Non-commutative Solitons and Integrable Equations

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(Tokyo U. present)
(Nagoya U. from Feb. 2004)

MH, ``Commuting Flows and Conservation Laws for NC Lax Hierarches,'' [hep-th/0311206]

cf. MH,``Noncommutative Solitons and D-branes,'' Ph.D thesis [hepth/0303256]

1. Introduction

- Noncommutative spaces are defined by noncommutativity of spatial coordinates:

$$[x^i, x^j] = i\theta^{ij}$$

This looks like CCR in QM:

$$[q, p] = i\hbar$$

$\downarrow$

``space-space uncertainly relation''

$\downarrow$

Resolution of singulality

(→ New physical objects)

e.g. resolution of small instanton singularity

(→ U(1) instantons)

NC gauge theories

|| → (real physics)

Commutative gauge theories in background of magnetic fields

- Gauge theories are realized on D-branes which are solitons in string theories
- In this contect, NC solitons are (lower-dim.) D-branes

Analysis of NC solitons (easy)

↕

Analysis of D-branes

Various successful applications

e.g. confirmation of Sen's conjecture on decay of D-branes

NC soliton theories are worthwhile !

Soliton equations in diverse dimensions

4	Anti-Self-Dual Yang-Mills eq. (instantons) $F_{\mu\nu} = -\tilde{F}_{\mu\nu}$ NC extension (Successful)
3	Bogomol'nyi eq. (monopoles) NC extension (Successful)
2 (+1)	KP eq. BCS eq. 2-dim. AKNS system … NC extension (This talk)
1 (+1)	KdV eq. mKdV eq Boussinesq eq. NLS eq. Burgers eq. sine-Gordon eq. Sawada-Kotera eq. … NC extension (This talk)

Dim. of space

Ward's observation:

Almost all integrable equations are reductions of the ASDYM eqs.

R.Ward,Phil.Trans.Roy.Soc.Lond.A315('85)451

ASDYM eq.

$\downarrow$ Reductions

KP eq. BCS eq.
KdV eq. Boussinesq eq.
NLS eq. mKdV eq.
sine-Gordon eq.
Burgers eq. …
(Almost all !)

e.g. [Mason & Woodhouse]

NC Ward's observation: Almost all NC integrable equations are reductions of the NC ASDYM eqs.

MH&K.Toda,PLA316('03)77[hepth/0211148]

NC ASDYM eq.

Successful

Reductions

NC KP eq. NC BCS eq.
NC KdV eq. NC Boussinesq eq.
NC NLS eq. NC mKdV eq.
NC sine-Gordon eq.
NC Burgers eq. …
(Almost all !?)

Successful?

Sato's theory may answer

Plan of this talk

1. Introduction
2. NC Gauge Theory
3. NC Sato’s Theory
4. Conservation Laws
5. Conclusion and Discussion

2. NC Gauge Theory

- NC gauge theories are equivalent to ordinary gauge theories in background of magnetic fields.
- They are obtained from ordinary commutative gauge theories by replacing products of fields with star-products.
- The star product:

$$f(x) * g(x) := f(x) \exp\left(\frac{i}{2} \theta^{ij} \overleftarrow{\partial}_i \overrightarrow{\partial}_j \right) g(x)$$

$$= f(x) g(x) + \frac{i}{2} \theta^{ij} \partial_i f(x) \partial_j g(x) + O(\theta^2)$$

A deformed product

$$f * (g * h) = (f * g) * h$$ Associative

$$[x^i, x^j]_* := x^i * x^j - x^j * x^i = i\theta^{ij}$$ NC !

(Ex.) 4-dim. (Euclidean) G=U(N) Yang-Mills theory

- Action

$$S = \int d^4x \, Tr \, F_{\mu\nu} F^{\mu\nu}$$

$$= \int d^4x \, Tr \left(F_{\mu\nu} F_{\mu\nu} + \tilde{F}_{\mu\nu} \tilde{F}_{\mu\nu} \right)$$

$$= \int d^4x \, Tr \left[\left(F_{\mu\nu} \mp \tilde{F}_{\mu\nu} \right)^2 \pm 2 F_{\mu\nu} \tilde{F}_{\mu\nu} \right]$$

- Eq. Of Motion: $(F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])$

$$[D^\nu, [D_\nu, D_\mu]] = 0$$

- BPS eq. (=(A)SD eq.)

