非可換ソリトンと Dブレイン

浜中　真志
(名古屋大学　多元数理)

MH,``NC Solitons and D-branes,''
Ph.D thesis (2003) [hep-th/0303256]

MH, ``Commuting Flows and Conservation Laws
for NC Lax Hierarches,'' [hep-th/0311206]

1. Introduction

- Non-Commutative (NC) spaces are defined by noncommutativity of the coordinates:

$$[x^i, x^j] = i\theta^{ij}$$

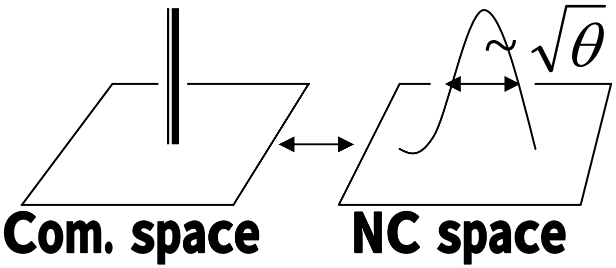
θ^{ij} **: NC parameter**

This looks like CCR in QM: $[q, p] = i\hbar$

(→ ``space-space uncertainty relation'' →)

Resolution of singulality

(→ New physical objects)



e.g. resolution of small instanton singularity

(→ U(1) instantons)

NC gauge theories ⟷ Com. gauge theories in background of magnetic fields

↓

(real physics)

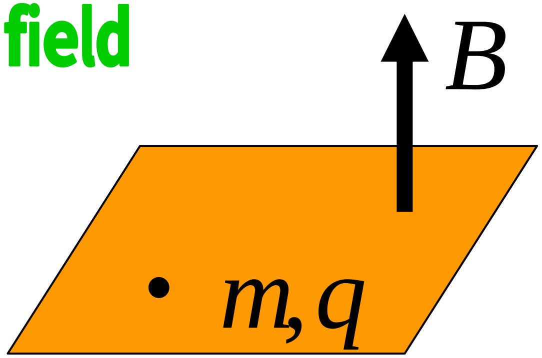
(Ex.) Motion of a charged particle in background magnetic field

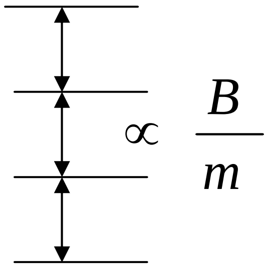
$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + qBx\dot{y}$$

$$\downarrow \quad m \to 0$$

$$L = qBx\dot{y}$$

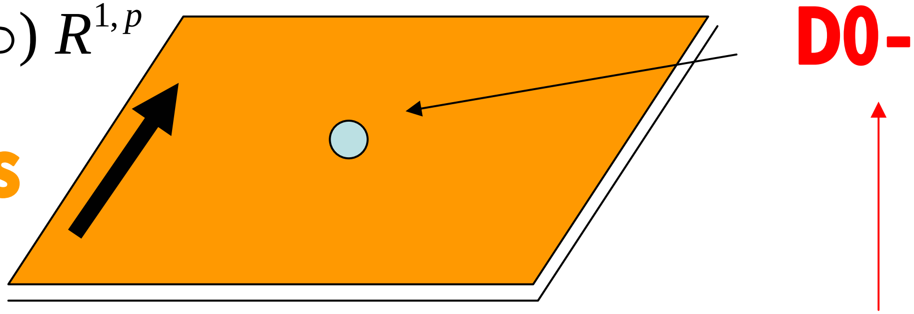
$$p_y = \frac{\partial L}{\partial \dot{y}} = qB\underline{x} \xrightarrow{[y, \underline{p_y}] = i} [x, y] = -i\frac{1}{qB}$$





Gauge theories are realized on D-branes
which are solitons in string theories

In this context, (NC) solitons are (lower-dim.) D-branes



$(R^{1,9} \supset)\ R^{1,p}$

B-fields

D0-brane

NC Soliton !

N Dp-branes → G=U(N) (p+1)-dim. NC Gauge theory

Analysis of NC solitons ↔ Analysis of D-branes

(easy to treat)

↓

Various applications

e.g. confirmation of Sen’s conjecture on decay of D-branes

Plan of this talk

1. Introduction
2. NC gauge theories
3. ADHM construction of (NC) instantons
4. Applications to D-brane dynamics

5. NC extension of soliton theories
6. NC Sato’s theories
7. Conservation Laws
8. Conclusion and Discussion

2. NC Gauge Theories

Here we discuss NC gauge theory of instantons.

(Ex.) 4-dim. (Euclidean) G=U(N) Yang-Mills theory

- Action

$$S=-\frac{1}{2}\int d^4x\, Tr\, F_{\mu\nu}F^{\mu\nu} \quad =-\frac{1}{4}\int d^4x\, Tr\left(F_{\mu\nu}F_{\mu\nu}+\tilde{F}_{\mu\nu}\tilde{F}_{\mu\nu}\right)$$

$$=-\frac{1}{4}\int d^4x\, Tr\left[\underbrace{\left(F_{\mu\nu}\mp\tilde{F}_{\mu\nu}\right)^2}_{=0\Leftrightarrow \text{ BPS}}\pm\underbrace{2F_{\mu\nu}\tilde{F}_{\mu\nu}}_{\leftrightarrow C_2}\right]$$

$$(F_{\mu\nu}:=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu])$$

- Eq. Of Motion:

$$[D^\nu,[D_\nu,D_\mu]]=0$$

- BPS eq. (=(A)SDYM eq.)

