ソリトン理論·可積分系の非可換空間への拡張

浜中　真志
(名古屋大学　多元数理)

Ref: MH, ``Commuting Flows and Conservation Laws for NC Lax Hierarchies,'' [hep-th/0311206]

cf. MH, ``Solitons on Non-Commutative spaces''
数研講究録 掲載予定 [アインシュタイン牧場]

URL: http://www2.yukawa.kyoto-u.ac.jp/~hamanaka

1. Introduction

Successful points on NC theories

- Appearance of new physical objects
- Description of real physics
- Various successful applications to D-brane dynamics etc.

NC Solitons play important roles (Integrable!)

Final goal: NC extension of all soliton theories

Ward's observation:

Almost all integrable equations are reductions of the ASDYM eqs.

R.Ward, Phil.Trans.Roy.Soc.Lond.A315('85)451

Anti-Self-Dual Yang-Mills eq.

$\downarrow$ Reductions

KP eq. BCS eq.
KdV eq. Boussinesq eq.
NLS eq. mKdV eq.
sine-Gordon eq. Burgers eq. …
(Almost all !)

e.g. [Mason&Woodhouse]

NC Ward's observation: Almost all NC integrable equations are reductions of the NC ASDYM eqs.

MH&K.Toda, PLA316('03)77[hep-th/0211148]

NC ASDYM eq.

Successful

↓ Reductions

Reductions

NC KP eq. NC BCS eq.
NC KdV eq. NC Boussinesq eq.
NC NLS eq. NC mKdV eq.
NC sine-Gordon eq. NC Burgers eq. …
(Almost all !?)

Successful?

A general framework is needed

Plan of this talk

1. Introduction
2. NC Sato’s Theory
3. Conservation Laws
4. Exact Solutions and Ward’s conjecture
5. Conclusion and Discussion

2. NC Sato's Theory

- Sato's Theory : one of the most beautiful theory of solitons
  - Based on the exsitence of hierarchies and tau-functions
- Sato's theory reveals essential aspects of solitons:
  - Construction of exact solutions
  - Structure of solution spaces
  - Infinite conserved quantities
  - Hidden infinite-dim. symmetry

Let's discuss NC extension of Sato's theory

Derivation of soliton equations

- Prepare a Lax operator which is a pseudo-differential operator

$$L := \partial_x + u_2 \partial_x^{-1} + u_3 \partial_x^{-2} + u_4 \partial_x^{-3} + \cdots \qquad u_k = u_k(\underline{x^1, x^2, x^3, \cdots})$$

Noncommutativity is introduced here:

$$[x^i, x^j] = i\theta^{ij}$$

- Introduce a differential operator

$$B_m := (\underbrace{L * \cdots * L}_{m\ times})_{\geq 0}$$

- Define NC (KP) hierarchy equation:

$$\boxed{\frac{\partial L}{\partial x^m} = [B_m, L]_*}$$

$$\partial_m u_2 \partial_x^{-1} + \partial_m u_3 \partial_x^{-2} + \partial_m u_4 \partial_x^{-3} + \cdots$$

$$F_{m2}(u) \partial_x^{-1} + F_{m3}(u) \partial_x^{-2} + F_{m4}(u) \partial_x^{-3} + \cdots \longrightarrow$$

Star product:

$$f(x) * g(x) := f(x) \exp\left(\frac{i}{2}\theta^{ij}\overleftarrow{\partial}_i \overrightarrow{\partial}_j\right) g(x)$$

which realizes the noncommutativity

Each coefficient yields a differential equation.

Closer look at NC (KP) hierarchy

For m=2

$$\partial_x^{-1})\quad \partial_2 u_2 = \underline{2u_3'} + u_2''$$

$$\partial_x^{-2})\quad \partial_2 u_3 = \underline{2u_4'} + u_3'' + 2u_2 * u_2' + 2[u_2, u_3]_*$$

$$\partial_x^{-3})\quad \partial_2 u_4 = \underline{2u_5'} + u_4'' + 4u_3 * u_2' - 2u_2 * u_2'' + 2[u_2, u_4]_*$$

$$\vdots$$

Infinite kind of fields are represented in terms of one kind of field $u_2 \equiv u$

MH&K.Toda, [hep-th/0309265]

$$u_x := \frac{\partial u}{\partial x}$$

$$\partial_x^{-1} := \int^x dx'$$

For m=3

$$\partial_x^{-1})\quad \partial_3 u_2 = u_2''' + 3u_3'' + 3u_4'' + 3u_2' * u_2 + 3u_2 * u_2'$$

etc.

$$\vdots$$

$$\downarrow$$

$$u_t = \frac{1}{4}u_{xxx} + \frac{3}{4}(u_x * u + u * u_x) + \frac{3}{4}\partial_x^{-1}u_{yy} + \frac{3}{4}[u, \partial_x^{-1}u_{yy}]_*$$

(2+1)-dim. NC KP equation

and other NC equations (NC hierarchy equations)

$$u = u(x^1, x^2, x^3, \cdots)$$

$$\begin{matrix} \updownarrow & \updownarrow & \updownarrow \\ x & y & t \end{matrix}$$

(KP hierarchy) $\xrightarrow{\text{reductions}}$ (various hierarchies.)

