Non-Commutative Solitons and Integrable Systems

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MH, ``Commuting Flows and Conservation Laws for NC Lax Hierarchies,'' [hep-th/0311206]

cf. MH,``NC Solitons and D-branes,''
Ph.D thesis (2003) [hep-th/0303256]

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1. Introduction

- Non-Commutative (NC) spaces are defined by noncommutativity of the coordinates:

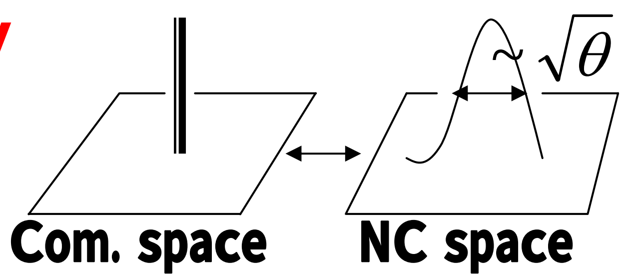
$$[x^i, x^j] = i\theta^{ij}$$

θ^{ij} **: NC parameter**

This looks like CCR in QM: $[q, p] = i\hbar$

(→ ``space-space uncertainty relation'' →)

Resolution of singularity



(→ New physical objects)

e.g. resolution of small instanton singularity

(→ U(1) instantons)

NC gauge theories $\leftrightarrow$ Com. gauge theories in background of magnetic fields

$\downarrow$

(real physics)

- Gauge theories are realized on D-branes which are solitons in string theories
- In this context, NC solitons are (lower-dim.) D-branes

Analysis of NC solitons $\leftrightarrow$ Analysis of D-branes

(easy)

$\downarrow$

Various successful applications

e.g. confirmation of Sen’s conjecture on decay of D-branes

NC extension of soliton theories are worth studying !

Soliton equations in diverse dimensions

4	Anti-Self-Dual Yang-Mills eq. (instantons) $F_{\mu\nu} = -\tilde{F}_{\mu\nu}$	NC extension (Successful)
3	Bogomol'nyi eq. (monopoles)	NC extension (Successful)
2 (+1)	KP eq. BCS eq. DS eq. …	NC extension (This talk)
1 (+1)	KdV eq. Boussinesq eq. NLS eq. Burgers eq. sine-Gordon eq. Sawada-Kotera eq	NC extension (This talk)

Dim. of space

Ward's observation:

Almost all integrable equations are reductions of the ASDYM eqs.

R.Ward, Phil.Trans.Roy.Soc.Lond.A315('85)451

ASDYM eq.

↓ Reductions

KP eq. BCS eq.

KdV eq. Boussinesq eq.

NLS eq. mKdV eq.

sine-Gordon eq. Burgers eq. …

(Almost all !)

e.g. [Mason&Woodhouse]

NC Ward's observation: Almost all NC integrable equations are reductions of the NC ASDYM eqs.

MH&K.Toda, PLA316('03)77[hep-th/0211148]

NC ASDYM eq.

Successful

Reductions

Reductions

NC KP eq. NC BCS eq.
NC KdV eq. NC Boussinesq eq.
NC NLS eq. NC mKdV eq.
NC sine-Gordon eq. NC Burgers eq. …
(Almost all !?)

Successful?

Sato's theory may answer

Plan of this talk

1. Introduction
2. NC Gauge Theory (Review)
3. NC Sato’s Theory
4. Conservation Laws
5. Conclusion and Discussion

2. NC Gauge Theory

Here we discuss NC gauge theory of instantons.

(Ex.) 4-dim. (Euclidean) G=U(N) Yang-Mills theory

- Action

$$S=-\frac{1}{2}\int d^4x\,Tr\,F_{\mu\nu}F^{\mu\nu}=-\frac{1}{4}\int d^4x\,Tr\left(F_{\mu\nu}F_{\mu\nu}+\tilde{F}_{\mu\nu}\tilde{F}_{\mu\nu}\right)$$

$$=-\frac{1}{4}\int d^4x\,Tr\left[\underbrace{\left(F_{\mu\nu}\mp\tilde{F}_{\mu\nu}\right)^2}_{=0\Leftrightarrow\ \textbf{BPS}}\pm\underbrace{2F_{\mu\nu}\tilde{F}_{\mu\nu}}_{\leftrightarrow C_2}\right]$$

$$(F_{\mu\nu}:=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu])$$

- Eq. Of Motion:

$$[D^\nu,[D_\nu,D_\mu]]=0$$

- BPS eq. (=(A)SDYM eq.)