$$F_{\mu\nu}=\pm\tilde{F}_{\mu\nu}$$ → instantons

$$(\Leftrightarrow F_{z_1\bar{z}_1}+F_{z_2\bar{z}_2}=0,\quad F_{z_1z_2}=0)$$

(i) Star-product formalism

NC theories are obtained from commutative gauge theories by replacing products of fields with star-products: $f(x)g(x)\rightarrow f(x)*g(x)$

- The star product:

$$f(x)*g(x):=f(x)\exp\left(\frac{i}{2}\theta^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right)g(x)$$

$$=f(x)g(x)+i\frac{\theta^{ij}}{2}\partial_i f(x)\partial_j g(x)+O(\theta^2)$$

A deformed product ↔ **Presence of background magnetic fields**

$$f*(g*h)=(f*g)*h$$

Associative

$$[x^i,x^j]_*:=x^i*x^j-x^j*x^i=i\theta^{ij}$$

NC !

(Ex.) 4-dim. NC (Euclidean) G=U(N) Yang-Mills theory (All products are star products)

- Action

$$S = -\frac{1}{2}\int d^4x\, Tr\, F_{\mu\nu} * F^{\mu\nu} \quad = -\frac{1}{4}\int d^4x\, Tr\left(F_{\mu\nu} * F_{\mu\nu} + \tilde{F}_{\mu\nu} * \tilde{F}_{\mu\nu}\right)$$

$$= -\frac{1}{4}\int d^4x\, Tr\Big[\underbrace{\left(F_{\mu\nu} \mp \tilde{F}_{\mu\nu}\right)_*^2}_{=0 \Leftrightarrow \textbf{BPS}} \pm \underbrace{2F_{\mu\nu} * \tilde{F}_{\mu\nu}}_{\leftrightarrow C_2}\Big]$$

$$(F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + \underline{[A_\mu, A_\nu]_*})$$

↑

Don't omit even for G=U(1)

$$(\because U(1) \cong U(\infty))$$

- Eq. Of Motion:

$$[D^\nu, [D_\nu, D_\mu]_*]_* = 0$$

- BPS eq. (=NC (A)SDYM eq.)

$$F_{\mu\nu} = \pm\tilde{F}_{\mu\nu}$$

→ NC instantons

$$(\Leftrightarrow F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0, \quad F_{z_1 z_2} = 0)$$

A deformed theory is obtained.

Notes

- Noncommutativity violates Lorentz symmetry.
- U(1) part of gauge groups play important roles:
  - It behaves as non-abelian
  - SU(N) gauge theories are basically forbidden

$$g_1, g_2 \in SU(N) \Rightarrow g_1 * g_2 \notin SU(N)$$

- Star-products contain infinite derivatives
  - Non-local (→ Quantum treatment is hard)
  - Impossible to solve??? (solvable !!!)

Today's discussions are all classical.

(ii) Operator formalism

- This time, we begin with noncommutativity of coords. and treat the coords.and functions as operators.
- For simplicity, let’s consider NC plane:

$$[\hat{x}^1, \hat{x}^2] = i\theta \qquad (\theta > 0)$$

$$\Big\downarrow \quad \hat{a} := \frac{1}{\sqrt{2\theta}}\hat{z},\ \hat{a}^* := \frac{1}{\sqrt{2\theta}}\hat{\bar{z}}, \qquad z := x^1 + ix^2$$

$$[\hat{a}, \hat{a}^*] = 1$$

occupation number basis

Ann. op. **Cre. Op.** **acting on Fock sp.** $H = \langle |0\rangle, |1\rangle, |2\rangle, \cdots \rangle$

- Fields are also operators in the Fock space:

$$\hat{f}(\hat{x}^1, \hat{x}^2) = \sum_{m,n=0}^{\infty} f_{mn} |m\rangle\langle n|$$

Infinite-size matrix

Equivalence of the two formalisms

(i) Star-product formalism	← Weyl trf. →	**(ii) Operator formalism**
$[x^1, x^2]_* = i\theta$	Noncommutativity	$[\hat{x}^1, \hat{x}^2] = i\theta$
$f(x^1, x^2)$	Fields	$\hat{f} = \sum_{m,n=0}^{\infty} f_{mn} \lvert m \rangle\langle n \rvert$
Star product: *	Products $(\widehat{f * g} = \hat{f} \circ \hat{g})$	**Matrix multiplication: $\circ$**
$\partial_i f$ $(\partial_i x^j = \delta_i^{\ j})$	Differentiation	$\partial_i \hat{f} := [\hat{\partial}_i, \hat{f}] := [-i(\theta^{-1})_{ij} \hat{x}^j, \hat{f}]$ $(\partial_i \hat{x}^j = \delta_i^{\ j})$
$\int dx^1 dx^2 f(x^1, x^2)$	Integration	$2\pi\theta \, Tr_H \, \hat{f}$
$2\exp\left\{-\frac{r^2}{\theta}\right\}$ **(Gaussian)**	A projection	$\lvert 0 \rangle\langle 0 \rvert$

4-dim. NC (Euclidean) G=U(N) Yang-Mills theory

∞

EOM: $$[\hat{D}^\nu,[\hat{D}_\nu,\hat{D}_\mu]]=0$$

BPS eq.: $$\hat{F}_{z_1\bar{z}_1}+\hat{F}_{z_2\bar{z}_2}=0, \quad \hat{F}_{z_1 z_2}=0$$

Noncommutativity: $$\theta^{\mu\nu}=\left[\begin{array}{cc|cc} 0 & \theta^1 & & \\ -\theta^1 & 0 & \multicolumn{2}{c}{O} \\ \hline & & 0 & \theta^2 \\ \multicolumn{2}{c|}{O} & -\theta^2 & 0 \end{array}\right] \begin{array}{l} \leftrightarrow H_1 \\ \\ \\ \leftrightarrow H_2 \end{array}$$

