

# 非可換ソリトンと可積分系

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MH, “Commuting Flows and Conservation Laws for NC Lax Hierarchies,” [[hep-th/0311206](https://arxiv.org/abs/hep-th/0311206)]

cf. MH, “NC Solitons and D-branes,”

Ph.D thesis (2003) [[hep-th/0303256](https://arxiv.org/abs/hep-th/0303256)]

URL: <http://www2.yukawa.kyoto-u.ac.jp/~hamanaka>

# 1. Introduction

- Non-Commutative (NC) spaces are defined by noncommutativity of the coordinates:

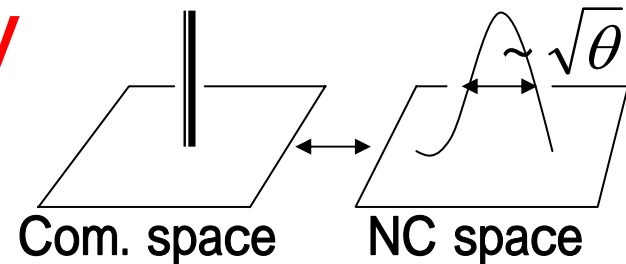
$$[x^i, x^j] = i\theta^{ij} \quad \theta^{ij} : \text{NC parameter}$$

This looks like CCR in QM:  $[q, p] = i\hbar$

( $\rightarrow$  "space-space uncertainty relation"  $\rightarrow$ )

## Resolution of singularity

( $\rightarrow$  New physical objects)



e.g. resolution of small instanton singularity

( $\rightarrow$  U(1) instantons)

NC gauge theories



Com. gauge theories

in background of  
magnetic fields

(real physics)

(Ex.) Motion of a charged particle  
in background magnetic field

$$L = \frac{m}{2} (\dot{x}^2 + \dot{y}^2) + qBx\dot{y}$$

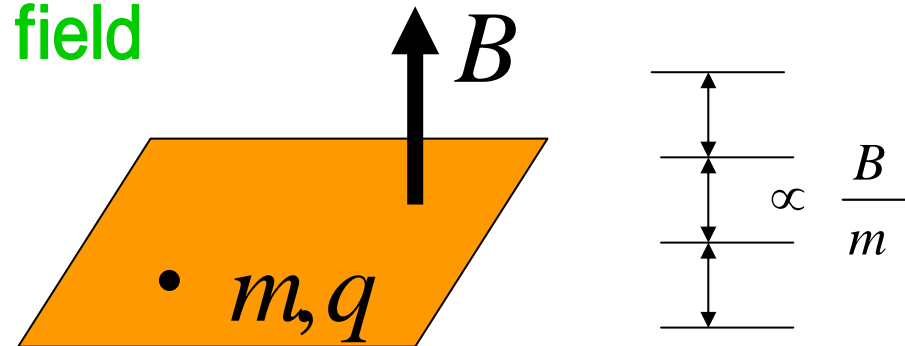
$$\downarrow m \rightarrow 0$$

$$L = qBx\dot{y}$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = qBx$$

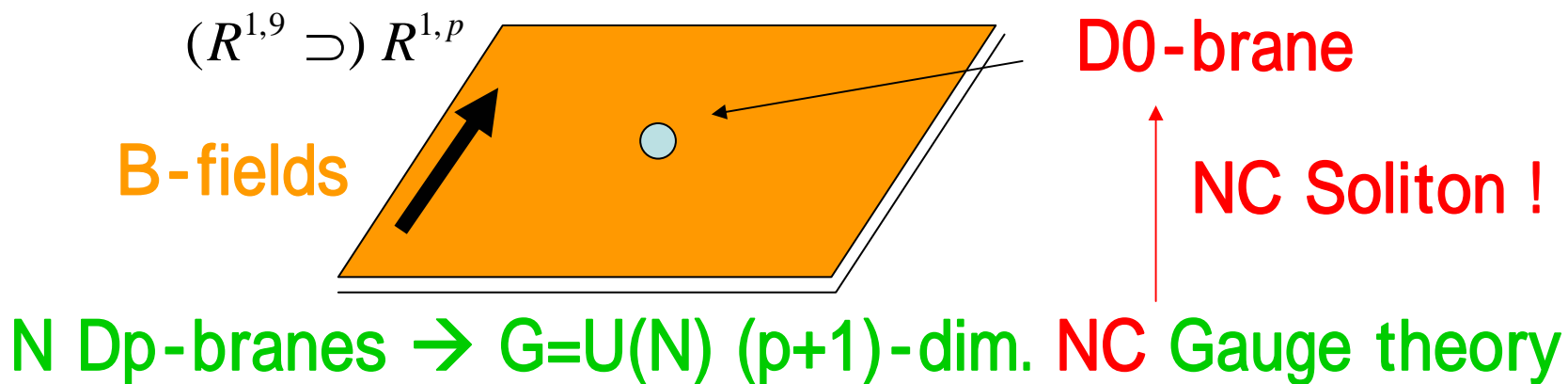
$$\text{---} \frac{[y, p_y] = i}{\text{---}} \rightarrow$$