$$F_{\mu\nu} = \pm \tilde{F}_{\mu\nu} \qquad \rightarrow \text{instantons}$$

$$\updownarrow$$

$$F_{z_1 \bar{z}_1} + F_{z_2 \bar{z}_2} = 0, \quad F_{z_1 z_2} = 0$$

(Ex.) 4-dim. NC (Euclidean) G=U(N) Yang-Mills theory
(All products are star products)

- Action

$$S = \int d^4x\, Tr\, F_{\mu\nu} * F^{\mu\nu}$$
$$= \int d^4x\, Tr\left(F_{\mu\nu} * F_{\mu\nu} + \tilde{F}_{\mu\nu} * \tilde{F}_{\mu\nu}\right)$$
$$= \int d^4x\, Tr\left[\left(F_{\mu\nu} \mp \tilde{F}_{\mu\nu}\right)_*^2 \pm 2F_{\mu\nu} * \tilde{F}_{\mu\nu}\right]$$

- Eq. Of Motion: $(F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + \underline{[A_\mu, A_\nu]_*})$

$$[D^\nu, [D_\nu, D_\mu]_*]_* = 0$$

Don't omit even for G=U(1)

$(\because U(1) \cong U(\infty))$

- BPS eq. (=(A)SD eq.)

$$F_{\mu\nu} = \pm \tilde{F}_{\mu\nu}$$

→ NC instantons

$$\updownarrow$$

$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0, \quad F_{z_1 z_2} = 0$$

ADHM construction of instantons

Atiyah-Drinfeld-Hitchin-Manin, PLA65('78)185

ADHM eq. (G=``U(k)"): k times k matrix eq.

$$[B_1, B_1^*]+[B_2, B_2^*]+I\,I^* - J^*J = 0$$
$$[B_1, B_2]+IJ = 0$$

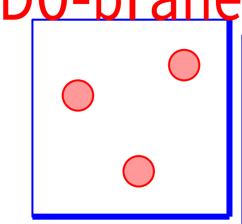
BPS

ADHM data $B_{1,2} : k\times k,$

$I : k\times N, \quad J : N\times k$

1:1

k D0-branes



N D4-branes

Instatntons $A_\mu : N\times N$

ASD eq. (G=U(N), C_2=-k): N times N PDE

BPS

$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$$
$$F_{z_1 z_2} = 0$$

ADHM construction of BPST instantons (N=2,k=1)

ADHM eq. (G=``U(1)'')

$$[B_1, B_1^*] + [B_2, B_2^*] + I I^* - J^* J = 0$$

$$[B_1, B_2] + IJ = 0$$

$$B_{1,2} = \alpha_{1,2}, \qquad I = (\rho, 0),\ J = \begin{pmatrix} 0 \\ \rho \end{pmatrix}$$

position

size

$$A_\mu = \frac{i(x-b)^\nu \eta^{(-)}_{\mu\nu}}{(x-b)^2 + \rho^2},\ F_{\mu\nu} = \frac{2i\rho^2}{((x-b)^2 + \rho^2)^2} \eta^{(-)}_{\mu\nu}$$

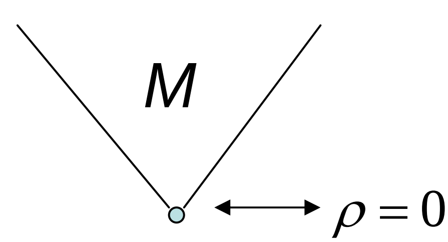
$$\downarrow \ \rho \to 0$$

singular

ASD eq. (G=U(2), C_2=-1)

$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$$

$$F_{z_1 z_2} = 0$$



Small instanton singulality

ADHM construction of NC BPST instantons (N=2,k=1)

ADHM eq. (G=``U(1)”)

$$[B_1, B_1^*] + [B_2, B_2^*] + I I^* - J^* J = \zeta$$
$$[B_1, B_2] + IJ = 0$$

$$B_{1,2} = \alpha_{1,2}, \qquad I = (\sqrt{\rho^2 + \underline{\zeta}}, 0),\ J = \begin{pmatrix} 0 \\ \rho \end{pmatrix}$$

position

Size → slightly fat?

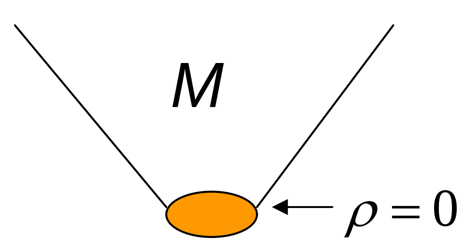
$A_\mu, F_{\mu\nu}$: something smooth

$$\downarrow \ \rho \to 0$$

ASD eq. (G=U(2), C_2=-1)

$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$$
$$F_{z_1 z_2} = 0$$

Regular!
(U(1) instanton!)