Fields: $$\hat{f}(\hat{x}^\mu)=\sum C_{m_1m_2n_1n_2}\left|m_1\right\rangle\left\langle n_1\right|\otimes\left|m_2\right\rangle\left\langle n_2\right|$$

3. ADHM construction of instantons

Atiyah-Drinfeld-Hitchin-Manin, PLA65('78)185

ADHM eq. (G=``U(k)"): k × k matrix eq.

$$[B_1,B_1^*]+[B_2,B_2^*]+I\,I^*-J^*J=0$$
$$[B_1,B_2]+IJ=0$$

ADHM data $B_{1,2}:k\times k,\quad I:k\times N,\quad J:N\times k$

1:1

Instantons $A_\mu:N\times N$

ASD eq. (G=U(N), C_2=-k): N × N PDE

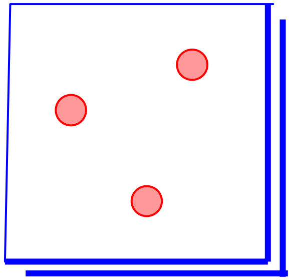
$$F_{z_1\bar{z}_1}+F_{z_2\bar{z}_2}=0$$
$$F_{z_1z_2}=0$$

D-brane's interpretation

Douglas, Witten

BPS

k D0-branes



N D4-branes

BPS

String theory is a treasure house of dualities

ADHM construction of BPST instanton (N=2,k=1)

ADHM eq. (G=``U(1)”)

$$[B_1,B_1^*]+[B_2,B_2^*]+I\,I^*-J^*J=0$$

$$[B_1,B_2]+IJ=0$$

$$B_{1,2}=\alpha_{1,2},\qquad I=(\rho,0),\ J=\begin{pmatrix}0\\ \rho\end{pmatrix}$$

↕ **position** ↕ ↕ **size**

$$A_\mu=\frac{i(x-b)^\nu\eta^{(-)}_{\mu\nu}}{(x-b)^2+\rho^2},\ F_{\mu\nu}=\frac{2i\rho^2}{((x-b)^2+\rho^2)^2}\eta^{(-)}_{\mu\nu}\ \xrightarrow{\rho\to 0}$$

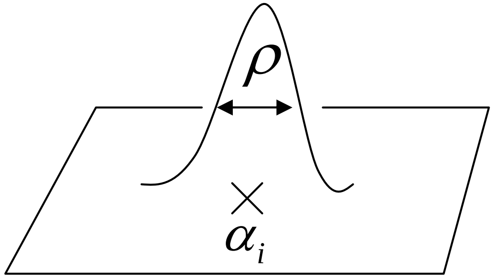
singular

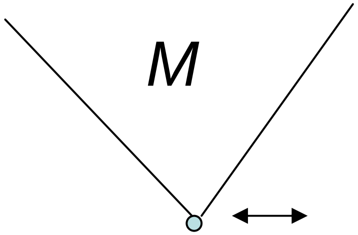
ASD eq. (G=U(2), C2=-1)

$$F_{z_1\bar{z}_1}+F_{z_2\bar{z}_2}=0$$

$$F_{z_1z_2}=0$$

Final remark: matrices B and coords. z always appear in pair: z-B





$\rho=0$

Small instanton singularity

ADHM construction of NC BPST instanton (N=2,k=1)

**Nekrasov&Schwarz,
CMP198('98)689
[hep-th/9802068]**

ADHM eq. (G=``U(1)'') 1 × 1 matrix eq.

$$[B_1,B_1^*]+[B_2,B_2^*]+II^*-J^*J=\zeta$$

$$[B_1,B_2]+IJ=0$$

$$B_{1,2}=\alpha_{1,2},\qquad I=(\sqrt{\rho^2\underline{+\zeta}},0),\ J=\begin{pmatrix}0\\ \rho\end{pmatrix}$$

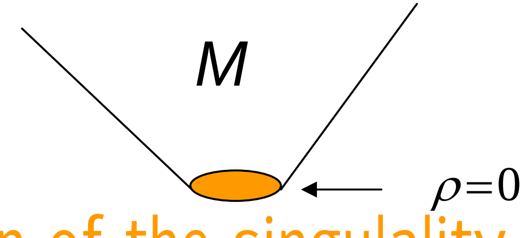
↕ position ↕ size → slightly fat?

$A_\mu, F_{\mu\nu}$: something smooth $\xrightarrow[\rho\to 0]{}$ Regular! (U(1) instanton!)

ASD eq. (G=U(2), C_2=-1)