$$F_{\mu\nu}=\pm\tilde{F}_{\mu\nu}\qquad\rightarrow\ \text{instantons}$$

$$(\Leftrightarrow F_{z_1\bar{z}_1}+F_{z_2\bar{z}_2}=0,\quad F_{z_1z_2}=0)$$

(Q) How we get NC version of the theories?

(A) They are obtained from ordinary commutative gauge theories by replacing products of fields with star-products: $f(x)g(x) \to f(x) * g(x)$

- The star product:

$$f(x) * g(x) := f(x) \exp\left(\frac{i}{2}\theta^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right) g(x) = f(x)g(x) + i\frac{\theta^{ij}}{2}\partial_i f(x)\partial_j g(x) + O(\theta^2)$$

A deformed product

$$f * (g * h) = (f * g) * h$$ Associative

$$[x^i, x^j]_* := x^i * x^j - x^j * x^i = i\theta^{ij}$$ NC !

Presence of background magnetic fields

In this way, we get NC-deformed theories with infinite derivatives in NC directions. (integrable???)

(Ex.) 4-dim. NC (Euclidean) G=U(N) Yang-Mills theory (All products are star products)

- Action

$$S = -\frac{1}{2}\int d^4x Tr\, F_{\mu\nu} * F^{\mu\nu} \quad = -\frac{1}{4}\int d^4x Tr\left(F_{\mu\nu} * F_{\mu\nu} + \tilde{F}_{\mu\nu} * \tilde{F}_{\mu\nu}\right)$$

$$= -\frac{1}{4}\int d^4x Tr\left[\underbrace{\left(F_{\mu\nu} \mp \tilde{F}_{\mu\nu}\right)_*^2}_{=0 \Leftrightarrow \text{ BPS}} \pm \underbrace{2F_{\mu\nu} * \tilde{F}_{\mu\nu}}_{\leftrightarrow C_2}\right]$$

$$(F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + \underline{[A_\mu, A_\nu]_*})$$

Don't omit even for G=U(1)

$$(\because U(1) \cong U(\infty))$$

- Eq. Of Motion:

$$[D^\nu, [D_\nu, D_\mu]_*]_* = 0$$

- BPS eq. (=NC (A)SDYM eq.)

$$F_{\mu\nu} = \pm\tilde{F}_{\mu\nu}$$

→ NC instantons

$$(\Leftrightarrow F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0, \quad F_{z_1z_2} = 0)$$

ADHM construction of instantons

Atiyah-Drinfeld-Hitchin-Manin, PLA65('78)185

ADHM eq. (G=``U(k)''): k times k matrix eq.

$$[B_1, B_1^*] + [B_2, B_2^*] + I I^* - J^* J = 0$$
$$[B_1, B_2] + IJ = 0$$

$\updownarrow$

ADHM data $B_{1,2} : k \times k, \quad I : k \times N, \quad J : N \times k$

1:1

Instantons $A_\mu : N \times N$

ASD eq. (G=U(N), C_2=-k): N times N PDE

$$F_{z_1 \bar{z}_1} + F_{z_2 \bar{z}_2} = 0$$
$$F_{z_1 z_2} = 0$$

ADHM construction of instantons

Atiyah-Drinfeld-Hitchin-Manin, PLA65('78)185

D-brane's interpretation

Douglas, Witten

ADHM eq. (G=``U(k)''): k times k matrix eq.

