ソリトン・インスタントン と非可換幾何学

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Ref: MH, ``Commuting Flows and Conservation Laws for NC Lax Hierarchies,''[hep-th/0311206]

cf. MH, ``Solitons on Non-Commutative spaces''
数研講究録 掲載予定 [アインシュタイン牧場]

URL: http://www2.yukawa.kyoto-u.ac.jp/~hamanaka

1. Introduction

- Non-Commutative (NC) spaces are defined by noncommutativity of the coordinates:

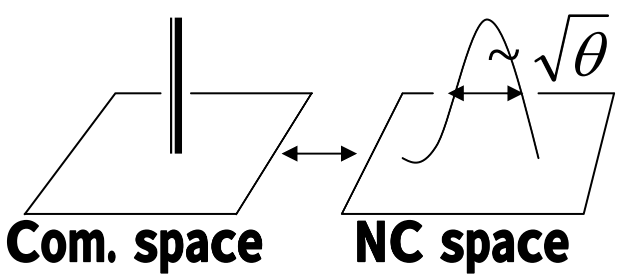
$$[x^i, x^j] = i\theta^{ij}$$

θ^{ij} **: NC parameter**

This looks like CCR in QM: $[q, p] = i\hbar$

(→ ``space-space uncertainty relation'' →)

Resolution of singulality



(→ New physical objects)

e.g. resolution of small instanton singularity

(→ U(1) instantons)

NC gauge theories ⟷ Com. gauge theories in background of magnetic fields

↓

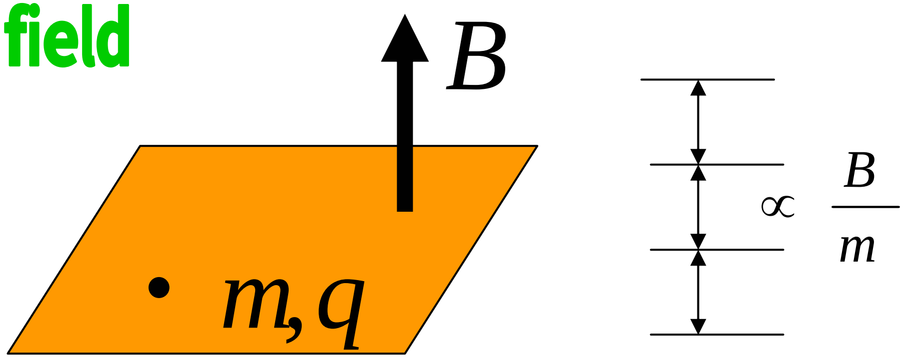
(real physics)

(Ex.) Motion of a charged particle in background magnetic field

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + qBx\dot{y}$$

$$\downarrow \quad m \to 0$$

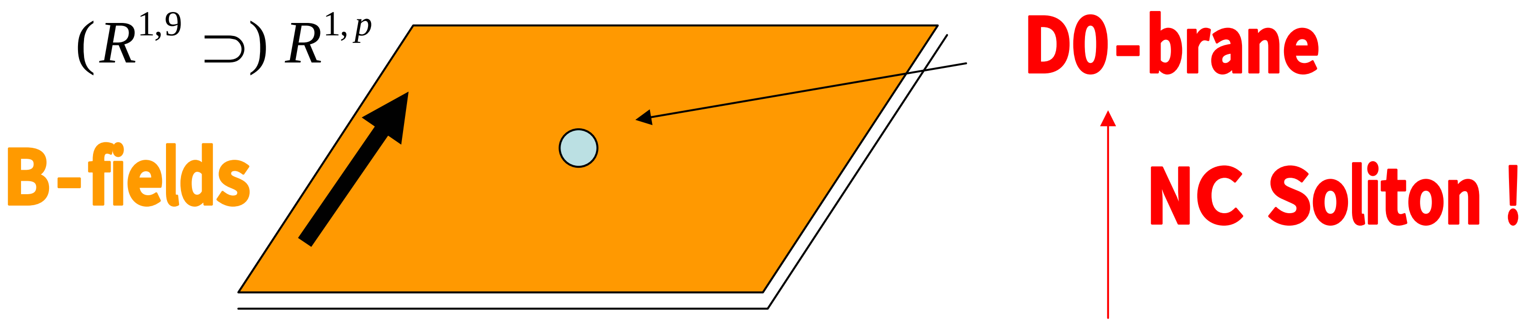
$$L = qBx\dot{y}$$



$$p_y = \frac{\partial L}{\partial \dot{y}} = qB\underline{x} \xrightarrow{[y,\, \underline{p_y}] = i} [x, y] = -i\frac{1}{qB}$$