$$F_{z_1\bar{z}_1}+F_{z_2\bar{z}_2}=0$$

$$F_{z_1z_2}=0$$

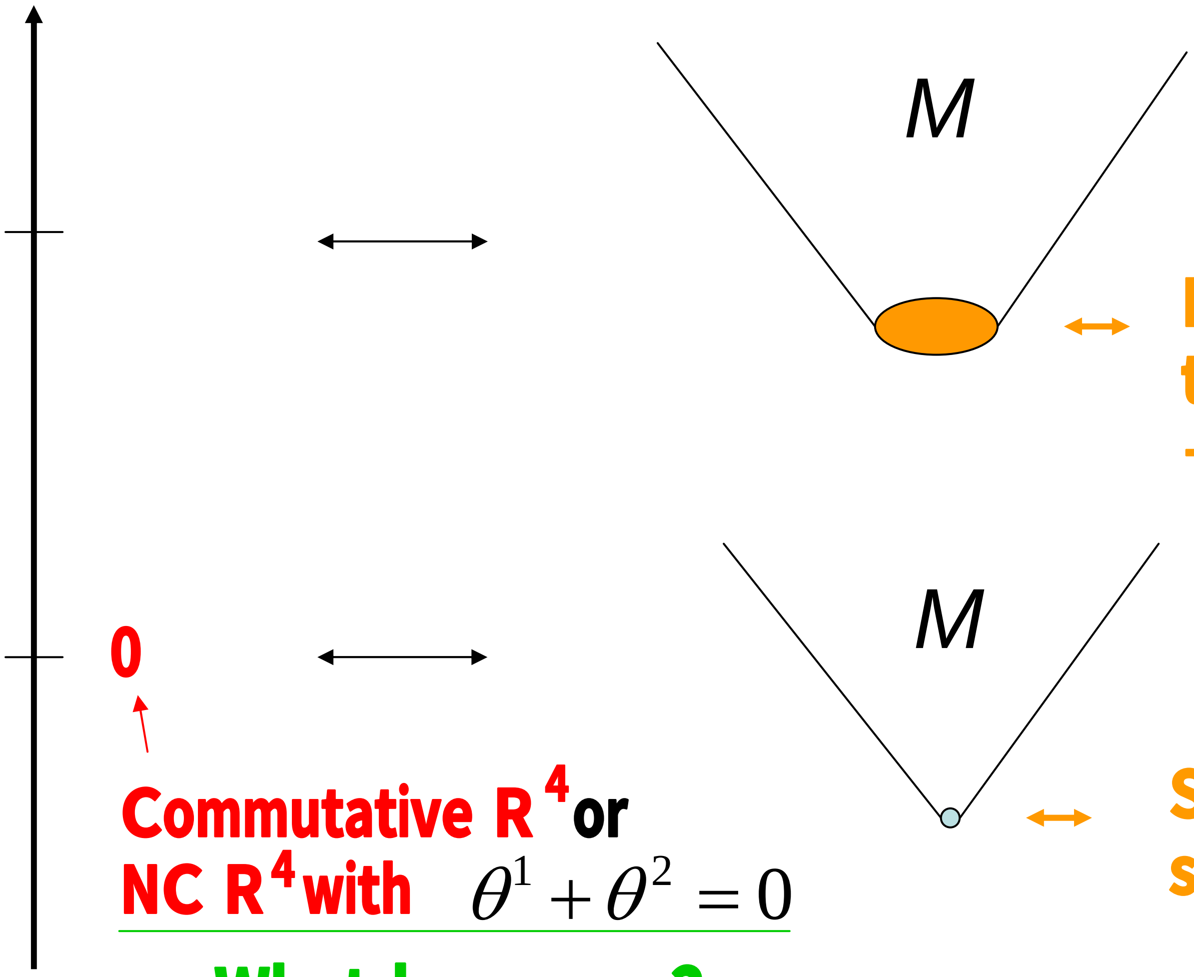


Resolution of the singularity

Comments on instanton moduli M_ζ

H. Nakajima

$$\zeta \propto \theta^1 + \theta^2$$



Resolution of the singularity → U(1) instanton

Small instanton singularity.

Commutative R^4 or NC R^4 with $\theta^1 + \theta^2 = 0$

What happens ?

ADHM construction of NC U(1) instanton w/ ζ =0

MH,
PRD65('02)085022
[hep-th/0109070]

ADHM eq. (G=``U(1)") 1 × 1 matrix eq.

$$[B_1, B_1^*]+[B_2, B_2^*]+I\,I^* - J^*J = 0$$
$$[B_1, B_2]+IJ = 0$$

$$B_{1,2} = \alpha_{1,2}, \; I = 0, \; J = 0$$

↕ ↕

position size → singular?

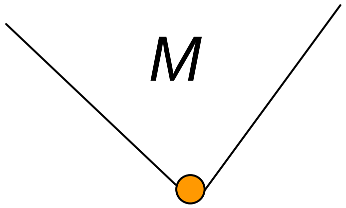
$$D_{z_i} = U_1^* \partial_{z_i} U_1 - \frac{\overline{\alpha}_i}{2\theta^i}|0\rangle\langle 0|, \quad F_{12} = -F_{34} = \frac{i}{\theta}|0\rangle\langle 0| \leftrightarrow \frac{2i}{\theta}\exp\left\{-\frac{r^2}{\theta}\right\}$$

Solution Generating Technique !

ASD eq. (G=U(1), C₂=-1)

$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$$
$$F_{z_1 z_2} = 0$$

$$U_1 U_1^* = 1,$$
$$U_1^* U_1 = 1 - |0\rangle\langle 0|$$



Small instanton singularity

Regular!

Inconsistent?

ADHM construction of NC U(1) instanton w/ ζ =0

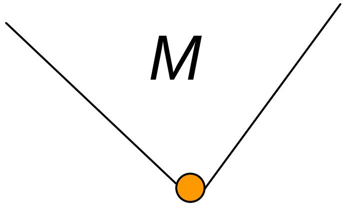
MH,
PRD65('02)085022
[hep-th/0109070]

ADHM eq. (G=``U(1)'') 1 × 1 matrix eq.

$$[B_1,B_1^*]+[B_2,B_2^*]+I\,I^*-J^*J=0$$
$$[B_1,B_2]+IJ=0$$

$$B_{1,2}=\alpha_{1,2},\ I=0,\ J=0$$

↕ position ↕ size → singular?