$$[x, y] = -i \frac{1}{qB}$$



Gauge theories are realized on D-branes  
which are solitons in string theories

In this context, (NC) solitons are (lower-dim.) D-branes



Analysis of NC solitons (easy to treat) ↔ Analysis of D-branes

↓  
Various applications

e.g. confirmation of Sen's conjecture on decay of D-branes

# Successful points on NC theories

- Appearance of **new** physical objects
- Description of **real** physics
- Various **successful applications** to D-brane dynamics etc.

NC Solitons play important roles

NC extension of soliton theories  
are worth studying !

# Soliton equations in diverse dimensions

4	Anti-Self-Dual Yang-Mills eq. (instantons) $F_{\mu\nu} = -\tilde{F}_{\mu\nu}$	NC extension (Successful)
3	Bogomol'nyi eq. (monopoles)	NC extension (Successful)
2 (+1)	KP eq. BCS eq. DS eq. ...	NC extension (This talk)
1 (+1)	KdV eq. Boussinesq eq. NLS eq. Burgers eq. sine-Gordon eq. Sawada-Kotera eq	NC extension (This talk)



Dim. of space

# Ward's observation:

Almost all integrable equations are reductions of the ASDYM eqs.

R.Ward, Phil.Trans.Roy.Soc.Lond.A315( 85)451

ASDYM eq.

↓ Reductions

KP eq. BCS eq.  
KdV eq. Boussinesq eq.  
NLS eq. mKdV eq.  
sine-Gordon eq. Burgers eq. ...

(Almost all ! )

e.g. [Mason&Woodhouse]

**NC** Ward's observation: Almost all  
**NC** integrable equations are  
reductions of the **NC** ASDYM eqs.

MH&K.Toda, PLA316( 03)77[hep-th/0211148]

**NC** ASDYM eq.

Successful

↓ Reductions

Reductions

**NC** KP eq. **NC** BCS eq.  
**NC** KdV eq. **NC** Boussinesq eq.  
**NC** NLS eq. **NC** mKdV eq.  
**NC** sine-Gordon eq. **NC** Burgers eq. ...

(Almost all !?)

Successful?

Sato's theory may answer



# Plan of this talk

1. Introduction
2. NC Gauge Theory (Review)
3. NC Sato's Theory
4. Conservation Laws
5. Conclusion and Discussion

# 2. NC Gauge Theory

Here we discuss NC gauge theory of **instantons**.  
 (Ex.) 4-dim. (Euclidean)  $G=U(N)$  Yang-Mills theory

- Action

$$\begin{aligned}
 S &= -\frac{1}{2} \int d^4x \text{Tr} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} \int d^4x \text{Tr} (F_{\mu\nu} F_{\mu\nu} + \tilde{F}_{\mu\nu} \tilde{F}_{\mu\nu}) \\
 &= -\frac{1}{4} \int d^4x \text{Tr} \left[ \underbrace{(F_{\mu\nu} \mp \tilde{F}_{\mu\nu})^2}_{=0 \Leftrightarrow \text{BPS}} \pm \underbrace{2F_{\mu\nu} \tilde{F}_{\mu\nu}}_{\Leftrightarrow C_2} \right]
 \end{aligned}$$

- Eq. Of Motion:

$$[D^\nu, [D_\nu, D_\mu]] = 0$$

$$(F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])$$

- BPS eq. (= (A)SDYM eq.)

$$F_{\mu\nu} = \pm \tilde{F}_{\mu\nu} \quad \rightarrow \text{instantons}$$

$$(\Leftrightarrow F_{z_1 \bar{z}_1} + F_{z_2 \bar{z}_2} = 0, \quad F_{z_1 z_2} = 0)$$

(Q) How we get NC version of the theories?