- (Ex.) KdV hierarchy

Reduction condition

$$L^2 = B_2 (=: \partial_x^2 + u)$$: 2-reduction

gives rise to NC KdV hierarchy

which includes (1+1)-dim. NC KdV eq.:

$$u_t = \frac{1}{4} u_{xxx} + \frac{3}{4} (u_x * u + u * u_x)$$

Note $\frac{\partial u}{\partial x_{2N}} = 0$: dimensional reduction in x_{2N} directions

KP : $u(x^1, x^2, x^3, x^4, x^5, \ldots)$; $x \; y \; t$ **: (2+1)-dim.**

$\downarrow$

KdV : $u(x^1, \; x^3, \; x^5, \ldots)$; $x \; t$ **: (1+1)-dim.**

l-reduction of NC KP hierarchy yields wide class of other NC hierarchies

- No-reduction → NC KP $(x, y, t) = (x^1, x^2, x^3)$
- 2-reduction → NC KdV $(x, t) = (x^1, x^3)$
- 3-reduction → NC Boussinesq $(x, t) = (x^1, x^2)$
- 4-reduction → NC Coupled KdV …
- 5-reduction → …
- 3-reduction of BKP → NC Sawada-Kotera
- 2-reduction of mKP → NC mKdV
- Special 1-reduction of mKP → NC Burgers
- …

Noncommutativity should be introduced into space-time coords

3. Conservation Laws

- Conservation laws: $\partial_t \sigma = \partial_i J^i$ σ : Conserved density

time   space

Then $Q := \int_{space} dx \sigma$ is a conserved quantity.

$$\because \partial_t Q = \int_{space} dx \partial_t \sigma = \int_{\substack{spatial \\ \inf inity}} dS_i J^i = 0$$

Infinite conservation laws for the hierarchies

$$\partial_m res_{-1} L^n = \partial_x J + \underline{\theta^{ij} \partial_j \Xi_i}$$

time   space

I have succeeded in the evaluation explicitly !

$res_{-r} L^n$: coefficient of ∂_x^{-r} in L^n

Noncommutativity should be introduced in space-time directions only. →

$t \equiv x^m$

∂_j **should be space or time derivative**

→ ordinary conservation laws !

Hot (old?) Results

Infinite conserved densities for NC hierarchy eqs. (n=1,2,..., ∞)

$$\sigma = res_{-1}L^n + \theta^{im}\sum_{k=0}^{m-1}\sum_{l=0}^{k}(-1)^{k-l}\binom{k}{l}(res_{-(l+1)}L^n)\lozenge(\partial_i\partial_x^{k-l}res_k L^m)$$

$t \equiv x^m$ $\quad$ $res_r L^n$: coefficient of ∂_x^r in L^n

$\lozenge$: Strachan's product (commutative and non-associative)

$$f(x)\lozenge g(x) := f(x)\left(\sum_{s=0}^{\infty}\frac{(-1)^s}{(2s+1)!}\left(\frac{1}{2}\theta^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right)^{2s}\right)g(x)$$

MH, [hep-th/0311206]

We can calculate the explicit forms of conserved densities for wide class of NC soliton equations.

- Space-Space noncommutativity:

  NC deformation is slight: $\sigma = res_{-1} L^n$

- Space-time noncommutativity

  NC deformation is drastical:

  – Example: NC KP and KdV equations $([t, x] = i\theta)$

$$\sigma = res_{-1} L^n - 3\theta((res_{-1} L^n) \Diamond u_3' + (res_{-2} L^n) \Diamond u_2')$$

meaningful ?

4. Exact Solutions and Ward's conjecture

- We have found exact N-soliton solutions.
- 1-soliton solutions are all the same as commutative ones because of

$$f(x-vt)*g(x-vt)=f(x-vt)g(x-vt)$$

- Multi-soliton solutions behave in almost the same way as commutative ones except for phase shifts.
- Noncommutativity affects the phase shifts

Exact N-soliton solutions of the NC KP hierarchy

$$L = \Phi \partial_x \Phi^{-1}$$ **solves the NC KP hierarchy !**

$$\Phi f := \left| W(y_1, \ldots, y_N, f) \right|_{N+1,N+1}$$ **quasi-determinant of Wronski matrix**

$$y_i = \exp \xi(x, \alpha_i) + a_i \exp \xi(x, \beta_i)$$

$$\xi(x, \alpha) = x_1 \alpha + x_2 \alpha^2 + x_3 \alpha^3 + \cdots$$

Etingof-Gelfand-Retakh, [q-alg/9701008]

The exact solutions are actually N-soliton solutions !