Small instanton singularity

consistent

$$D_{z_i}=U_1\hat{\partial}_{z_i}U_1^*-\frac{\overline{\alpha}_i}{2\theta^i}|0\rangle\langle 0|,\qquad F_{12}=-F_{34}=\frac{i}{\theta}|0\rangle\langle 0|\leftrightarrow\frac{2i}{\theta}\exp\left\{-\frac{r^2}{\theta}\right\}$$

<NC side>

Regular!

↓ **SW map**

ASD eq. (G=U(1), C₂=-1)

$$F_{z_1\bar{z}_1}+F_{z_2\bar{z}_2}=0$$
$$F_{z_1z_2}=0$$

<Com. side>

$$J_{D0}(x)=\underline{\frac{2}{\theta^2}}+\underline{\delta^{(2)}(z_1-\alpha_1)\delta^{(2)}(z_2-\alpha_2)}$$
$$\| $$
$$F_{\mu\nu}\tilde{F}^{\mu\nu}$$

D4 (=infinite D0s)

a D0 (singular)

4. Applications to D-brane Dynamics

- Solution Generating Technique (SGT)

based on an auto-Backlund transformation:

$$D_{z_i} \to U_k^* D_{z_i} U_k - \sum_{m=1}^{k} \frac{\overline{\alpha}_i^{(m)}}{2\theta^i} | p_m \rangle\langle p_m |$$

: Almost gauge transformation

$$U_k U_k^* = 1, \qquad \langle p_m | p_n \rangle = \delta_{mn} ,$$

$$U_k^* U_k = 1 - P_k = 1 - \sum_{m=1}^{k} | p_m \rangle\langle p_m |$$

Almost unitary operator

Harvey-Kraus-Larsen

which leaves EOM as it is: $\dfrac{\delta L}{\delta \phi} \to U_k^* \dfrac{\delta L}{\delta \phi} U_k$

$$e.g. \quad [D^\nu,[D_\nu,D_\mu]] \to [U_k^* D^\nu U_k,[U_k^* D_\nu U_k, U_k^* D_\mu U_k]] \qquad (U_k | p_m \rangle = 0)$$

$$= [U_k^* D^\nu,[D_\nu, D_\mu U_k]] = U_k^* [D^\nu,[D_\nu,D_\mu]] U_k$$

- SGT is independent of the details of actions
- SGT generates non-trivial soliton solutions from trivial solutions.
- (Ex.) Confirmation of Sen’s conjecture on decay of D-branes via tachyon condensation

Effective action of D25-brane in SFT:

$$S = \frac{c}{G_s} \int d^{24} x (2\pi\theta Tr_H) L$$

$$L = -V(T-1)\sqrt{\det(G_{\mu\nu} + 2\pi\alpha'(F-B)_{\mu\nu})} + \cdots$$

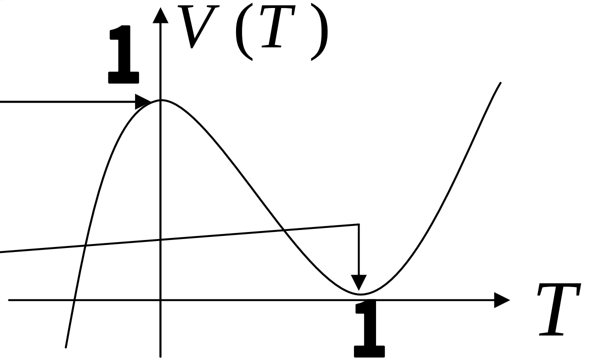
Unstable D25-brane

$$V(T=0)=1$$

$$c = T_{D25} g_s$$

Closed string vacuum

$$V(T=1)=0$$



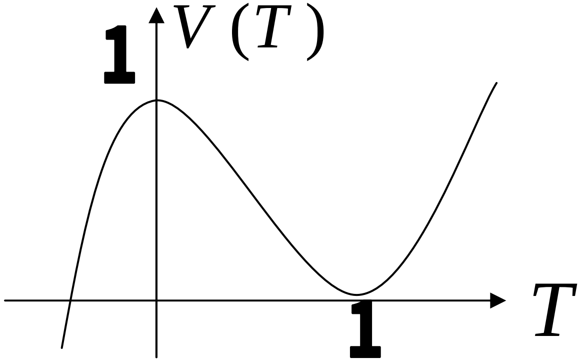
- SGT is independent of the details of actions
- SGT generates non-trivial soliton solutions from trivial solutions.
- (Ex.) Confirmation of Sen's conjecture on decay of D-branes via tachyon condensation

  Effective action of D25-brane in SFT:

$$S = \frac{c}{G_s}\int d^{24}x(2\pi\theta Tr_H)L$$

$$L = -V(T-1)\sqrt{\det(G_{\mu\nu} + 2\pi\alpha'(F-B)_{\mu\nu})} + \cdots$$

Sen's conjecture:
The EOM has lower-dim.
D-brane solutions.
Solve it ! → impossible???



- SGT is independent of the details of actions
- SGT generates non-trivial soliton solutions from trivial solutions.
- (Ex.) Confirmation of Sen's conjecture on decay of D-branes via tachyon condensation

Effective action of D25-brane in SFT:

$$S = \frac{c}{G_s}\int d^{24}x(2\pi\theta Tr_H)L$$

$L =$ **(complicated but gauge invariant !)**

$T=1, D_z=\hat{\partial}_z, A_t=0$ **: closed string vacuum**

$\downarrow$ **SGT**

$T=1-P_k, D_z=U_k^*\hat{\partial}_z U_k, A_t=0 \xrightarrow[\text{fluctuation}]{\text{tension}}$ **k D23-branes !!!** **(exact result !)**

5. NC Extension of Soliton Theories

Soliton equations in diverse dimensions

4	Anti-Self-Dual Yang-Mills eq. (instantons) $F_{\mu\nu} = -\tilde{F}_{\mu\nu}$	NC extension (Successful)
3	Bogomol’nyi eq. (monopoles)	NC extension (Successful)
2 (+1)	KP eq. BCS eq. DS eq. …	NC extension (This talk)
1 (+1)	KdV eq. Boussinesq eq. NLS eq. Burgers eq. sine-Gordon eq. Sawada-Kotera eq	NC extension (This talk)

↑ Dim. of space

Ward's observation:

Almost all integrable equations are reductions of the ASDYM eqs.

