{z_1 z_2} = 0$$

BPS

String theory is a treasure house of dualities

ADHM construction of BPST instanton (N=2,k=1)

ADHM eq. (G=U(1))

$$[B_1, B_1^*] + [B_2, B_2^*] + I I^* - J^* J = 0$$

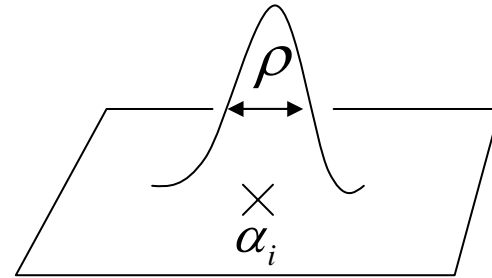
$$[B_1, B_2] + I J = 0$$

$$B_{1,2} = \alpha_{1,2}, \quad I = (\rho, 0), \quad J = \begin{pmatrix} 0 \\ \rho \end{pmatrix}$$

\updownarrow \updownarrow \updownarrow
position **size**

$$A_\mu = \frac{i(x-b)^\nu \eta_{\mu\nu}^{(-)}}{(x-b)^2 + \rho^2}, \quad F_{\mu\nu} = \frac{2i\rho^2}{((x-b)^2 + \rho^2)^2} \eta_{\mu\nu}^{(-)}$$

Final remark: matrices B and coords. z always appear in pair: z-B

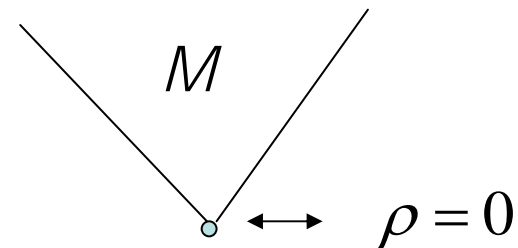


$\rho \rightarrow 0$ → singular

ASD eq. (G=U(2), C2=-1)

$$F_{z_1 \bar{z}_1} + F_{z_2 \bar{z}_2} = 0$$

$$F_{z_1 z_2} = 0$$



Small instanton singularity

ADHM construction of NC BPST instanton (N=2, k=1)

Nekrasov&Schwarz,
CMP198(98)689
[hep-th/9802068]

ADHM eq. (G=U(1)) 1 times 1 matrix eq.

$$[B_1, B_1^*] + [B_2, B_2^*] + I I^* - J^* J = \zeta$$

$$[B_1, B_2] + I J = 0$$

$$B_{1,2} = \alpha_{1,2}, \quad I = (\sqrt{\rho^2 + \zeta}, 0), \quad J = \begin{pmatrix} 0 \\ \rho \end{pmatrix}$$

position size \rightarrow slightly fat?

$A_\mu, F_{\mu\nu}$: something smooth

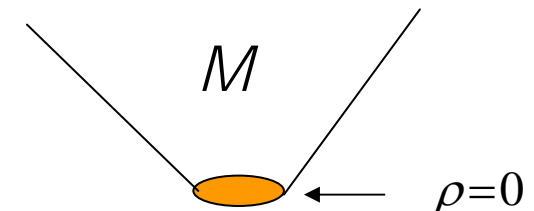
\longrightarrow
 $\rho \rightarrow 0$

Regular!
(U(1) instanton!)

ASD eq. (G=U(2), C₂=-1)