(A) They are obtained from ordinary commutative gauge theories by replacing products of fields

with star-products:  $f(x)g(x) \rightarrow f(x) * g(x)$

• The star product:

$$f(x) * g(x) := f(x) \exp\left(\frac{i}{2} \theta^{ij} \overleftarrow{\partial}_i \overrightarrow{\partial}_j\right) g(x) = f(x)g(x) + i \frac{\theta^{ij}}{2} \partial_i f(x) \partial_j g(x) + O(\theta^2)$$

$$f * (g * h) = (f * g) * h \quad \text{Associative}$$

$$[x^i, x^j]_* := x^i * x^j - x^j * x^i = i\theta^{ij} \quad \text{NC !}$$

A deformed product



Presence of  
background  
magnetic fields

In this way, we get NC-deformed theories  
with infinite derivatives in NC directions. (integrable???)

# (Ex.) 4-dim. NC (Euclidean) $G=U(N)$ Yang-Mills theory

(All products are star products)

- Action

$$\begin{aligned}
 S &= -\frac{1}{2} \int d^4x \text{Tr} F_{\mu\nu} * F^{\mu\nu} = -\frac{1}{4} \int d^4x \text{Tr} (F_{\mu\nu} * F_{\mu\nu} + \tilde{F}_{\mu\nu} * \tilde{F}_{\mu\nu}) \\
 &= -\frac{1}{4} \int d^4x \text{Tr} \left[ \underbrace{(F_{\mu\nu} \mp \tilde{F}_{\mu\nu})^2}_{=0 \Leftrightarrow \text{BPS}} \pm \underbrace{2F_{\mu\nu} * \tilde{F}_{\mu\nu}}_{\Leftrightarrow C_2} \right] \\
 &\quad (F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + \underbrace{[A_\mu, A_\nu]_*})
 \end{aligned}$$

- Eq. Of Motion:

$$[D^\nu, [D_\nu, D_\mu]_*]_* = 0$$

Don't omit even for  $G=U(1)$

- BPS eq. (=NC (A)SDYM eq.)

( $\because U(1) \cong U(\infty)$ )

$$F_{\mu\nu} = \pm \tilde{F}_{\mu\nu} \quad \rightarrow \text{NC instantons}$$

$$(\Leftrightarrow F_{z_1 \bar{z}_1} + F_{z_2 \bar{z}_2} = 0, \quad F_{z_1 z_2} = 0)$$

# ADHM construction of instantons

Atiyah-Drinfeld-Hitchin-Manin, PLA65( 78)185

ADHM eq. ( $G=U(k)$ ):  $k$  times  $k$  matrix eq.

D-brane s  
interpretation

Douglas, Witten

$$[B_1, B_1^*] + [B_2, B_2^*] + I I^* - J^* J = 0$$

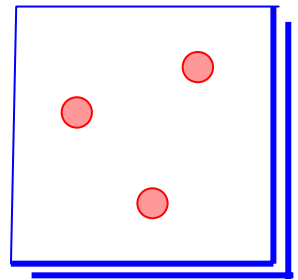
$$[B_1, B_2] + I J = 0$$

**BPS**

ADHM data  $B_{1,2} : k \times k, I : k \times N, J : N \times k$

$k$  D0-branes

1:1



Instantons  $A_\mu : N \times N$

$N$  D4-branes

ASD eq. ( $G=U(N), C_2=-k$ ):  $N$  times  $N$  PDE

$$F_{z_1 \bar{z}_1} + F_{z_2 \bar{z}_2} = 0$$

$$F_{z_1 z_2} = 0$$

**BPS**

String theory is a treasure house of dualities

# ADHM construction of BPST instanton (N=2,k=1)

ADHM eq. (G=U(1))