R.Ward, Phil.Trans.Roy.Soc.Lond.A315('85)451

ASDYM eq.

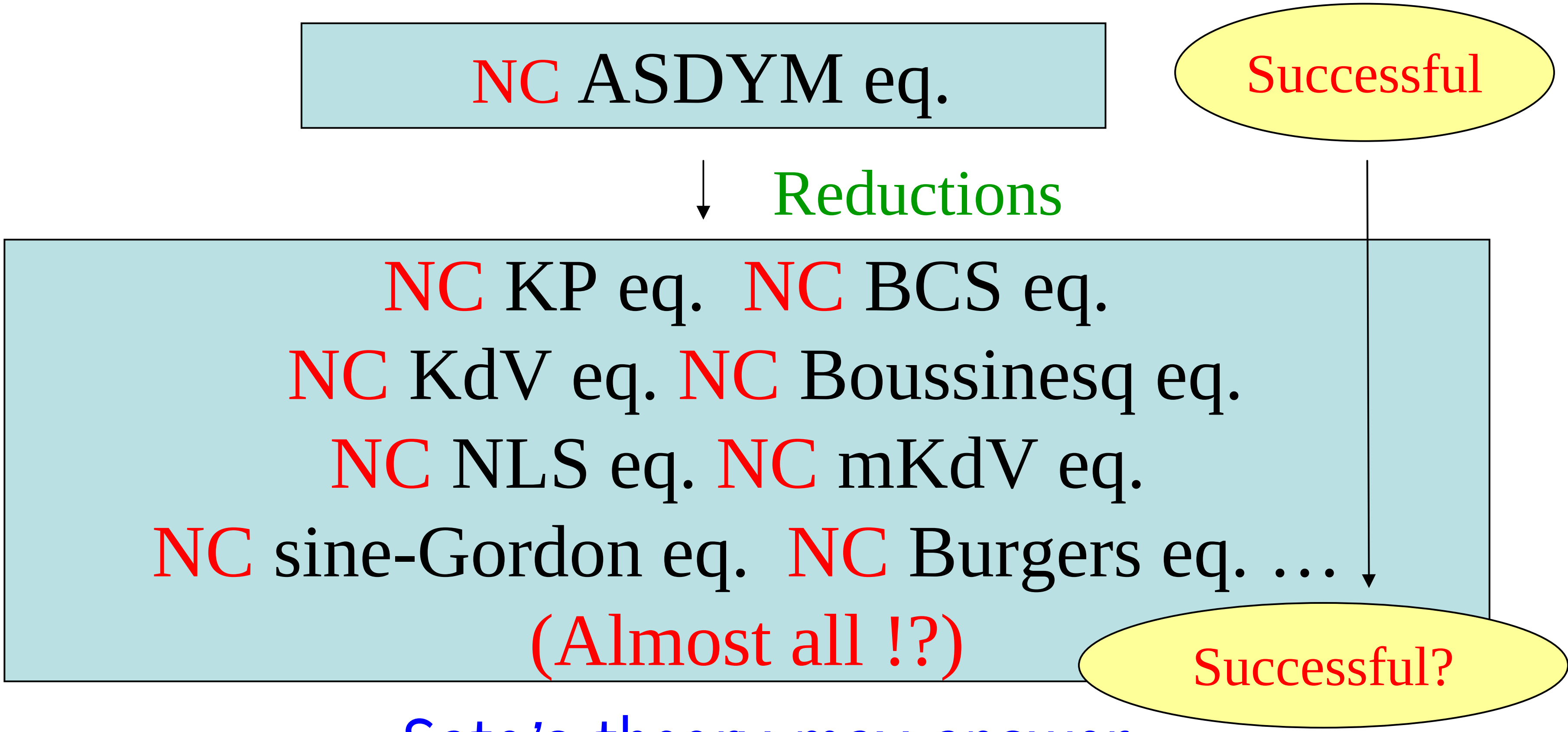
↓ Reductions

KP eq.   BCS eq.
KdV eq.   Boussinesq eq.
NLS eq. mKdV eq.
sine-Gordon eq.   Burgers eq. …
(Almost all !)

e.g. [Mason&Woodhouse]

NC Ward's observation: Almost all NC integrable equations are reductions of the NC ASDYM eqs.

MH&K.Toda, PLA316('03)77[hep-th/0211148]



NC ASDYM eq.

Successful

Reductions

NC KP eq. NC BCS eq.
NC KdV eq. NC Boussinesq eq.
NC NLS eq. NC mKdV eq.
NC sine-Gordon eq. NC Burgers eq. …
(Almost all !?)

Successful?