$$F_{z_1 \bar{z}_1} + F_{z_2 \bar{z}_2} = 0$$

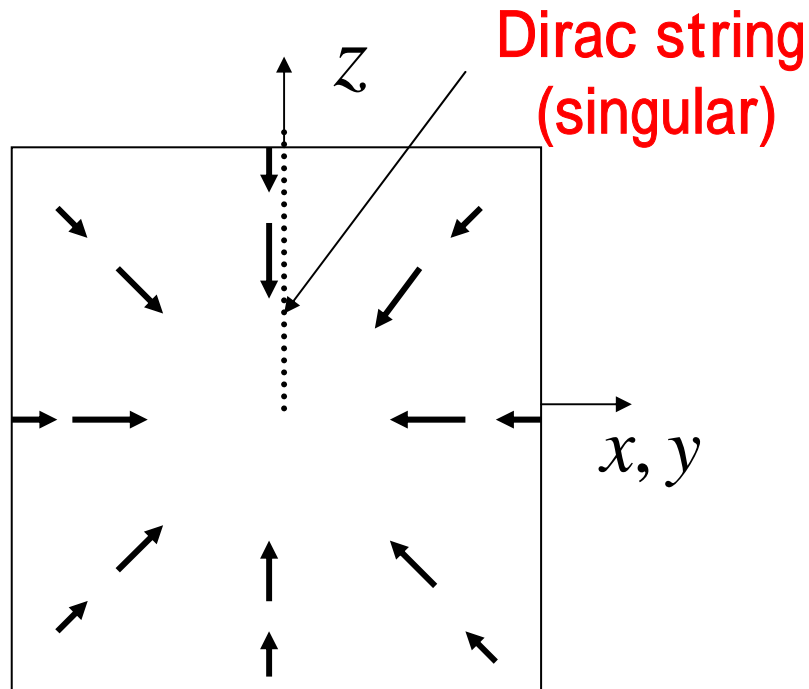
$$F_{z_1 z_2} = 0$$



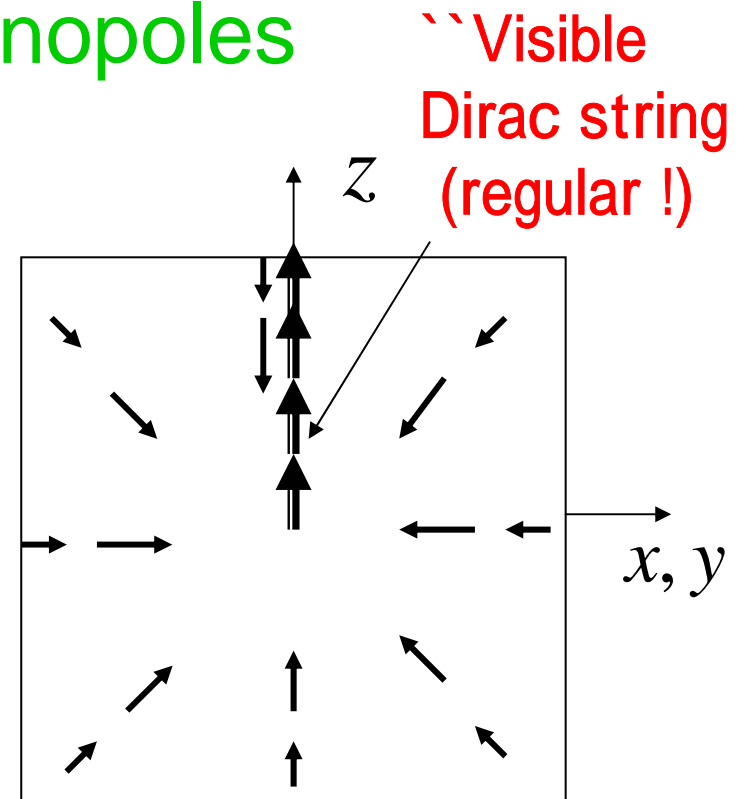
Resolution of the singularity

NC monopoles are also interesting

- Magnetic flux of Dirac monopoles



On commutative space



On NC space

ADHMN construction works well.

Moduli space is the same as commutative one.

Gross&Nekrasov, JHEP
[hep-th/0005204]

3. NC Sato's Theory

- Sato's Theory : one of the most beautiful theory of solitons
 - Based on the existence of hierarchies and tau-functions
- Sato's theory reveals essential aspects of solitons:
 - Construction of exact solutions
 - Structure of solution spaces
 - Infinite conserved quantities
 - Hidden infinite-dim. Symmetry

Let's discuss NC extension of Sato's theory

NC (KP) Hierarchy:

$$\frac{\partial L}{\partial x^m} = [B_m, L]_*$$

$$\begin{aligned} \partial_m u_2 \partial_x^{-1} + & F_{m2}(u) \partial_x^{-1} + \\ \partial_m u_3 \partial_x^{-2} + & F_{m3}(u) \partial_x^{-2} + \\ \partial_m u_4 \partial_x^{-3} + \dots & F_{m4}(u) \partial_x^{-3} + \dots \end{aligned}$$

Huge amount of ``NC evolution equations'' ($m=1,2,3,\dots$)

$$\begin{aligned} L &:= \partial_x + u_2 \partial_x^{-1} + u_3 \partial_x^{-2} + u_4 \partial_x^{-3} + \dots & B_1 &= L_{\geq 0} = \partial_x \\ B_m &:= (L * \dots * L)_{\geq 0} & B_2 &= L_{\geq 0}^2 = \partial_x^2 + 2u_2 \\ &\quad m \text{ times} & B_3 &= L_{\geq 0}^3 = \partial_x^3 + 3u_2 \partial_x + 3(u_3 + u_2') \\ u_k &= u_k(x^1, x^2, x^3, \dots) \end{aligned}$$