$$[B_1, B_1^*] + [B_2, B_2^*] + I I^* - J^* J = 0$$

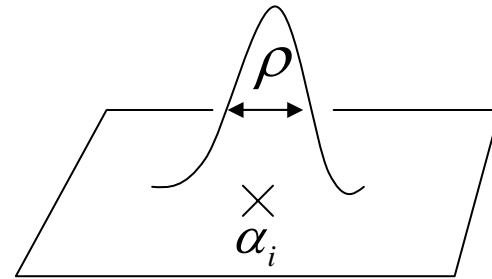
$$[B_1, B_2] + I J = 0$$

$$B_{1,2} = \alpha_{1,2}, \quad I = (\rho, 0), \quad J = \begin{pmatrix} 0 \\ \rho \end{pmatrix}$$

$\updownarrow$                        $\updownarrow$                        $\updownarrow$   
**position**                      **size**

$$A_\mu = \frac{i(x-b)^\nu \eta_{\mu\nu}^{(-)}}{(x-b)^2 + \rho^2}, \quad F_{\mu\nu} = \frac{2i\rho^2}{((x-b)^2 + \rho^2)^2} \eta_{\mu\nu}^{(-)}$$

**Final remark:** matrices B and coords. z always appear in pair: z-B

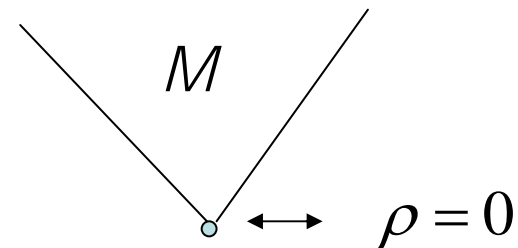


$\rho \rightarrow 0$  → singular

ASD eq. (G=U(2), C<sub>2</sub>=-1)

$$F_{z_1 \bar{z}_1} + F_{z_2 \bar{z}_2} = 0$$

$$F_{z_1 z_2} = 0$$



Small instanton singularity

# ADHM construction of NC BPST instanton

(N=2, k=1)

Nekrasov&Schwarz,  
CMP198( 98)689  
[hep-th/9802068]

ADHM eq. (G=U(1)) 1 times 1 matrix eq.

$$[B_1, B_1^*] + [B_2, B_2^*] + I I^* - J^* J = \zeta$$

$$[B_1, B_2] + I J = 0$$

$$B_{1,2} = \alpha_{1,2}, \quad I = (\sqrt{\rho^2 + \zeta}, 0), \quad J = \begin{pmatrix} 0 \\ \rho \end{pmatrix}$$

position ↔ size → slightly fat?

$A_\mu, F_{\mu\nu}$  : something smooth

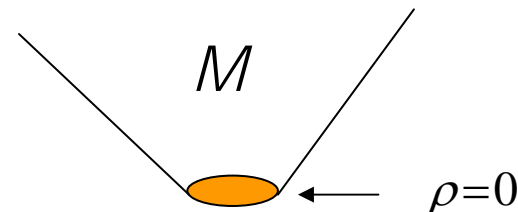
→  
 $\rho \rightarrow 0$

Regular!  
(U(1) instanton!)

ASD eq. (G=U(2), C<sub>2</sub>=-1)