Resolution of the singularity

3. NC Sato's Theory

- Sato's Theory : one of the most beautiful theory of solitons
  - Based on the exsitence of hierarchies and tau-functions
- Sato's theory reveals essential aspects of solitons:
  - Construction of exact solutions
  - Structure of solution spaces
  - Infinite conserved quantities
  - Hidden infinite-dim. symmetry

Is it possible to extend it

to NC spaces ? → YES!

NC (KP) Hierarchy:

$$\boxed{\frac{\partial L}{\partial x^m} = [B_m, L]_*}$$

$$\partial_m u_2 \partial_x^{-1} + \qquad F_{m2}(u)\partial_x^{-1} +$$

$$\partial_m u_3 \partial_x^{-2} + \qquad F_{m3}(u)\partial_x^{-2} +$$

$$\partial_m u_4 \partial_x^{-3} + \cdots \qquad F_{m4}(u)\partial_x^{-3} + \cdots$$

Huge amount of ```NC evolution equations''

$$L := \partial_x + u_2\partial_x^{-1} + u_3\partial_x^{-2} + u_4\partial_x^{-3} + \cdots$$

$$B_m := (\underbrace{L * \cdots * L}_{m \text{ times}})_{\geq 0}$$

$$B_1 = \partial_x$$

$$B_2 = \partial_x^2 + 2u_2$$

$$B_3 = \partial_x^3 + 3u_2\partial_x + 3(u_3 + u_2')$$

$$u_k = u_k(\underline{x^1, x^2, x^3, \cdots}) \qquad [x^i, x^j] = i\theta^{ij}$$

Noncommutativity is introduced here

Negative powers of differential operators

$$\partial_x^n \circ f := \sum_{j=0}^{\infty} \binom{n}{j} (\partial_x^j f)\partial_x^{n-j}$$

$$\binom{n}{j} \equiv \frac{n(n-1)(n-2)\cdots(n-(j-1))}{j(j-1)(j-2)\cdots 1}$$

$$\partial_x^3 \circ f = f\partial_x^3 + 3f'\partial_x^2 + 3f''\partial_x^1 + f'''$$

$$\partial_x^2 \circ f = f\partial_x^2 + 2f'\partial_x + f''$$

$$\partial_x^{-1} \circ f = f\partial_x^{-1} - f'\partial_x^{-2} + f''\partial_x^{-3} - \cdots$$

$$\partial_x^{-2} \circ f = f\partial_x^{-2} - 2f'\partial_x^{-3} + 3f''\partial_x^{-4} - \cdots$$

Star product

$$f(x) * g(x) := f(x)\exp\left(\frac{i}{2}\theta^{ij}\overleftarrow{\partial}_i \overrightarrow{\partial}_j\right) g(x)$$

which makes theories ``noncommutative'':

$$[x^i, x^j]_* := x^i * x^j - x^j * x^i = i\theta^{ij}$$

Closer look at NC (KP) hierarchy

For m=2

$$\partial_x^{-1})\quad \partial_2 u_2 = 2\underline{u_3'} + u_2''$$

$$\partial_x^{-2})\quad \partial_2 u_3 = 2\underline{u_4'} + u_3'' + 2u_2 * u_2' + 2[u_2, u_3]_*$$

$$\partial_x^{-3})\quad \partial_2 u_4 = 2\underline{u_5'} + u_4'' + 4u_3 * u_2' - 2u_2 * u_2'' + 2[u_2, u_4]_*$$

$$\vdots$$

Infinite kind of fields are represented in terms of one kind of field $u_2 \equiv u$

MH&K.Toda, [hep-th/0309265]

For m=3

$$\partial_x^{-1})\quad \partial_3 u_2 = u_2''' + 3u_3'' + 3u_4'' + 3u_2' * u_2 + 3u_2 * u_2'$$

$$\vdots$$

$$\downarrow$$

$$u_t = \frac{1}{4}u_{xxx} + \frac{3}{4}(u * u)_x + \frac{3}{4}\partial_x^{-1}u_{yy} + \frac{3}{4}[u, \partial_x^{-1}u_{yy}]_*$$

NC KP equation

$$u_x := \frac{\partial u}{\partial x} \qquad u = u(x^1, x^2, x^3, \cdots)$$

$$\partial_x^{-1} := \int^x dx' \quad etc. \qquad x \quad y \quad t$$

(KP hie.) → (various hies.)