Noncommutativity is introduced here: $[x^i, x^j] = i\theta^{ij}$

Negative powers of differential operators

$$\partial_x^n \circ f := \sum_{j=0}^{\infty} \binom{n}{j} (\partial_x^j f) \partial_x^{n-j}$$

$$\frac{n(n-1)(n-2)\cdots(n-(j-1))}{j(j-1)(j-2)\cdots 1}$$

: binomial coefficient
which can be extended
to negative n
→ negative power of
differential operator
(well-defined !)

$$\partial_x^3 \circ f = f\partial_x^3 + 3f\partial_x^2 + 3f\partial_x + f''''$$

$$\partial_x^2 \circ f = f\partial_x^2 + 2f\partial_x + f''$$

$$\partial_x^{-1} \circ f = f\partial_x^{-1} - f\partial_x^{-2} + f''\partial_x^{-3} - \dots$$

$$\partial_x^{-2} \circ f = f\partial_x^{-2} - 2f\partial_x^{-3} + 3f''\partial_x^{-4} - \dots$$

Star product:

$$f(x) * g(x) := f(x) \exp\left(\frac{i}{2} \theta^{ij} \overleftarrow{\partial}_i \overrightarrow{\partial}_j\right) g(x)$$

which makes theories "noncommutative":

$$[x^i, x^j]_* := x^i * x^j - x^j * x^i = i\theta^{ij}$$

Closer look at NC (KP) hierarchy

For $m=2$

$$\partial_x^{-1}) \quad \partial_2 u_2 = \underline{2u_3'} + u_2''$$

$$\partial_x^{-2}) \quad \partial_2 u_3 = \underline{2u_4'} + u_3'' + 2u_2 * u_2' + 2[u_2, u_3]_*$$

$$\partial_x^{-3}) \quad \partial_2 u_4 = \underline{2u_5'} + u_4'' + 4u_3 * u_2' - 2u_2 * u_2'' + 2[u_2, u_4]_*$$

⋮

Infinite kind of fields are represented
in terms of one kind of field $u_2 \equiv u$

MH&K.Toda, [hep-th/0309265]

$$u_x := \frac{\partial u}{\partial x}$$

$$\partial_x^{-1} := \int^x dx'$$

For $m=3$

$$\partial_x^{-1}) \quad \partial_3 u_2 = u_2''' + 3u_3'' + 3u_4'' + 3u_2' * u_2 + 3u_2 * u_2'$$

⋮



$$u_t = \frac{1}{4} u_{xxx} + \frac{3}{4} (u_x * u + u * u_x) + \frac{3}{4} \partial_x^{-1} u_{yy} + \frac{3}{4} [u, \partial_x^{-1} u_{yy}]_*$$

(2+1)-dim.
NC KP equation

and other NC equations
(NC hierarchy equations)

$$u = u(x^1, x^2, x^3, \dots)$$

$$\begin{array}{ccc} \updownarrow & \updownarrow & \updownarrow \\ x & y & t \end{array}$$

etc.

(KP hierarchy) \rightarrow (various hierarchies.)

- (Ex.) KdV hierarchy

Reduction condition

$$L^2 = B_2 (=:\partial_x^2 + u) \quad : \text{2-reduction}$$

gives rise to NC KdV hierarchy

which includes (1+1)-dim. NC KdV eq.:

$$u_t = \frac{1}{4}u_{xxx} + \frac{3}{4}(u_x * u + u * u_x)$$

Note $\frac{\partial u}{\partial x_{2N}} = 0$: dimensional reduction in x_{2N} directions

/-reduction yields wide class of other NC hierarchies which include NC Boussinesq, coupled KdV, Sawada-Kotera, mKdV hierarchies and so on.

NC Burgers hierarchy

MH&K.Toda,JPA36(03)11981[hep-th/0301213]

- NC (1+1)-dim. Burgers equation:

$$\dot{u} = u'' + 2u * u' : \text{Non-linear \&}$$

Infinite order diff. eq. w.r.t. time ! (Integrable?)

NC Cole-Hopf transformation

$$u = \tau^{-1} * \tau' \quad (\xrightarrow{\theta \rightarrow 0} \partial_x \log \tau)$$

(NC) Diffusion equation:

$$\dot{\tau} = \tau'' : \text{Linear \& first order diff. eq. w.r.t. time}$$

(Integrable !)

NC Burgers eq. can be derived from $G=U(1)$ NC ASDYM eq.
(One example of NC Ward's observation)

NC Ward's observation (NC NLS eq.)