Sato's theory may answer

6. NC Sato’s Theories

- Sato’s Theory : one of the most beautiful theory of solitons
  - Based on the exsitence of hierarchies and tau-functions
- Sato’s theory reveals essential aspects of solitons:
  - Construction of exact solutions
  - Structure of solution spaces
  - Infinite conserved quantities
  - Hidden infinite-dim. Symmetry

Let’s discuss NC extension of Sato’s theory

NC (KP) Hierarchy:

$$\boxed{\frac{\partial L}{\partial x^m} = [B_m, L]_*}$$

$$\begin{array}{ll}
\partial_m u_2 \partial_x^{-1} + & F_{m2}(u)\partial_x^{-1} + \\
\partial_m u_3 \partial_x^{-2} + & F_{m3}(u)\partial_x^{-2} + \\
\partial_m u_4 \partial_x^{-3} + \cdots & F_{m4}(u)\partial_x^{-3} + \cdots
\end{array}$$

Huge amount of ``NC evolution equations'' (m=1,2,3,…)

$$L := \partial_x + u_2 \partial_x^{-1} + u_3 \partial_x^{-2} + u_4 \partial_x^{-3} + \cdots$$

$$B_m := (\underset{m\ times}{L * \cdots * L})_{\geq 0}$$

$$u_k = u_k(\underline{x^1, x^2, x^3, \cdots})$$

$$B_1 = L_{\geq 0} = \partial_x$$

$$B_2 = L_{\geq 0}^2 = \partial_x^2 + 2u_2$$

$$B_3 = L_{\geq 0}^3 = \partial_x^3 + 3u_2\partial_x + 3(u_3 + u_2')$$

Noncommutativity is introduced here: $[x^i, x^j] = i\theta^{ij}$

Negative powers of differential operators

$$\partial_x^n \circ f := \sum_{j=0}^{\infty} \underset{\overset{|||}{\frac{n(n-1)(n-2)\cdots(n-(j-1))}{j(j-1)(j-2)\cdots 1}}}{\begin{pmatrix} n \\ j \end{pmatrix}} (\partial_x^j f)\partial_x^{n-j}$$

: binomial coefficient
which can be extended
to negative n
→ negative power of
differential operator
(well-defined !)

$$\partial_x^3 \circ f = f\partial_x^3 + 3f'\partial_x^2 + 3f''\partial_x^1 + f'''$$

$$\partial_x^2 \circ f = f\partial_x^2 + 2f'\partial_x + f''$$

$$\partial_x^{-1} \circ f = f\partial_x^{-1} - f'\partial_x^{-2} + f''\partial_x^{-3} - \cdots$$

$$\partial_x^{-2} \circ f = f\partial_x^{-2} - 2f'\partial_x^{-3} + 3f''\partial_x^{-4} - \cdots$$

Star product:

$$f(x) * g(x) := f(x) \exp\left(\frac{i}{2}\theta^{ij}\overleftarrow{\partial}_i \overrightarrow{\partial}_j\right) g(x)$$

which makes theories ``noncommutative'':

$$[x^i, x^j]_* := x^i * x^j - x^j * x^i = i\theta^{ij}$$

Closer look at NC (KP) hierarchy

For m=2

$$\partial_x^{-1})\quad \partial_2 u_2 = \underline{2u_3'} + u_2''$$

$$\partial_x^{-2})\quad \partial_2 u_3 = \underline{2u_4'} + u_3'' + 2u_2 * u_2' + 2[u_2, u_3]_*$$

$$\partial_x^{-3})\quad \partial_2 u_4 = \underline{2u_5'} + u_4'' + 4u_3 * u_2' - 2u_2 * u_2'' + 2[u_2, u_4]_*$$

$$\vdots$$

Infinite kind of fields are represented in terms of one kind of field $u_2 \equiv u$

MH&K.Toda, [hep-th/0309265]

$$u_x := \frac{\partial u}{\partial x}$$

$$\partial_x^{-1} := \int^x dx'$$

etc.

For m=3

$$\partial_x^{-1})\quad \partial_3 u_2 = u_2''' + 3u_3'' + 3u_4'' + 3u_2' * u_2 + 3u_2 * u_2'$$

$$\vdots$$

$$\downarrow$$

$$u_t = \frac{1}{4}u_{xxx} + \frac{3}{4}(u_x * u + u * u_x) + \frac{3}{4}\partial_x^{-1}u_{yy} + \frac{3}{4}[u, \partial_x^{-1}u_{yy}]_*$$

(2+1)-dim. NC KP equation

and other NC equations (NC hierarchy equations)

$$u = u(\underset{\substack{\updownarrow \\ x}}{x^1}, \underset{\substack{\updownarrow \\ y}}{x^2}, \underset{\substack{\updownarrow \\ t}}{x^3}, \cdots)$$

(KP hierarchy) → (various hierarchies.)

- (Ex.) KdV hierarchy

Reduction condition

$$L^2 = B_2 (=: \partial_x^2 + u)$$: 2-reduction

gives rise to NC KdV hierarchy

which includes (1+1)-dim. NC KdV eq.:

$$u_t = \frac{1}{4} u_{xxx} + \frac{3}{4} (u_x * u + u * u_x)$$

Note $\frac{\partial u}{\partial x_{2N}} = 0$: dimensional reduction in x_{2N} directions

l -reduction yields wide class of other NC hierarchies which include NC Boussinesq, coupled KdV, Sawada-Kotera, mKdV hierarchies and so on.

NC Burgers hierarchy

MH&K.Toda,JPA36(‘03)11981[hepth/0301213]

- NC (1+1)-dim. Burgers equation:

$$\dot{u} = u'' + 2u * u'$$: Non-linear &

Infinite order diff. eq. w.r.t. time ! (Integrable?)