$$F_{z_1 \bar{z}_1} + F_{z_2 \bar{z}_2} = 0$$

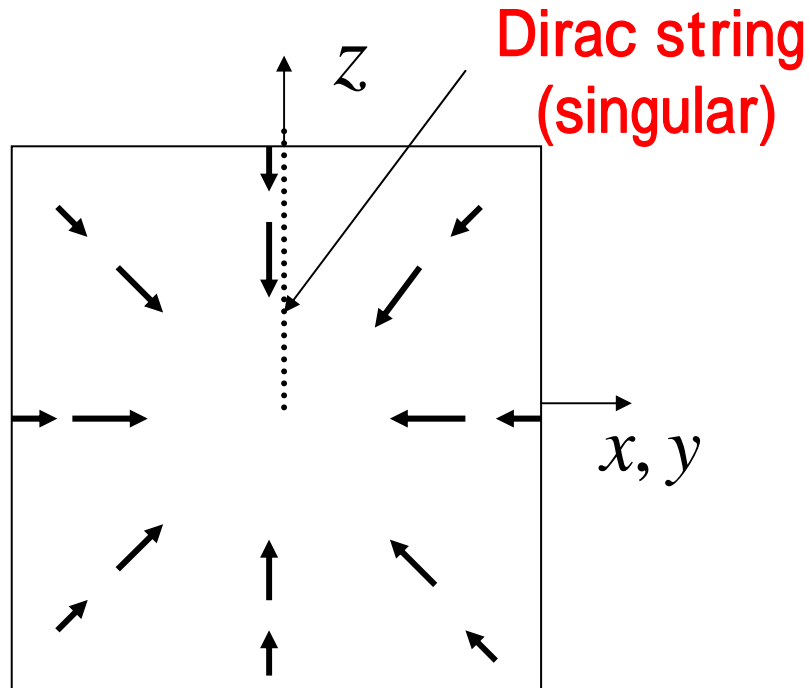
$$F_{z_1 z_2} = 0$$



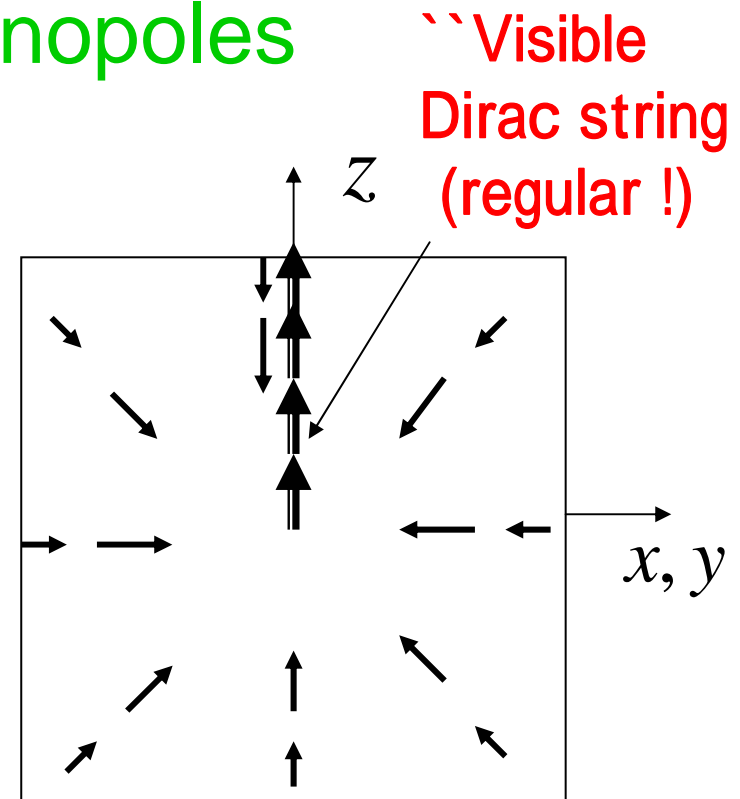
Resolution of the singularity

# NC monopoles are also interesting

- Magnetic flux of Dirac monopoles



On commutative space



On NC space

ADHMN construction works well.

Moduli space is the same as commutative one.

Gross&Nekrasov, JHEP  
[hep-th/0005204]



# 3. NC Sato's Theory

- Sato's Theory : one of the most beautiful theory of solitons
  - Based on the existence of hierarchies and tau-functions
- Sato's theory reveals essential aspects of solitons:
  - Construction of exact solutions
  - Structure of solution spaces
  - Infinite conserved quantities
  - Hidden infinite-dim. symmetry

Let's discuss NC extension of Sato's theory

# NC (KP) Hierarchy:

$$\frac{\partial L}{\partial x^m} = [B_m, L]_*$$

$$\begin{aligned} \partial_m u_2 \partial_x^{-1} + & F_{m2}(u) \partial_x^{-1} + \\ \partial_m u_3 \partial_x^{-2} + & F_{m3}(u) \partial_x^{-2} + \\ \partial_m u_4 \partial_x^{-3} + \dots & F_{m4}(u) \partial_x^{-3} + \dots \end{aligned}$$