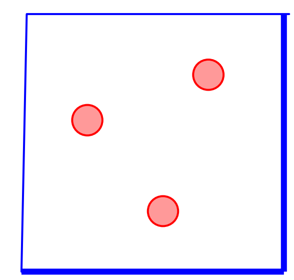
$$[B_1, B_1^*]+[B_2, B_2^*]+I\,I^*-J^*J=0$$
$$[B_1, B_2]+IJ=0$$

BPS

k D0-branes

ADHM data $B_{1,2}: k\times k,\quad I: k\times N,\quad J: N\times k$

1:1



Instantons $A_\mu : N\times N$

N D4-branes

ASD eq. (G=U(N), C_2=-k): N times N PDE

$$F_{z_1\bar{z}_1}+F_{z_2\bar{z}_2}=0$$
$$F_{z_1 z_2}=0$$

BPS

String theory is a treasure house of dualities

ADHM construction of BPST instanton (N=2,k=1)

ADHM eq. (G=``U(1)'')

$$[B_1,B_1^*]+[B_2,B_2^*]+II^*-J^*J=0$$
$$[B_1,B_2]+IJ=0$$

$$B_{1,2}=\alpha_{1,2},\qquad I=(\rho,0),\ J=\begin{pmatrix}0\\ \rho\end{pmatrix}$$

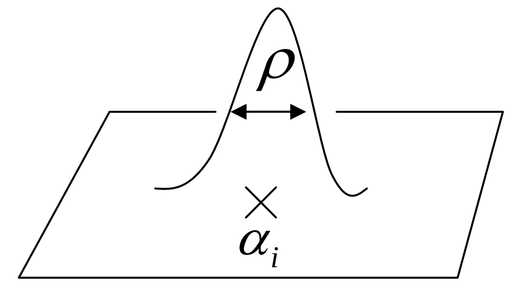
position **size**

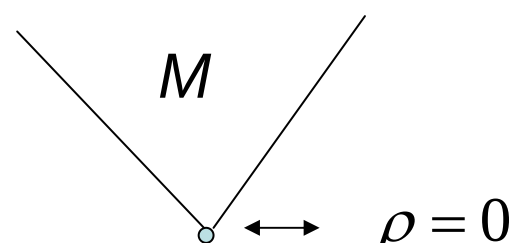
$$A_\mu=\frac{i(x-b)^\nu\eta^{(-)}_{\mu\nu}}{(x-b)^2+\rho^2},\ F_{\mu\nu}=\frac{2i\rho^2}{((x-b)^2+\rho^2)^2}\eta^{(-)}_{\mu\nu}\ \xrightarrow{\ \rho\to 0\ }\ \text{singular}$$

ASD eq. (G=U(2), C_2=-1)

$$F_{z_1\bar z_1}+F_{z_2\bar z_2}=0$$
$$F_{z_1 z_2}=0$$

Final remark: matrices B and coords. z always appear in pair: z-B





Small instanton singularity

ADHM construction of NC BPST instanton (N=2,k=1)

Nekrasov&Schwarz, CMP198('98)689 [hep-th/9802068]

ADHM eq. (G=``U(1)'') 1 times 1 matrix eq.

$$[B_1,B_1^*]+[B_2,B_2^*]+I\,I^*-J^*J=\zeta$$
$$[B_1,B_2]+IJ=0$$

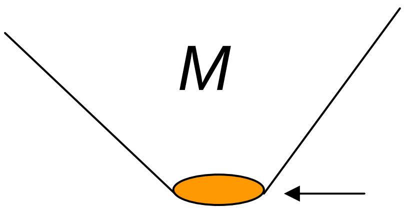
$$B_{1,2}=\alpha_{1,2},\qquad I=(\sqrt{\rho^2+\zeta},0),\ J=\begin{pmatrix}0\\ \rho\end{pmatrix}$$

$\updownarrow$ position $\qquad$ $\updownarrow$ size → slightly fat?

$A_\mu, F_{\mu\nu}$: something smooth $\quad \xrightarrow{\rho\to 0}$ Regular! (U(1) instanton!)