NC Cole-Hopf transformation

$$u = \tau^{-1} * \tau' \quad \left(\xrightarrow{\theta \to 0} \partial_x \log \tau\right)$$

(NC) Diffusion equation:

$$\dot{\tau} = \tau''$$: Linear & first order diff. eq. w.r.t. time

(Integrable !)

NC Burgers eq. can be derived from G=U(1) NC ASDYM eq.
(One example of NC Ward’s observation)

NC Ward's observation (NC NLS eq.)

- Reduced ASDYM eq.: $x^{\mu} \to (t, x)$

Legare,
[hep-th/0012077]

$$(i) \quad B' = 0$$

$$(ii) \quad C' - \dot{A} + [A, C]_{*} = 0$$

$$(iii) \quad A' - \dot{B} + [C, B]_{*} = 0$$

A, B, C: 2 × 2 matrices (gauge fields)

Further Reduction:

$$A = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}, \ B = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ C = -i\underline{\begin{pmatrix} q * \bar{q} & q' \\ q' & -\bar{q} * q \end{pmatrix}}$$

NOT traceless

$$(ii) \Rightarrow \begin{pmatrix} 0 & i\dot{q} - q'' - 2q * \bar{q} * q \\ i\dot{\bar{q}} + \bar{q}'' + 2\bar{q} * q * \bar{q} & 0 \end{pmatrix} = 0$$

$$i\dot{q} = q'' + 2q * \bar{q} * q$$ **: NC NLS eq.**

Note: $A, B, C \in gl(2, C) \xrightarrow{\theta \to 0} sl(2, C)$

U(1) part is important

NC Ward's observation (NC Burgers eq.)

- Reduced ASDYM eq.: $x^{\mu} \rightarrow (t, x)$

MH&K.Toda, JPA
[hep-th/0301213]

G=U(1)

$$(i) \quad \dot{A} + \underline{[B, A]}_{*} = 0$$

$$(ii) \quad \dot{C} - B' + \underline{[B, C]}_{*} = 0$$

A, B, C: 1 × 1 matrices (gauge fields)

should remain

Further Reduction: $A = 0,\ B = u' - u^{2},\ C = u$

$$(ii) \Rightarrow \quad \dot{u} = u'' + 2u' * u$$

: NC Burgers eq.

Note: Without the commutators [,], (ii) yields:

$$\dot{u} = u'' + \underline{u' * u + u * u'}$$

: neither linearizable nor Lax form

symmetric

7. Conservation Laws

- We have obtained wide class of NC hierarchies and NC (soliton) equations.

- Are they integrable or special

from viewpoints of soliton theories? → YES !

Now we show the existence of infinite number of conserved quatities which suggests

a hidden infinite-dimensional symmetry.

Conservation Laws

- Conservation laws: $\partial_t \sigma = \partial_i J^i$ σ : Conserved density

time **space**

Then $Q := \int_{space} dx\sigma$ is a conserved quantity.

$$\because \partial_t Q = \int_{space} dx \partial_t \sigma = \int_{\substack{spatial \\ \inf inity}} dS_i J^i = 0$$

Conservation laws for the hierarchies

$res_{-r}L^n$:

coefficient of ∂_x^{-r} in L^n

Follwing G.Wilson's approach, we have:

$$\partial_m res_{-1}L^n = \partial_x J + \underline{[A,B]_*} = \partial_x J + \underline{\theta^{ij}\partial_j \Xi_i}$$

troublesome

I have succeeded in the evaluation explicitly !

∂_j **should be space or time derivative**

↓

Noncommutativity should be introduced in space-time directions only.

Main Results

Infinite conserved densities for NC hierarchy eqs. (n=1,2,…)

$$\sigma = res_{-1} L^n + \theta^{mi} \sum_{k=1}^{m} \sum_{l=n+1}^{n+1+m-k} (-1)^{m+n-k-l} \begin{pmatrix} m-k \\ l-n-1 \end{pmatrix} a_{n-l} \Diamond \partial_i b_{m-k}^{(m+n-k-l+1)}$$

$$t \equiv x^m \qquad L^n = \partial_x^n + \sum_{l=1}^{\infty} a_{n-l} \partial_x^{n-l}, \quad B_m = \partial_x^m + \sum_{k=1}^{m} b_{m-k} \partial_x^{m-k}$$

$\Diamond$: Strachan’s product

$$f(x) \Diamond g(x) := f(x) \left(\sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)!} \left(\frac{1}{2} \theta^{ij} \overleftarrow{\partial}_i \overrightarrow{\partial}_j \right)^{2s} \right) g(x)$$

MH, [hep-th/0311206]

Example: NC KP and KdV equations $([t, x] = i\theta)$

$$\sigma = res_{-1} L^n - 3\theta((res_{-1} L^n) \Diamond u_3' + (res_{-2} L^n) \Diamond u_2')$$

: meaningful ?

8. Conclusion and Discussion

- In this talk, we discussed NC solitons and D-branes.
- We saw

  Successful !
  - Resolution of singularities→New physical objects
  - Equivalence between noncommutativity and magnetic fields
  - Easy treatment →Various applications to D-brane dynamics
- As a further direction, we proposed NC extension of soliton theories motivated by Ward's conjecture, which is expected to pioneer new study area of integrable systems.

  Fruitful !?
- We proved the existence of hierarchies and infinite conserved densities which would lead to integrabilities.
- Other applications would be possible:

  Welcome !

  e.g. QHE, quantum gravity, cosmology, etc.

アインシュタイン牧場

- URL:http://www2.yukawa.kyoto-u.ac.jp/~hamanaka
- このページの上の方に非可換関連、ソリトン理論の文献リストをアップしております。
- 他にも何かと便利です：
  - 情報
  - 検索
  - リンク
  - …