Huge amount of "NC evolution equations" ( $m=1,2,3,\dots$ )

$$\begin{aligned} L &:= \partial_x + u_2 \partial_x^{-1} + u_3 \partial_x^{-2} + u_4 \partial_x^{-3} + \dots & B_1 &= L_{\geq 0} = \partial_x \\ B_m &:= (L * \dots * L)_{\geq 0} & B_2 &= L_{\geq 0}^2 = \partial_x^2 + 2u_2 \\ & \quad m \text{ times} & B_3 &= L_{\geq 0}^3 = \partial_x^3 + 3u_2 \partial_x + 3(u_3 + u_2') \\ u_k &= u_k(x^1, x^2, x^3, \dots) \end{aligned}$$

Noncommutativity is introduced here:  $[x^i, x^j] = i\theta^{ij}$

# Negative powers of differential operators

$$\partial_x^n \circ f := \sum_{j=0}^{\infty} \binom{n}{j} (\partial_x^j f) \partial_x^{n-j}$$

$$\frac{n(n-1)(n-2)\cdots(n-(j-1))}{j(j-1)(j-2)\cdots 1}$$

: binomial coefficient  
which can be extended  
to negative n  
→ negative power of  
differential operator  
(well-defined !)

$$\partial_x^3 \circ f = f\partial_x^3 + 3f\partial_x^2 + 3f\partial_x + f''''$$

$$\partial_x^2 \circ f = f\partial_x^2 + 2f\partial_x + f''$$

$$\partial_x^{-1} \circ f = f\partial_x^{-1} - f\partial_x^{-2} + f\partial_x^{-3} - \dots$$

$$\partial_x^{-2} \circ f = f\partial_x^{-2} - 2f\partial_x^{-3} + 3f\partial_x^{-4} - \dots$$

Star product: 
$$f(x) * g(x) := f(x) \exp\left(\frac{i}{2} \theta^{ij} \overleftarrow{\partial}_i \overrightarrow{\partial}_j\right) g(x)$$

which makes theories "noncommutative":

$$[x^i, x^j]_* := x^i * x^j - x^j * x^i = i\theta^{ij}$$

# Closer look at NC (KP) hierarchy

For  $m=2$

$$\partial_x^{-1}) \quad \partial_2 u_2 = \underline{2u_3'} + u_2''$$

$$\partial_x^{-2}) \quad \partial_2 u_3 = \underline{2u_4'} + u_3'' + 2u_2 * u_2' + 2[u_2, u_3]_*$$

$$\partial_x^{-3}) \quad \partial_2 u_4 = \underline{2u_5'} + u_4'' + 4u_3 * u_2' - 2u_2 * u_2'' + 2[u_2, u_4]_*$$

⋮

Infinite kind of fields are represented  
in terms of one kind of field  $u_2 \equiv u$

MH&K.Toda, [hep-th/0309265]

$$u_x := \frac{\partial u}{\partial x}$$

$$\partial_x^{-1} := \int^x dx'$$

For  $m=3$

$$\partial_x^{-1}) \quad \partial_3 u_2 = u_2''' + 3u_3'' + 3u_4'' + 3u_2' * u_2 + 3u_2 * u_2'$$

⋮



$$u_t = \frac{1}{4} u_{xxx} + \frac{3}{4} (u_x * u + u * u_x) + \frac{3}{4} \partial_x^{-1} u_{yy} + \frac{3}{4} [u, \partial_x^{-1} u_{yy}]_*$$

(2+1)-dim.  
NC KP equation

and other NC equations  
(NC hierarchy equations)

$$u = u(x^1, x^2, x^3, \dots)$$

$$\begin{array}{ccc} \updownarrow & \updownarrow & \updownarrow \\ x & y & t \end{array}$$

*etc.*

(KP hierarchy) <sup>reductions</sup>  $\rightarrow$  (various hierarchies.)