- Reduced ASDYM eq.: $x^\mu \rightarrow (t, x)$

Legare,
[hep-th/0012077]

$$(i) \quad B' = 0$$

$$(ii) \quad C' - \dot{A} + [A, C]_* = 0$$

$$(iii) \quad A' - \dot{B} + [C, B]_* = 0$$

A, B, C: 2 times 2
matrices (gauge fields)



Further
Reduction:

$$A = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}, B = \frac{i}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, C = -i \begin{pmatrix} q * \bar{q} & q' \\ q' & -\bar{q} * q \end{pmatrix}$$

$$(ii) \Rightarrow \begin{pmatrix} 0 & i\dot{q} - q'' - 2q * \bar{q} * q \\ i\dot{\bar{q}} + \bar{q}'' + 2\bar{q} * q * \bar{q} & 0 \end{pmatrix} = 0$$

NOT traceless

$$i\dot{q} = q'' + 2q * \bar{q} * q \quad : \text{NC NLS eq.}$$

Note: $A, B, C \in gl(2, C) \xrightarrow{\theta \rightarrow 0} sl(2, C)$

**U(1) part is
important**

NC Ward's observation (NC Burgers eq.)

- Reduced ASDYM eq.: $x^\mu \rightarrow (t, x)$

MH&K.Toda, JPA
[hep-th/0301213]

$$G=U(1)$$

$$(i) \quad \dot{A} + \underline{[B, A]}_* = 0$$

$$(ii) \quad \dot{C} - B' + \underline{[B, C]}_* = 0$$

A, B, C: 1 times 1
matrices (gauge fields)



Further

Reduction: $A = 0, B = u' - u^2, C = u$

should remain

$$(ii) \Rightarrow \dot{u} = u'' + 2u' * u \quad : \text{NC Burgers eq.}$$

Note: Without the commutators $[,]$, (ii) yields:

$$\dot{u} = u'' + \underline{u' * u + u * u'} \quad : \text{neither linearizable nor Lax form}$$

symmetric

4. Conservation Laws

- We have obtained wide class of NC hierarchies and NC (soliton) equations.
- Are they **integrable** or **special** from viewpoints of soliton theories? → **YES !**

Now we show **the existence of infinite number of conserved quantities** which suggests a hidden infinite-dimensional symmetry.

Conservation Laws

- Conservation laws: $\overset{\text{time}}{\partial_t} \sigma = \overset{\text{space}}{\partial_i} J^i$ σ : Conserved density

Then $Q := \int_{space} dx \sigma$ is a conserved quantity.

$$\because \partial_t Q = \int_{space} dx \partial_t \sigma = \int_{spatial} dS_i J^i = 0$$

Conservation laws for the hierarchies

Following G.Wilson's approach, we have:

$$\partial_m res_{-1} L^n = \partial_x J + \underbrace{[A, B]_*}_{\text{troublesome}} = \partial_x J + \underbrace{\theta^{ij} \partial_j \Xi_i}_{\text{I have succeeded in the evaluation explicitly!}}$$

$res_{-r} L^n$:
coefficient
of ∂_x^{-r} in L^n

↑
troublesome

↑
I have succeeded in the evaluation explicitly !

∂_j should be space or time derivative



Noncommutativity should be introduced in space-time directions only.

Main Results

Infinite conserved densities for NC hierarchy eqs. ($n=1,2,\dots$)

$$\sigma = \text{res}_{-1} L^n + \theta^{mi} \sum_{k=1}^m \sum_{l=n+1}^{n+1+m-k} (-1)^{m+n-k-l} \binom{m-k}{l-n-1} a_{n-l} \diamond \partial_i b_{m-k}^{(m+n-k-l+1)}$$

$$t \equiv x^m \quad L^n = \partial_x^n + \sum_{l=1}^{\infty} a_{n-l} \partial_x^{n-l}, \quad B_m = \partial_x^m + \sum_{k=1}^{\infty} b_{m-k} \partial_x^{m-k}$$

\diamond : Strachan's product

$$f(x) \diamond g(x) := f(x) \left(\sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)!} \left(\frac{1}{2} \theta^{ij} \bar{\partial}_i \bar{\partial}_j \right)^{2s} \right) g(x)$$

MH, [hep-th/0311206]

Example: NC KP and KdV equations ($[t, x] = i\theta$)

$$\sigma = \text{res}_{-1} L^n - 3\theta((\text{res}_{-1} L^n) \diamond u'_3 + (\text{res}_{-2} L^n) \diamond u'_2) \quad : \text{meaningful ?}$$

5. Conclusion and Discussion

- We proved the existence of infinite conserved quantities for wide class of NC hierarchies and gave the infinite conserved densities explicitly.
- Our results strongly suggest that infinite-dim. symmetry would be hidden in NC (soliton) equations.
What is it ? ← theories of tau-functions are needed
(via e.g. Hirota's bilinearization $u = \partial_x^2 \log \tau$)
→ the completion of NC Sato's theory
- What is space-time noncommutativity ?
- What is the twistor descriptions ? → NC twistor theory
e.g. Kapustin&Kuznetsov&Orlov, Hannabuss, Hannover group,...

There are many things to be seen.