- KdV hierarchy

Reduction condition

$$L^2 = B_2 (=: \partial_x^2 + u) \quad \text{: 2-reduction}$$

gives rise to NC KdV hierarchy

which includes NC KdV eq.:

$$u_t = \frac{1}{4} u_{xxx} + \frac{3}{4} (u * u)_x$$

Note

$$\frac{\partial u}{\partial x_{2l}} = 0 \quad \text{: dimensional reduction}$$

l -reduction yields other NC hierarchies which include NC Boussinesq, coupled KdV, Sawada-Kotera, mKdV hierarchies and so on.

NC Burgers hierarchy

MH&K.Toda,JPA36('03)11981[hepth/0301213]

- NC (1+1)-dim. Burgers equation:

$$\dot{u} = u'' + 2u * u'$$

: Non-linear&

Infinite order diff. eq. w.r.t. time !

NC Cole-Hopf transformation

$$u = \tau^{-1} * \tau' \quad (\xrightarrow{\theta \to 0} \partial_x \log \tau)$$

$$\dot{\tau} = \tau''$$

: Linear &

first order diff. eq. w.r.t. time

Integrable !

NC Burgers eq. can be derived from
G=U(1) NC ASDYM eq. (One example of
NC Ward's observation)

4. Conservation Laws

- We have obtained wide class of NC hierarchies and NC (soliton) equations.

- Are they integrable or special from viewpoints of soliton theories? → YES !

Now we show the existence of infinite number of conserved quatities which suggests a hidden infinite-dimensional symmetry.

Conservation Laws

- Conservation laws:

$$\partial_t \sigma = \partial_i J^i \qquad \sigma$$: Conserved density

Then $Q := \int_{space} dx\,\sigma$ is a conserved quantity.

$$\because\ \partial_t Q = \int_{space} dx\,\partial_t \sigma = \int_{\substack{spatial\\ \inf inity}} dS_i J^i = 0$$

Conservation laws for the hierarchies

Follwing G.Wilson's approach, we have:

$$\partial_m res_{-1} L^n = \partial_x J + \underbrace{[A, B]_*}_{\text{troublesome}} = \partial_x J + \theta^{ij} \partial_j \Xi_i$$

I have succeeded in the evaluation explicitly !

Noncommutativity should be introduced
in space-time directions only.

$res_{-r} L^n$: coefficient of ∂_x^{-r} in L^n

Main Results

Infinite conserved densities for NC hierarchy eqs. (n=1,2,…)

$$\sigma = res_{-1}L^n + \theta^{mi} \sum_{k=1}^{m} \sum_{l=n+1}^{n+1+m-k} (-1)^{m+n-k-l} \binom{m-k}{l-n-1} \times a_{n-l} \Diamond \partial_i b_{m-k}^{(m+n-k-l+1)}$$

$$t \equiv x^m$$

$$L^n = \partial_x^n + \sum_{l=1}^{\infty} a_{n-l} \partial_x^{n-l}$$

$$B_m = \partial_x^m + \sum_{k=1}^{\infty} b_{m-k} \partial_x^{m-k}$$

$\Diamond$: Strachan's product

$$f(x) \Diamond g(x) := f(x) \left(\sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)!} \left(\frac{1}{2} \theta^{ij} \overleftarrow{\partial}_i \overrightarrow{\partial}_j \right)^{2s} \right) g(x)$$

MH, [hep-th/0311206]

Example: NC KP and KdV equations $([t,x] = i\theta)$

$$\sigma = res_{-1}L^n - 3\theta((res_{-1}L^n) \Diamond u_3' + (res_{-2}L^n) \Diamond u_2')$$

5. Conclusion and Discussion

- We proved the existence of infinite conserved quantities for wide class of NC hierarchies and gave the infinite conserved densities explicitly.
- Our results strongly suggest that infinite-dim. symmetry would be hidden in NC (soliton) equations.

  What is it ?

  → theories of tau-functions are needed

  (via e.g. Hirota's bilinearization)

  → the completion of NC Sato's theory
- The interpretation of space-time noncommutativity should be clarified.
- What is the twistor descriptions ?

  e.g. Kapustin&Kuznetsov&Orlov, Hannabuss,…

There are many things to be seen.