ASD eq. (G=U(2), C_2=-1)

$$F_{z_1\bar{z}_1}+F_{z_2\bar{z}_2}=0$$
$$F_{z_1z_2}=0$$

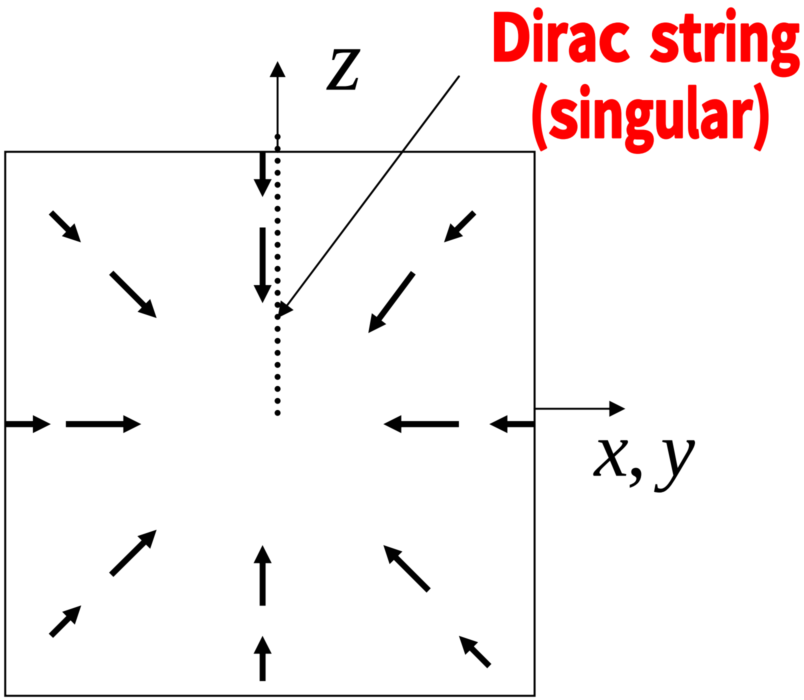


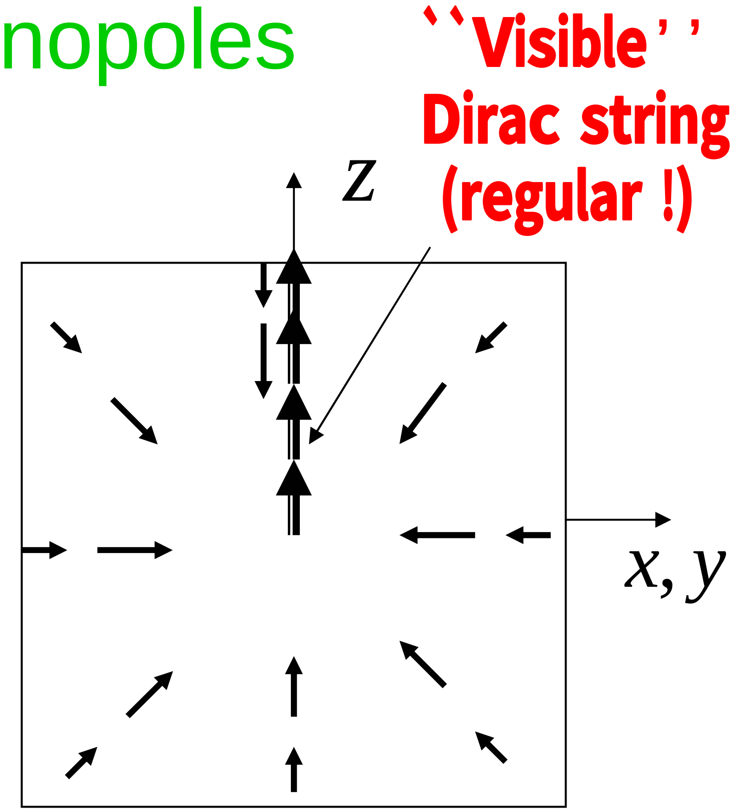
$\rho=0$

Resolution of the singularity

NC monopoles are also interesting

- Magnetic flux of Dirac monopoles





On commutative space

On NC space

Gross&Nekrasov, JHEP [hep-th/0005204]

ADHMN construction works well.
Moduli space is the same as commutative one.