- (Ex.) KdV hierarchy

Reduction condition

$$L^2 = B_2 (=:\partial_x^2 + u) \quad : \text{2-reduction}$$

gives rise to NC KdV hierarchy

which includes (1+1)-dim. NC KdV eq.:

$$u_t = \frac{1}{4}u_{xxx} + \frac{3}{4}(u_x * u + u * u_x)$$

Note  $\frac{\partial u}{\partial x_{2N}} = 0$  : dimensional reduction in  $x_{2N}$  directions

KP :	$u(x^1, x^2, x^3, x^4, x^5, \dots)$	
	$x \quad y \quad t$	: (2+1)-dim.
$\downarrow$		$\downarrow$
KdV :	$u(x^1, x^3, x^5, \dots)$	
	$x \quad t$	: (1+1)-dim.

$\ell$ -reduction yields wide class of other NC hierarchies which include NC Boussinesq, coupled KdV, Sawada-Kotera, mKdV hierarchies and so on.

- 2-reduction  $\rightarrow$  NC KdV
- 3-reduction  $\rightarrow$  NC Boussinesq
- 4-reduction  $\rightarrow$  NC Coupled KdV
- 5-reduction  $\rightarrow$  ...
- 3-reduction of BKP  $\rightarrow$  NC Sawada-Kotera
- 2-reduction of mKP  $\rightarrow$  NC mKdV
- Special 1-reduction of mKP  $\rightarrow$  NC Burgers
- ...

# NC Burgers hierarchy

MH&K.Toda,JPA36( 03)11981[hep-th/0301213]

- NC (1+1)-dim. Burgers equation:

$$\dot{u} = u'' + 2u * u' : \text{Non-linear \&}$$

Infinite order diff. eq. w.r.t. time ! (Integrable?)

NC Cole-Hopf transformation

$$u = \tau^{-1} * \tau' \quad (\xrightarrow{\theta \rightarrow 0} \partial_x \log \tau)$$

(NC) Diffusion equation:

$$\dot{\tau} = \tau'' : \text{Linear \& first order diff. eq. w.r.t. time}$$

(Integrable !)

NC Burgers eq. can be derived from G=U(1) NC ASDYM eq.  
(One example of NC Ward's observation)

# NC Ward's observation (NC NLS eq.)

- Reduced ASDYM eq.:  $x^\mu \rightarrow (t, x)$

Legare,  
[hep-th/0012077]

(i)  $B' = 0$

(ii)  $C' - \dot{A} + [A, C]_* = 0$

(iii)  $A' - \dot{B} + [C, B]_* = 0$

A, B, C: 2 times 2  
matrices (gauge fields)



Further  
Reduction:

$$A = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}, B = \frac{i}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, C = -i \begin{pmatrix} q * \bar{q} & q' \\ q' & -\bar{q} * q \end{pmatrix}$$

$$(ii) \Rightarrow \begin{pmatrix} 0 & i\dot{q} - q'' - 2q * \bar{q} * q \\ i\dot{\bar{q}} + \bar{q}'' + 2\bar{q} * q * \bar{q} & 0 \end{pmatrix} = 0$$

**NOT traceless**

$$i\dot{q} = q'' + 2q * \bar{q} * q \quad : \text{NC NLS eq.}$$

**Note:**  $A, B, C \in gl(2, C) \xrightarrow{\theta \rightarrow 0} sl(2, C)$

**U(1) part is  
important**



# NC Ward's observation (NC Burgers eq.)

- Reduced ASDYM eq.:  $x^\mu \rightarrow (t, x)$

MH&K.Toda, JPA  
[hep-th/0301213]

$$G=U(1)$$

$$(i) \quad \dot{A} + \underline{[B, A]}_* = 0$$

$$(ii) \quad \dot{C} - B' + \underline{[B, C]}_* = 0$$

A, B, C: 1 times 1  
matrices (gauge fields)



Further

Reduction:  $A = 0, B = u' - u^2, C = u$

should remain

$$(ii) \Rightarrow \dot{u} = u'' + 2u' * u \quad : \text{NC Burgers eq.}$$

Note: Without the commutators  $[ , ]$ , (ii) yields:

$$\dot{u} = u'' + \underline{u' * u + u * u'} \quad : \text{neither linearizable nor Lax form}$$

symmetric

# 4. Conservation Laws

- We have obtained wide class of NC hierarchies and NC (soliton) equations.
- Are they **integrable** or **special** from viewpoints of soliton theories? → **YES !**

Now we show **the existence of infinite number of conserved quantities** which suggests a hidden infinite-dimensional symmetry.