Gauge theories are realized on D-branes
which are solitons in string theories

In this context, (NC) solitons are (lower-dim.) D-branes



N Dp-branes → G=U(N) (p+1)-dim. NC Gauge theory

Analysis of NC solitons ⟷ Analysis of D-branes

(easy to treat)

↓

Various applications

e.g. confirmation of Sen's conjecture on decay of D-branes

Plan of this talk

1. Introduction
2. NC gauge theories
3. ADHM construction of (NC) instantons
4. Applications to D-brane dynamics

ゲージ理論

5. NC extension of soliton theories
6. NC Sato's theories
7. Conservation Laws
8. Exact Solutions and Ward's conjecture
9. Conclusion and Discussion

ソリトン理論

2. NC Gauge Theories

Here we discuss NC gauge theory of instantons.

(Ex.) 4-dim. (Euclidean) G=U(N) Yang-Mills theory

- Action

$$S=-\frac{1}{2}\int d^4x\,Tr\,F_{\mu\nu}F^{\mu\nu}=-\frac{1}{4}\int d^4x\,Tr\left(F_{\mu\nu}F_{\mu\nu}+\tilde{F}_{\mu\nu}\tilde{F}_{\mu\nu}\right)$$

$$=-\frac{1}{4}\int d^4x\,Tr\left[\underbrace{\left(F_{\mu\nu}\mp\tilde{F}_{\mu\nu}\right)^2}_{=0\,\Leftrightarrow\,\text{BPS}}\pm\underbrace{2F_{\mu\nu}\tilde{F}_{\mu\nu}}_{\leftrightarrow C_2}\right]$$

$$(F_{\mu\nu}:=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu])$$

- Eq. Of Motion:

$$[D^\nu,[D_\nu,D_\mu]]=0$$

- BPS eq. (=(A)SDYM eq.)

$$F_{\mu\nu}=\pm\tilde{F}_{\mu\nu}\qquad\rightarrow\ \text{instantons}$$

$$(\Leftrightarrow F_{z_1\bar{z}_1}+F_{z_2\bar{z}_2}=0,\quad F_{z_1z_2}=0)$$

(i) Star-product formalism

NC theories are obtained from commutative gauge theories by replacing products of fields with star-products: $f(x)g(x) \to f(x)*g(x)$

- The star product:

$$f(x)*g(x) := f(x)\exp\left(\frac{i}{2}\theta^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right)g(x)$$

$$= f(x)g(x) + i\frac{\theta^{ij}}{2}\partial_i f(x)\partial_j g(x) + O(\theta^2)$$

A deformed product $\longleftrightarrow$ **Presence of background magnetic fields**

$f*(g*h) = (f*g)*h$ Associative

$[x^i, x^j]_* := x^i * x^j - x^j * x^i = i\theta^{ij}$ NC !

(Ex.) 4-dim. NC (Euclidean) G=U(N) Yang-Mills theory (All products are star products)

- Action

$$S = -\frac{1}{2}\int d^4x\, Tr\, F_{\mu\nu} * F^{\mu\nu} \quad = -\frac{1}{4}\int d^4x\, Tr\left(F_{\mu\nu} * F_{\mu\nu} + \tilde{F}_{\mu\nu} * \tilde{F}_{\mu\nu}\right)$$

$$= -\frac{1}{4}\int d^4x\, Tr\Big[\underbrace{\left(F_{\mu\nu} \mp \tilde{F}_{\mu\nu}\right)_*^2}_{=0 \Leftrightarrow \text{BPS}} \pm \underbrace{2F_{\mu\nu} * \tilde{F}_{\mu\nu}}_{\leftrightarrow C_2}\Big]$$

$$(F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + \underline{[A_\mu, A_\nu]_*})$$

Don't omit even for G=U(1)

$$(\because U(1) \cong U(\infty))$$

- Eq. Of Motion:

$$[D^\nu, [D_\nu, D_\mu]_*]_* = 0$$

- BPS eq. (=NC (A)SDYM eq.)

$$F_{\mu\nu} = \pm\tilde{F}_{\mu\nu}$$

→ NC instantons

$$(\Leftrightarrow F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0, \quad F_{z_1z_2} = 0)$$

A deformed theory is obtained.

