非可換ソリトン方程式の無限個の保存量

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Ref: MH, ``Commuting Flows and Conservation Laws for NC Lax Hierarchies,''[hep-th/0311206]

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1. Introduction

Successful points on NC theories

- Appearance of new physical objects
- Description of real physics
- Various successful applications to D-brane dynamics etc.

NC Solitons play important roles (Integrable!)

Final goal: NC extension of all soliton theroies

Ward's observation:

Almost all integrable equations are reductions of the ASDYM eqs.

R.Ward, Phil.Trans.Roy.Soc.Lond.A315('85)451

Anti-Self-Dual Yang-Mills eq.

↓ Reductions

KP eq. BCS eq.
KdV eq. Boussinesq eq.
NLS eq. mKdV eq.
sine-Gordon eq. Burgers eq. …
(Almost all !)

e.g. [Mason&Woodhouse]

NC Ward's observation: Almost all NC integrable equations are reductions of the NC ASDYM eqs.

MH&K.Toda, PLA316('03)77[hep-th/0211148]

NC ASDYM eq.

Successful

↓ Reductions

Reductions

NC KP eq. NC BCS eq.

NC KdV eq. NC Boussinesq eq.

NC NLS eq. NC mKdV eq.

NC sine-Gordon eq. NC Burgers eq. …

(Almost all !?)

Successful?

A general framework is needed

2. NC Sato's Theory

- Sato's Theory : one of the most beautiful theory of solitons
  - Based on the exsitence of hierarchies and tau-functions
- Sato's theory reveals essential aspects of solitons:
  - Construction of exact solutions
  - Structure of solution spaces
  - Infinite conserved quantities
  - Hidden infinite-dim. symmetry

Let's discuss NC extension of Sato's theory

Derivation of soliton equations

- Give a Lax operator which is a pseudo-differential operator

$$L := \partial_x + u_2 \partial_x^{-1} + u_3 \partial_x^{-2} + u_4 \partial_x^{-3} + \cdots$$

$$u_k = u_k(x^1, x^2, x^3, \cdots)$$

Noncommutativity is introduced here:

$$[x^i, x^j] = i\theta^{ij}$$

- Define a differential operator

$$B_m := (\underbrace{L * \cdots * L}_{m \text{ times}})_{\geq 0}$$

- Introduce NC (KP) hierarchy equation:

$$\frac{\partial L}{\partial x^m} = [B_m, L]_*$$

$$\partial_m u_2 \partial_x^{-1} + \partial_m u_3 \partial_x^{-2} + \partial_m u_4 \partial_x^{-3} + \cdots$$

$$F_{m2}(u) \partial_x^{-1} + F_{m3}(u) \partial_x^{-2} + F_{m4}(u) \partial_x^{-3} + \cdots$$

Star product:

$$f(x) * g(x) := f(x) \exp\left(\frac{i}{2} \theta^{ij} \overleftarrow{\partial}_i \overrightarrow{\partial}_j \right) g(x)$$

which realizes the noncommutativity

→ **Each coefficient yields a differential equation.**

Closer look at NC (KP) hierarchy

For m=2

$$\partial_x^{-1})\quad \partial_2 u_2 = \underline{2u_3'} + u_2''$$

$$\partial_x^{-2})\quad \partial_2 u_3 = \underline{2u_4'} + u_3'' + 2u_2 * u_2' + 2[u_2, u_3]_*$$

$$\partial_x^{-3})\quad \partial_2 u_4 = \underline{2u_5'} + u_4'' + 4u_3 * u_2' - 2u_2 * u_2'' + 2[u_2, u_4]_*$$

$\vdots$

Infinite kind of fields are represented in terms of one kind of field $u_2 \equiv u$

MH&K.Toda, [hep-th/0309265]

$$u_x := \frac{\partial u}{\partial x}$$

$$\partial_x^{-1} := \int^x dx'$$

For m=3

$$\partial_x^{-1})\quad \partial_3 u_2 = u_2''' + 3u_3'' + 3u_4'' + 3u_2' * u_2 + 3u_2 * u_2'$$

etc.

$\vdots$

$\downarrow$

$$u_t = \frac{1}{4}u_{xxx} + \frac{3}{4}(u_x * u + u * u_x) + \frac{3}{4}\partial_x^{-1}u_{yy} + \frac{3}{4}[u, \partial_x^{-1}u_{yy}]_*$$

(2+1)-dim. NC KP equation

and other NC equations (NC hierarchy equations)

$$u = u(x^1, x^2, x^3, \cdots)$$

$$\updownarrow \quad \updownarrow \quad \updownarrow$$

$$x \quad y \quad t$$

(KP hierarchy) $\xrightarrow{\text{reductions}}$ (various hierarchies.)