3. NC Sato's Theory

- Sato's Theory : one of the most beautiful theory of solitons
  - Based on the exsitence of hierarchies and tau-functions
- Sato's theory reveals essential aspects of solitons:
  - Construction of exact solutions
  - Structure of solution spaces
  - Infinite conserved quantities
  - Hidden infinite-dim. Symmetry

Let's discuss NC extension of Sato's theory

NC (KP) Hierarchy:

$$\boxed{\frac{\partial L}{\partial x^m} = [B_m, L]_*}$$

$$\begin{aligned} &\partial_m u_2 \partial_x^{-1} + && F_{m2}(u)\partial_x^{-1} + \\ &\partial_m u_3 \partial_x^{-2} + && F_{m3}(u)\partial_x^{-2} + \\ &\partial_m u_4 \partial_x^{-3} + \cdots && F_{m4}(u)\partial_x^{-3} + \cdots \end{aligned}$$

Huge amount of ``NC evolution equations'' (m=1,2,3,…)

$$L := \partial_x + u_2 \partial_x^{-1} + u_3 \partial_x^{-2} + u_4 \partial_x^{-3} + \cdots$$

$$B_m := (\underbrace{L * \cdots * L}_{m \text{ times}})_{\geq 0}$$

$$u_k = u_k(x^1, x^2, x^3, \cdots)$$

$$B_1 = L_{\geq 0} = \partial_x$$

$$B_2 = L_{\geq 0}^2 = \partial_x^2 + 2u_2$$

$$B_3 = L_{\geq 0}^3 = \partial_x^3 + 3u_2 \partial_x + 3(u_3 + u_2')$$

Noncommutativity is introduced here: $[x^i, x^j] = i\theta^{ij}$

Negative powers of differential operators

$$\partial_x^n \circ f := \sum_{j=0}^{\infty} \binom{n}{j} (\partial_x^j f) \partial_x^{n-j}$$

$$\binom{n}{j} = \frac{n(n-1)(n-2)\cdots(n-(j-1))}{j(j-1)(j-2)\cdots 1}$$

: binomial coefficient which can be extended to negative n → negative power of differential operator (well-defined !)

$$\partial_x^3 \circ f = f\partial_x^3 + 3f'\partial_x^2 + 3f''\partial_x^1 + f'''$$

$$\partial_x^2 \circ f = f\partial_x^2 + 2f'\partial_x + f''$$

$$\partial_x^{-1} \circ f = f\partial_x^{-1} - f'\partial_x^{-2} + f''\partial_x^{-3} - \cdots$$

$$\partial_x^{-2} \circ f = f\partial_x^{-2} - 2f'\partial_x^{-3} + 3f''\partial_x^{-4} - \cdots$$

Star product: $$f(x) * g(x) := f(x) \exp\left(\frac{i}{2} \theta^{ij} \overleftarrow{\partial}_i \overrightarrow{\partial}_j \right) g(x)$$

which makes theories "noncommutative":

$$[x^i, x^j]_* := x^i * x^j - x^j * x^i = i\theta^{ij}$$

Closer look at NC (KP) hierarchy

For m=2

$$\partial_x^{-1})\quad \partial_2 u_2 = \underline{2u_3'} + u_2''$$

$$\partial_x^{-2})\quad \partial_2 u_3 = \underline{2u_4'} + u_3'' + 2u_2 * u_2' + 2[u_2, u_3]_*$$

$$\partial_x^{-3})\quad \partial_2 u_4 = \underline{2u_5'} + u_4'' + 4u_3 * u_2' - 2u_2 * u_2'' + 2[u_2, u_4]_*$$

$$\vdots$$

Infinite kind of fields are represented in terms of one kind of field $u_2 \equiv u$

MH&K.Toda, [hep-th/0309265]

$$u_x := \frac{\partial u}{\partial x}$$

$$\partial_x^{-1} := \int^x dx'$$

For m=3

$$\partial_x^{-1})\quad \partial_3 u_2 = u_2''' + 3u_3'' + 3u_4'' + 3u_2' * u_2 + 3u_2 * u_2'$$

etc.

$$\vdots$$

$$\downarrow$$

$$u_t = \frac{1}{4}u_{xxx} + \frac{3}{4}(u_x * u + u * u_x) + \frac{3}{4}\partial_x^{-1}u_{yy} + \frac{3}{4}[u, \partial_x^{-1}u_{yy}]_*$$

(2+1)-dim. NC KP equation

and other NC equations (NC hierarchy equations)

$$u = u(x^1, x^2, x^3, \cdots)$$

$$\updownarrow \quad \updownarrow \quad \updownarrow$$

$$x \quad y \quad t$$

(KP hierarchy) $\xrightarrow{\text{reductions}}$ (various hierarchies.)