# Conservation Laws

- Conservation laws:  $\overset{\text{time}}{\partial_t} \sigma = \overset{\text{space}}{\partial_i} J^i$      $\sigma$  : Conserved density

Then  $Q := \int_{space} dx \sigma$  is a conserved quantity.

$$\because \partial_t Q = \int_{space} dx \partial_t \sigma = \int_{spatial}^{inf\ inity} dS_i J^i = 0$$

## Conservation laws for the hierarchies

Following G.Wilson's approach, we have:

$$\partial_m res_{-1} L^n = \partial_x J + \underbrace{[A, B]_*}_{\text{troublesome}} = \partial_x J + \underbrace{\theta^{ij} \partial_j \Xi_i}$$

↑  
troublesome

$res_{-r} L^n$  :  
coefficient  
of  $\partial_x^{-r}$  in  $L^n$

↑  
I have succeeded in the evaluation explicitly !

$\partial_j$  should be space or time derivative



Noncommutativity should be introduced in space-time directions only.

# Hot Results

## Infinite conserved densities for NC hierarchy eqs. ( $n=1,2,\dots$ )

$$\sigma = \text{res}_{-1} L^n + \theta^{mi} \sum_{k=1}^m \sum_{l=n+1}^{n+1+m-k} (-1)^{m+n-k-l} \binom{m-k}{l-n-1} a_{n-l} \diamond \partial_i b_{m-k}^{(m+n-k-l+1)}$$

$$t \equiv x^m \quad L^n = \partial_x^n + \sum_{l=1}^{\infty} a_{n-l} \partial_x^{n-l}, \quad B_m = \partial_x^m + \sum_{k=1}^m b_{m-k} \partial_x^{m-k}$$

$\diamond$  : Strachan's product

$$f(x) \diamond g(x) := f(x) \left( \sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)!} \left( \frac{1}{2} \theta^{ij} \bar{\partial}_i \bar{\partial}_j \right)^{2s} \right) g(x)$$

We can calculate the explicit forms of conserved densities for wide class of NC soliton equations.

- Space-Space noncommutativity:

NC deformation is slight:  $\sigma = \text{res}_{-1} L^n$

- Space-time noncommutativity

NC deformation is drastical:

– Example: NC KP and KdV equations  $([t, x] = i\theta)$

$$\sigma = \text{res}_{-1} L^n - 3\theta((\text{res}_{-1} L^n) \diamond u'_3 + (\text{res}_{-2} L^n) \diamond u'_2)$$

meaningful ?

# 5. Conclusion and Discussion

- In this talk, we discussed integrability and various aspects of NC soliton eqs. in diverse dimensions.
- In higher dimensions, we saw how resolutions of singularity occur and new physical objects appear from the viewpoint of ADHM construction. **Successful !**
- In lower dimensions, we proved the existence of infinite conserved quantities for wide class of NC hierarchies and gave the infinite conserved densities explicitly from the viewpoint of Sato's theory, which suggests that infinite-dim. symmetry would be hidden in the NC (soliton) equations. **Going well !**
- Of course, there are still many things to be seen.

# Further directions

- Completion of NC Sato's theory
  - Hirota's bilinearization and tau-functions  $u = \partial_x^2 \log \tau$   
→ hidden symmetry (deformed affine Lie algebras?)  
Cf. Date-Jimbo-Kashiwara-Miwa,...
  - Geometrical descriptions from NC extension of the theories of Krichever, Mulase and Segal-Wilson and so on.
- Confirmation of NC Ward's conjecture
  - NC twistor theory  
e.g. Kapustin&Kuznetsov&Orlov, Hannabuss, Hannover group,...
  - D-brane interpretations → physical meanings
- Foundation of Hamiltonian formalism with space-time noncommutativity
  - Initial value problems, Liouville's theorem, Noether's thm,...

You are welcome !!!