- (Ex.) KdV hierarchy

Reduction condition

$$L^2 = B_2 (=: \partial_x^2 + u)$$: 2-reduction

gives rise to NC KdV hierarchy

which includes (1+1)-dim. NC KdV eq.:

$$u_t = \frac{1}{4} u_{xxx} + \frac{3}{4} (u_x * u + u * u_x)$$

Note $\frac{\partial u}{\partial x_{2N}} = 0$: dimensional reduction in x_{2N} directions

KP : $u(x^1, x^2, x^3, x^4, x^5, \ldots)$ with $x^1 = x$, $x^2 = y$, $x^3 = t$ **: (2+1)-dim.**

↓

KdV : $u(x^1, x^3, x^5, \ldots)$ with $x^1 = x$, $x^3 = t$ **: (1+1)-dim.**

l-reduction of NC KP hierarchy yields wide class of other NC hierarchies

- No-reduction → NC KP $(x,y,t)=(x^1,x^2,x^3)$
- 2-reduction → NC KdV $(x,t)=(x^1,x^3)$
- 3-reduction → NC Boussinesq $(x,t)=(x^1,x^2)$
- 4-reduction → NC Coupled KdV …
- 5-reduction → …
- 3-reduction of BKP → NC Sawada-Kotera
- 2-reduction of mKP → NC mKdV
- Special 1-reduction of mKP → NC Burgers
- …

Noncommutativity should be introduced into space-time coords

4. Conservation Laws

- Conservation laws: $\partial_t \sigma = \partial_i J^i$ σ : Conserved density

time → ∂_t, space → ∂_i

Then $Q := \int_{space} dx \sigma$ is a conserved quantity.

$$\because \partial_t Q = \int_{space} dx \partial_t \sigma = \int_{\substack{spatial \\ \text{inf}\, inity}} dS_i J^i = 0$$

Conservation laws for the hierarchies

$$\partial_m res_{-1} L^n = \partial_x J + \theta^{ij} \partial_j \Xi_i$$

time → ∂_m, space → ∂_x

I have found the explicit form ! (for $\theta^{ij} \partial_j \Xi_i$)

$res_{-r} L^n$: coefficient of ∂_x^{-r} in L^n

$t \equiv x^m$

Noncommutativity should be introduced in space-time directions only. →

∂_j should be space or time derivative

→ ordinary conservation laws !

Hot Results

Infinite conserved densities for NC hierarchy eqs. (n=1,2,…)

$$\sigma = res_{-1}L^n + \theta^{im}\sum_{k=0}^{m-1}\sum_{l=0}^{k}(-1)^{k-l}\binom{k}{l}(res_{-(l+1)}L^n)\Diamond(\partial_i\partial_x^{k-l}res_k L^m)$$

$t \equiv x^m$ $\qquad res_r L^n$: coefficient of ∂_x^r in L^n

$\Diamond$: Strachan's product (commutative and non-associative)

$$f(x)\Diamond g(x) := f(x)\left(\sum_{s=0}^{\infty}\frac{(-1)^s}{(2s+1)!}\left(\frac{1}{2}\theta^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right)^{2s}\right)g(x)$$

MH, [hep-th/0311206]

We can calculate the explicit forms of conserved densities for wide class of NC soliton equations.

- Space-Space noncommutativity:

  NC deformation is slight: $\sigma = res_{-1} L^n$

- Space-time noncommutativity

  NC deformation is drastical:

  – Example: NC KP and KdV equations $([t, x] = i\theta)$

$$\sigma = res_{-1} L^n - 3\theta((res_{-1} L^n) \lozenge u_3' + (res_{-2} L^n) \lozenge u_2')$$

meaningful ?

4. Conclusion and Discussion

- We proved the existence of infinite conserved quantities for wide class of NC Lax hierarchies.
- We gave the infinite conserved densities explicitly from the viewpoint of Sato's theory, which suggests that infinite-dim. symmetry would be hidden in the NC (soliton) equations.
- The results show that NC soliton eqs. have very special properties though they include non-linear terms and infinite (time-!) derivatives.
- Of course, there are still many things to be seen in order to confirm the integrability.

Further directions

- Completion of NC Sato's theory
  - Hirota's bilinearization and tau-functions $u=\partial_x^2 \log\tau$
    → hidden symmetry (deformed affine Lie algebras?)

    Cf. Dimakis&Mueller-Hoissen, Wang&Wadati…
  - Geometrical descriptions from NC extension of the theories of Krichever, Mulase and Segal-Wilson and so on.
- Confirmation of NC Ward's conjecture
  - NC twistor theory

    e.g. Kapustin&Kuznetsov&Orlov, Hannabuss, Hannover group,…
  - D-brane interpretations → physical meanings
- Foundation of Hamiltonian formalism with space-time noncommutativity
  - Initial value problems, Liouville's theorem, Noether's thm,…