Notes

- Noncommutativity violates Lorentz symmetry.
- U(1) part of gauge groups play important roles:
  - It behaves as non-abelian
  - SU(N) gauge theories are basically forbidden

$$g_1, g_2 \in SU(N) \Rightarrow g_1 * g_2 \notin SU(N)$$

- Star-products contain infinite derivatives
  - Non-local (→ Quantum treatment is hard)
  - Impossible to solve??? (solvable !!!)

Today's discussions are all classical.

(ii) Operator formalism

- This time, we begin with noncommutativity of coords. and treat the coords.and functions as operators.
- For simplicity, let's consider NC plane:

$$[\hat{x}^1, \hat{x}^2] = i\theta \qquad (\theta > 0)$$

$$\downarrow \quad \hat{a} := \frac{1}{\sqrt{2\theta}}\hat{z},\ \hat{a}^* := \frac{1}{\sqrt{2\theta}}\hat{\bar{z}}, \quad z := x^1 + ix^2$$

$$[\hat{a}, \hat{a}^*] = 1$$

occupation number basis

Ann. op. Cre. Op. acting on Fock sp. $H = \langle |0\rangle, |1\rangle, |2\rangle, \cdots \rangle$

- Fields are also operators in the Fock space:

$$\hat{f}(\hat{x}^1, \hat{x}^2) = \sum_{m,n=0}^{\infty} f_{mn} |m\rangle\langle n|$$

Infinite-size matrix

Equivalence of the two formalisms

(i) Star-product formalism	← Weyl trf. →	(ii) Operator formalism
$[x^1, x^2]_* = i\theta$	Noncommutativity	$[\hat{x}^1, \hat{x}^2] = i\theta$
$f(x^1, x^2)$	Fields	$\hat{f} = \sum_{m,n=0}^{\infty} f_{mn} \lvert m\rangle\langle n\rvert$
Star product: *	Products $(\widehat{f * g} = \hat{f} \circ \hat{g})$	Matrix multiplication: $\circ$
$\partial_i f$ $(\partial_i x^j = \delta_i^{\ j})$	Differentiation	$\partial_i \hat{f} := [\hat{\partial}_i, \hat{f}] := [-i(\theta^{-1})_{ij} \hat{x}^j, \hat{f}]$ $(\partial_i \hat{x}^j = \delta_i^{\ j})$
$\int dx^1 dx^2 f(x^1, x^2)$	Integration	$2\pi\theta\, Tr_H \hat{f}$
$2\exp\left\{-\frac{r^2}{\theta}\right\}$ (Gaussian)	A projection	$\lvert 0\rangle\langle 0\rvert$

4-dim. NC (Euclidean) G=U(N) Yang-Mills theory

EOM: $[\hat{D}^{\nu},[\hat{D}_{\nu},\hat{D}_{\mu}]]=0$ $\qquad\infty$

BPS eq.: $\hat{F}_{z_1\bar{z}_1}+\hat{F}_{z_2\bar{z}_2}=0,\quad \hat{F}_{z_1z_2}=0$

Noncommutativity:

$$\theta^{\mu\nu}=\begin{array}{cc}\left[\begin{array}{cc|cc} 0 & \theta^1 & & \\ -\theta^1 & 0 & \multicolumn{2}{c}{O} \\ \hline & & 0 & \theta^2 \\ \multicolumn{2}{c|}{O} & -\theta^2 & 0 \end{array}\right] & \begin{array}{l} \leftrightarrow H_1 \\ \\ \leftrightarrow H_2 \end{array}\end{array}$$

Fields: $\hat{f}(\hat{x}^{\mu})=\sum C_{m_1m_2n_1n_2}|m_1\rangle\langle n_1|\otimes|m_2\rangle\langle n_2|$

3. ADHM construction of (NC) instantons

Atiyah-Drinfeld-Hitchin-Manin, PLA65('78)185

ADHM eq. (G=``U(k)''): k × k matrix eq.

