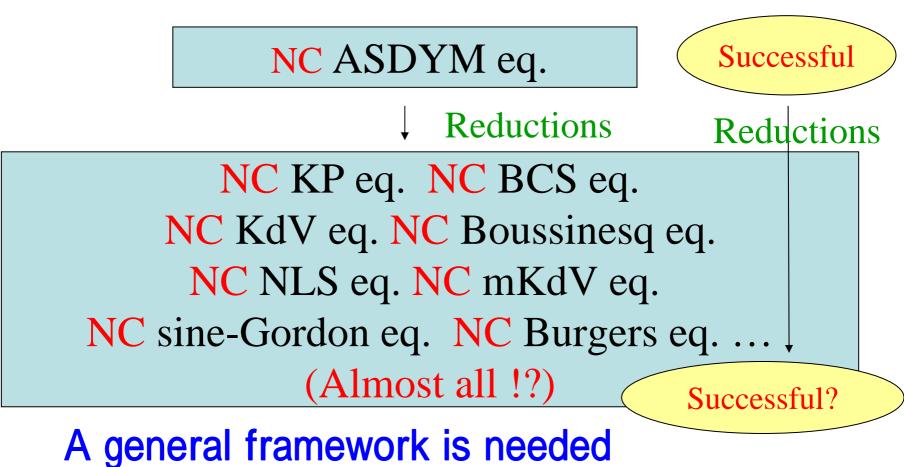


Ref: MH, ``Commuting Flows and Conservation Laws for NC Lax Hierarchies,"[hep-th/0311206]

Cf: MH, ``NC Solitons and Integrable Systems, Proc. of NCGP04 [hep-th/041mnnn]

NC Ward's observation: Almost all NC integrable equations are reductions of the NC ASDYM eqs. MH&K.Toda, PLA316(03)77[hep-th/0211148]



2. NC Sato's Theory

- Sato's Theory : one of the most beautiful theory of solitons
 - Based on the exsitence of

hierarchies and tau-functions

- Sato's theory reveals essential aspects of solitons:
 - Construction of exact solutions
 - Structure of solution spaces
 - Infinite conserved quantities
 - Hidden infinite-dim. symmetry

Let's discuss NC extension of Sato's theory

Derivation of soliton equations

• Give a Lax operator which is a pseudodifferential operator

$$L := \partial_x + u_2 \partial_x^{-1} + u_3 \partial_x^{-2} + u_4 \partial_x^{-3} + \cdots$$

• Define a differential operator

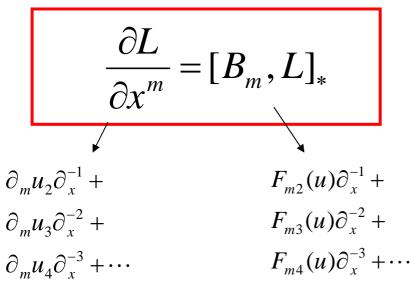
$$B_m \coloneqq (L \ast \cdots \ast L)_{\geq 0}$$

m times

$$u_{k} = u_{k}(x^{1}, x^{2}, x^{3}, \cdots)$$
Noncommutativity
is introduced here:

$$[x^{i}, x^{j}] = i\theta^{i}$$

Introduce NC (KP) hierarchy equations:



Star product: $f(x) * g(x) \coloneqq f(x) \exp\left(\frac{i}{2}\theta^{ij}\overline{\partial}_i\overline{\partial}_j\right)g(x)$

which realizes the noncommutativity

Each coefficient yields a differential equation.

NC KP hierarchy equations Infinite kind of fields u_k can be represented in terms of $u_2 \equiv u$

$$\frac{\partial u}{\partial x^3} = \frac{1}{4}u_{xxx} + \frac{3}{4}(u_x * u + u * u_x) + \frac{3}{4}\partial_x^{-1}u_{yy} + \frac{3}{4}[u, \partial_x^{-1}u_{yy}]_*$$

$$u_{x} \coloneqq \frac{\partial u}{\partial x} \quad \partial_{x}^{-1} \coloneqq \int^{x} dx' \quad \ell t C. \qquad (2+1)-\dim. \text{ NC KP equation}$$

$$\frac{\partial u}{\partial x^{4}} = f(u, u_{x}, u_{y}, \cdots) \qquad \qquad \begin{array}{l} u = u(x^{1}, x^{2}, x^{3}, \cdots) \\ \downarrow & \downarrow & \downarrow \\ x & y & t \end{array}$$

$$\frac{\partial u}{\partial x^{5}} = g(u, u_{x}, u_{y}, \cdots)$$

$$\frac{\partial u}{\partial x^{6}} = h(u, u_{x}, u_{y}, \cdots)$$
NC KP hierarchy equations

(KP hierarchy) \rightarrow (various hierarchies.)