- (Ex.) KdV hierarchy

  Reduction condition

$$L^2 = B_2 (=: \partial_x^2 + u)$$: 2-reduction

  gives rise to NC KdV hierarchy

  which includes (1+1)-dim. NC KdV eq.:

$$u_t = \frac{1}{4} u_{xxx} + \frac{3}{4} (u_x * u + u * u_x)$$

Note $\dfrac{\partial u}{\partial x_{2N}} = 0$: dimensional reduction in x_{2N} directions

$$\begin{array}{lll}
\textbf{KP :} & u(x^1, x^2, x^3, x^4, x^5, \ldots) & \textbf{: (2+1)-dim.} \\
 & \quad\; x \quad\; y \quad\; t & \\
\downarrow & & \downarrow \\
\textbf{KdV :} & u(x^1, \quad x^3, \quad x^5, \ldots) & \textbf{: (1+1)-dim.} \\
 & \quad\; x \qquad\; t &
\end{array}$$

l-reduction yields wide class of other NC hierarchies which include NC Boussinesq, coupled KdV, Sawada-Kotera, mKdV hierarchies and so on.

- 2-reduction → NC KdV
- 3-reduction → NC Boussinesq
- 4-reduction → NC Coupled KdV
- 5-reduction → ...
- 3-reduction of BKP → NC Sawada-Kotera
- 2-reduction of mKP → NC mKdV
- Special 1-reduction of mKP → NC Burgers
- ...

NC Burgers hierarchy

MH&K.Toda,JPA36('03)11981[hep-th/0301213]

- NC (1+1)-dim. Burgers equation:

$$\dot{u} = u'' + 2u * u'$$: Non-linear &

Infinite order diff. eq. w.r.t. time ! (Integrable?)

NC Cole-Hopf transformation

$$u = \tau^{-1} * \tau' \quad \left(\xrightarrow{\theta \to 0} \partial_x \log \tau \right)$$

(NC) Diffusion equation:

$$\dot{\tau} = \tau''$$: Linear & first order diff. eq. w.r.t. time

(Integrable !)

NC Burgers eq. can be derived from G=U(1) NC ASDYM eq.
(One example of NC Ward's observation)

NC Ward's observation (NC NLS eq.)

- Reduced ASDYM eq.: $x^{\mu} \to (t, x)$

Legare, [hep-th/0012077]

$$(i) \quad B' = 0$$

$$(ii) \quad C' - \dot{A} + [A, C]_* = 0$$

$$(iii) \quad A' - \dot{B} + [C, B]_* = 0$$

A, B, C: 2 times 2 matrices (gauge fields)

Further Reduction:

$$A = \begin{pmatrix} 0 & q \\ -\overline{q} & 0 \end{pmatrix}, \; B = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; C = -i\begin{pmatrix} q * \overline{q} & q' \\ q' & -\overline{q} * q \end{pmatrix}$$

NOT traceless

$$(ii) \Rightarrow \begin{pmatrix} 0 & i\dot{q} - q'' - 2q * \overline{q} * q \\ i\dot{\overline{q}} + \overline{q}'' + 2\overline{q} * q * \overline{q} & 0 \end{pmatrix} = 0$$

$$i\dot{q} = q'' + 2q * \overline{q} * q$$ **: NC NLS eq.**

Note: $A, B, C \in gl(2, C) \xrightarrow{\theta \to 0} sl(2, C)$

U(1) part is important

NC Ward's observation (NC Burgers eq.)

- Reduced ASDYM eq.: $x^\mu \to (t, x)$

MH&K.Toda, JPA [hep-th/0301213]

G=U(1)

$$(i) \quad \dot{A} + \underline{[B, A]}_* = 0$$

$$(ii) \quad \dot{C} - B' + \underline{[B, C]}_* = 0$$

A, B, C: 1 times 1 matrices (gauge fields)

should remain

Further Reduction: $A = 0,\ B = u' - u^2,\ C = u$

$$(ii) \Rightarrow \quad \dot{u} = u'' + 2u' * u$$

: NC Burgers eq.