Noncommutativity might affect the phase shift by $\theta^{ij} \omega_i k_j$

[MH, work in progress]

$$\because) \exp(\omega_i t - k_i x) * \exp(\omega_j t - k_j x)$$

$$\cong \exp([t, x]_* \omega_{[i} k_{j]}) \exp((\omega_i + \omega_j) t - (k_i + k_j) x)$$

Exactly solvable!

4. Exact Solutions and Ward's conjecture

- We have found exact N-soliton solutions for the wide class of NC hierarchies.
- 1-soliton solutions are all the same as commutative ones because of

$$f(x-vt)*g(x-vt)=f(x-vt)g(x-vt)$$

- Multi-soliton solutions behave in almost the same way as commutative ones except for phase shifts.
- Noncommutativity might affect the phase shifts

NC Burgers hierarchy

MH&K.Toda,JPA36('03)11981[hep-th/0301213]

- NC (1+1)-dim. Burgers equation:

$$\dot{u} = u'' + 2u * u'$$: Non-linear &

Infinite order diff. eq. w.r.t. time ! (Integrable?)

NC Cole-Hopf transformation

$$u = \tau^{-1} * \tau' \quad \left(\xrightarrow{\theta \to 0} \partial_x \log \tau \right)$$

(NC) Diffusion equation:

$$\dot{\tau} = \tau''$$: Linear & first order diff. eq. w.r.t. time

(Integrable !)

A solution : $$\tau = 1 + \sum_{l=1}^{N} e^{k_l^2 t} * e^{\pm k_l x} = 1 + \sum_{l=1}^{N} \underline{e^{\frac{i}{2} k_l^3 \theta}} e^{k_l^2 t \pm k_l x}$$

Deformed!

NC Ward's conjecture (NC NLS eq.)

- Reduced ASDYM eq.: $x^{\mu} \to (t, x)$

Legare, [hep-th/0012077]

$$(i) \quad B' = 0$$

$$(ii) \quad C' - \dot{A} + [A, C]_* = 0$$

$$(iii) \quad A' - \dot{B} + [C, B]_* = 0$$

A, B, C: 2 times 2 matrices (gauge fields)

Further Reduction:

$$A = \begin{pmatrix} 0 & q \\ -\overline{q} & 0 \end{pmatrix}, \; B = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; C = -i\begin{pmatrix} q * \overline{q} & q' \\ q' & -\overline{q} * q \end{pmatrix}$$

NOT traceless

$$(ii) \Rightarrow \begin{pmatrix} 0 & i\dot{q} - q'' - 2q * \overline{q} * q \\ i\dot{\overline{q}} + \overline{q}'' + 2\overline{q} * q * \overline{q} & 0 \end{pmatrix} = 0$$

$$i\dot{q} = q'' + 2q * \overline{q} * q$$: NC NLS eq.

Note: $A, B, C \in gl(2, C) \xrightarrow{\theta \to 0} sl(2, C)$

U(1) part is important

NC Ward's conjecture (NC Burgers eq.)

- Reduced ASDYM eq.: $x^\mu \to (t, x)$

MH&K.Toda, JPA [hep-th/0301213]

G=U(1)

$$(i) \quad \dot{A} + \underline{[B, A]_*} = 0$$

$$(ii) \quad \dot{C} - B' + \underline{[B, C]_*} = 0$$

should remain

A, B, C: 1 times 1 matrices (gauge fields)

Further Reduction: $A = 0,\ B = u' - u^2,\ C = u$

$$(ii) \Rightarrow \quad \dot{u} = u'' + 2u' * u$$

: NC Burgers eq.

Note: Without the commutators [,], (ii) yields:

$$\dot{u} = u'' + \underline{u' * u + u * u'}$$

: neither linearizable nor Lax form

symmetric

5. Conclusion and Discussion

- We proved the existence of infinite conserved quantities and exact N-soliton solutions for wide class of NC Lax hierarchies.
- We gave the infinite conserved densities explicitly from the viewpoint of Sato's theory, which suggests that infinite-dim. symmetry would be hidden in the NC (soliton) equations.
- The results show that NC soliton eqs. have very special properties though they include non-linear terms and infinite (time!) derivatives.

Further directions

- Completion of NC Sato's theory
  - Theory of tau-functions → hidden symmetry (deformed affine Lie algebras?)

    Cf. Dimakis&Mueller-Hoissen, Wadati group, …
  - Geometrical descriptions from NC extension of the theories of Krichever, Mulase and Segal-Wilson and so on.
- Confirmation of NC Ward's conjecture
  - NC twistor theory

    e.g. Kapustin&Kuznetsov&Orlov, Hannabuss, Hannover group,…
  - D-brane interpretations → physical meanings
- Foundation of Hamiltonian formalism with space-time noncommutativity
  - Initial value problems, Liouville's theorem, Noether's thm,…