- No-reduction \rightarrow NC KP
- 2-reduction → NC KdV
- 3-reduction → NC Boussinesq
- 4-reduction → NC Coupled KdV
- 5-reduction $\rightarrow \dots$
- 3-reduction of BKP → NC Sawada-Kotera
- 2-reduction of mKP \rightarrow NC mKdV
- Special 1-reduction of mKP → NC Burgers

Noncommutativity should be introduced into space-time coords

 $(x, y, t) = (x^{1}, x^{2}, x^{3})$ $(x, t) = (x^{1}, x^{3})$

 $(x,t) = (x^1, x^2)$

4. Conservation Laws

- Conservation laws: $\partial_t \sigma = \partial_i J^i = \sigma$: Conserved density time space Then $Q \coloneqq \int dx \sigma$ is a conserved quantity. $\because \partial_t Q = \int_{space} dx \partial_t \sigma = \int_{spatial} dS_i J^i = 0$ **Conservation laws for the hierarchies** $\partial_m res_{-1}L^n = \partial_x J + \theta^{ij}\partial_j \Xi_i$ time space I have found the explicit form ! $res_{-r}L^n$: coefficient Noncommutativity should be introduced of ∂_x^{-r} in L^n in space-time directions only. \rightarrow $t \equiv x^m$
 - ∂_j should be space or time derivative \rightarrow ordinary conservation laws !

Hot Results ______ Infinite conserved densities for NC hierarchy eqs. (n=1,2,...)

$$\sigma = \operatorname{res}_{-1}L^{n} + \theta^{\operatorname{im}}\sum_{k=0}^{m-1}\sum_{l=0}^{k}(-1)^{k-l}\binom{k}{l}(\operatorname{res}_{-(l+1)}L^{n}) \diamond(\partial_{i}\partial_{x}^{k-l}\operatorname{res}_{k}L^{m})$$

 $t \equiv x^m$ $res_r L^n$: coefficient of ∂_x^r in L^n

Strachan's product (commutative and non-associative)

$$f(x) \diamond g(x) \coloneqq f(x) \left(\sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)!} \left(\frac{1}{2} \theta^{ij} \overline{\partial}_i \overline{\partial}_j \right)^{2s} \right) g(x)$$

MH, [hep-th/0311206]

We can calculate the explicit forms of conserved densities for wide class of NC soliton equations.

- Space-Space noncommutativity: NC deformation is slight: $\sigma = res_{-1}L^n$
- Space-time noncommutativity
 NC deformation is drastical:
 - Example: NC KP and KdV equations $([t,x]=i\theta)$

$$\sigma = \operatorname{res}_{-1}L^n - 3\theta((\operatorname{res}_{-1}L^n) \diamond u'_3 + (\operatorname{res}_{-2}L^n) \diamond u'_2)$$

meaningful ?

4. Conclusion and Discussion

- We proved the existence of infinite conserved quantities for wide class of NC Lax hierarchies.
- We gave the infinite conserved densities explicitly from the viewpoint of Sato's theory, which suggests that infinite-dim. symmetry would be hidden in the NC (soliton) equations.
- The results show that NC soliton eqs. have very special properties though they include non-linear terms and infinite (time-!) derivatives.
- Of course, there are still many things to be seen in order to confirm the integrability.

Further directions

- Completion of NC Sato's theory
 - Hirota's bilinearization and tau-functions $u = \partial_x^2 \log \tau$
 - → hidden symmetry (deformed affine Lie algebras?)

Cf. Dimakis&Mueller-Hoissen, Wang&Wadati...

- Geometrical descriptions from NC extension of the theories of Krichever, Mulase and Segal-Wilson and so on.
- Confirmation of NC Ward's conjecture
 - NC twistor theory
 - e.g. Kapustin&Kuznetsov&Orlov, Hannabuss, Hannover group,...
 - D-brane interpretations → physical meanings
- Foundation of Hamiltonian formalism with space-time noncommutativity
 - Initial value problems, Liouville's theorem, Noether's thm,...