Note: Without the commutators [,], (ii) yields:

$$\dot{u} = u'' + \underbrace{u' * u + u * u'}_{\text{symmetric}}$$

: neither linearizable nor Lax form

4. Conservation Laws

- We have obtained wide class of NC hierarchies and NC (soliton) equations.

- Are they integrable or special
from viewpoints of soliton theories? → **YES !**

Now we show the existence of infinite number of conserved quatities which suggests
a hidden infinite-dimensional symmetry.

Conservation Laws

- Conservation laws: $\partial_t \sigma = \partial_i J^i$ σ : Conserved density

time space

Then $Q := \int_{space} dx \sigma$ is a conserved quantity.

$$\because \partial_t Q = \int_{space} dx \partial_t \sigma = \int_{\substack{spatial \\ \inf inity}} dS_i J^i = 0$$

Conservation laws for the hierarchies

Follwing G.Wilson's approach, we have:

$$\partial_m res_{-1} L^n = \partial_x J + \underline{[A, B]_*} = \partial_x J + \underline{\theta^{ij} \partial_j \Xi_i}$$

troublesome

I have succeeded in the evaluation explicitly !

$res_{-r} L^n$: coefficient of ∂_x^{-r} in L^n

∂_j **should be space or time derivative**

↓

Noncommutativity should be introduced in space-time directions only.

Hot Results

Infinite conserved densities for NC hierarchy eqs. (n=1,2,…)

$$\sigma = res_{-1} L^n + \theta^{mi} \sum_{k=1}^{m} \sum_{l=n+1}^{n+1+m-k} (-1)^{m+n-k-l} \begin{pmatrix} m-k \\ l-n-1 \end{pmatrix} a_{n-l} \lozenge \partial_i b_{m-k}^{(m+n-k-l+1)}$$

$$t \equiv x^m \qquad L^n = \partial_x^n + \sum_{l=1}^{\infty} a_{n-l} \partial_x^{n-l}, \quad B_m = \partial_x^m + \sum_{k=1}^{m} b_{m-k} \partial_x^{m-k}$$

$\lozenge$: Strachan's product

$$f(x) \lozenge g(x) := f(x) \left(\sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)!} \left(\frac{1}{2} \theta^{ij} \overleftarrow{\partial}_i \overrightarrow{\partial}_j \right)^{2s} \right) g(x)$$

MH, [hep-th/0311206]

We can calculate the explicit forms of conserved densities for wide class of NC soliton equations.

- Space-Space noncommutativity:

  NC deformation is slight: $\sigma = res_{-1} L^n$

- Space-time noncommutativity

  NC deformation is drastical:

  – Example: NC KP and KdV equations $([t, x] = i\theta)$

$$\sigma = res_{-1} L^n - 3\theta((res_{-1} L^n) \Diamond u_3' + (res_{-2} L^n) \Diamond u_2')$$

meaningful ?

5. Conclusion and Discussion

- In this talk, we discussed integrability and various aspects of NC soliton eqs. in diverse dimensions.
- In higher dimensions, we saw how resolutions of singularity occur and new physical objects appear from the viewpoint of ADHM construction. **Successful !**
- In lower dimensions, we proved the existence of infinite conserved quantities for wide class of NC hierarchies and gave the infinite conserved densities explicitly from the viewpoint of Sato's theory, which suggests that infinite-dim. symmetry would be hidden in the NC (soliton) equations. **Going well !**
- Of course, there are still many things to be seen.

Further directions

- Completion of NC Sato's theory
  - Hirota's bilinearization and tau-functions $u=\partial_x^2 \log \tau$
    → hidden symmetry (deformed affine Lie algebras?)

    Cf. Date-Jimbo-Kashiwara-Miwa,···
  - Geometrical descriptions from NC extension of the theories of Krichever, Mulase and Segal-Wilson and so on.
- Confirmation of NC Ward's conjecture
  - NC twistor theory

    e.g. Kapustin&Kuznetsov&Orlov, Hannabuss, Hannover group,···
  - D-brane interpretations → physical meanings
- Foundation of Hamiltonian formalism with space-time noncommutativity
  - Initial value problems, Liouville's theorem, Noether's thm,…

You are welcome !!!