$$[B_1,B_1^*]+[B_2,B_2^*]+I\,I^*-J^*J=0$$
$$[B_1,B_2]+IJ=0$$

ADHM data $B_{1,2}:k\times k,\quad I:k\times N,\quad J:N\times k$

1:1

Instantons $A_\mu:N\times N$

ASD eq. (G=U(N), C_2=-k): N × N PDE

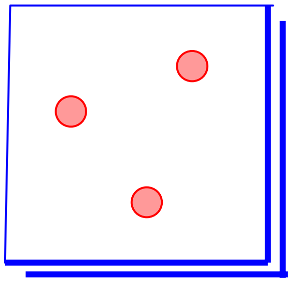
$$F_{z_1\bar{z}_1}+F_{z_2\bar{z}_2}=0$$
$$F_{z_1z_2}=0$$

D-brane's interpretation

Douglas, Witten

BPS

k D0-branes



N D4-branes

BPS

[cf. MH, 素粒子論研究 106 (2002) 1]

String theory is a treasure house of dualities

ADHM construction of BPST instanton (N=2,k=1)

ADHM eq. (G=``U(1)'')

$$[B_1, B_1^*] + [B_2, B_2^*] + I I^* - J^* J = 0$$

$$[B_1, B_2] + IJ = 0$$

$$B_{1,2} = \alpha_{1,2}, \quad I = (\rho, 0),\ J = \begin{pmatrix} 0 \\ \rho \end{pmatrix}$$

position **size**

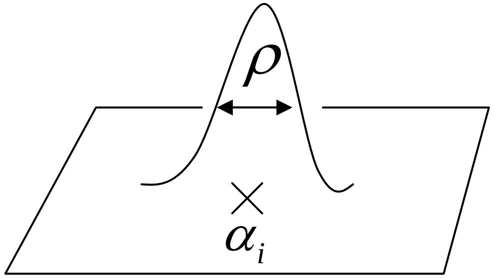
$$A_\mu = \frac{i(x-b)^\nu \eta^{(-)}_{\mu\nu}}{(x-b)^2 + \rho^2}, \quad F_{\mu\nu} = \frac{2i\rho^2}{((x-b)^2+\rho^2)^2}\eta^{(-)}_{\mu\nu} \xrightarrow{\rho \to 0} \text{singular}$$

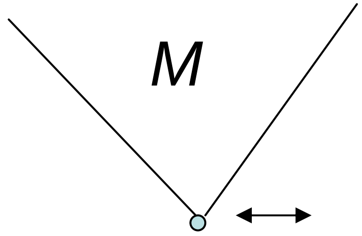
ASD eq. (G=U(2), C_2=-1)

$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$$

$$F_{z_1 z_2} = 0$$

Final remark: matrices B and coords. z always appear in pair: z-B





$\rho = 0$

Small instanton singularity

ADHM construction of NC BPST instanton (N=2,k=1)

Nekrasov&Schwarz, CMP198('98)689 [hep-th/9802068]

ADHM eq. (G=``U(1)'') 1 × 1 matrix eq.

$$[B_1, B_1^*]+[B_2, B_2^*]+I\,I^* - J^*J = \zeta$$

$$[B_1, B_2]+IJ = 0$$

$$B_{1,2} = \alpha_{1,2}, \qquad I = (\sqrt{\rho^2 + \underline{\zeta}}, 0),\ J = \begin{pmatrix} 0 \\ \rho \end{pmatrix}$$

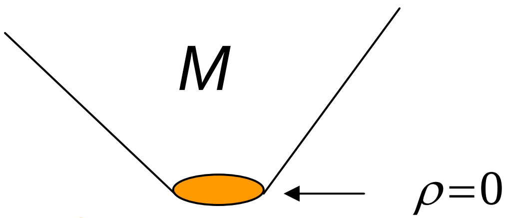
$\updownarrow$ position $\qquad$ $\updownarrow$ size → slightly fat?

$A_\mu, F_{\mu\nu}$: something smooth $\xrightarrow[\rho \to 0]{}$ Regular! (U(1) instanton!)

ASD eq. (G=U(2), C_2=-1)

$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$$

$$F_{z_1 z_2} = 0$$

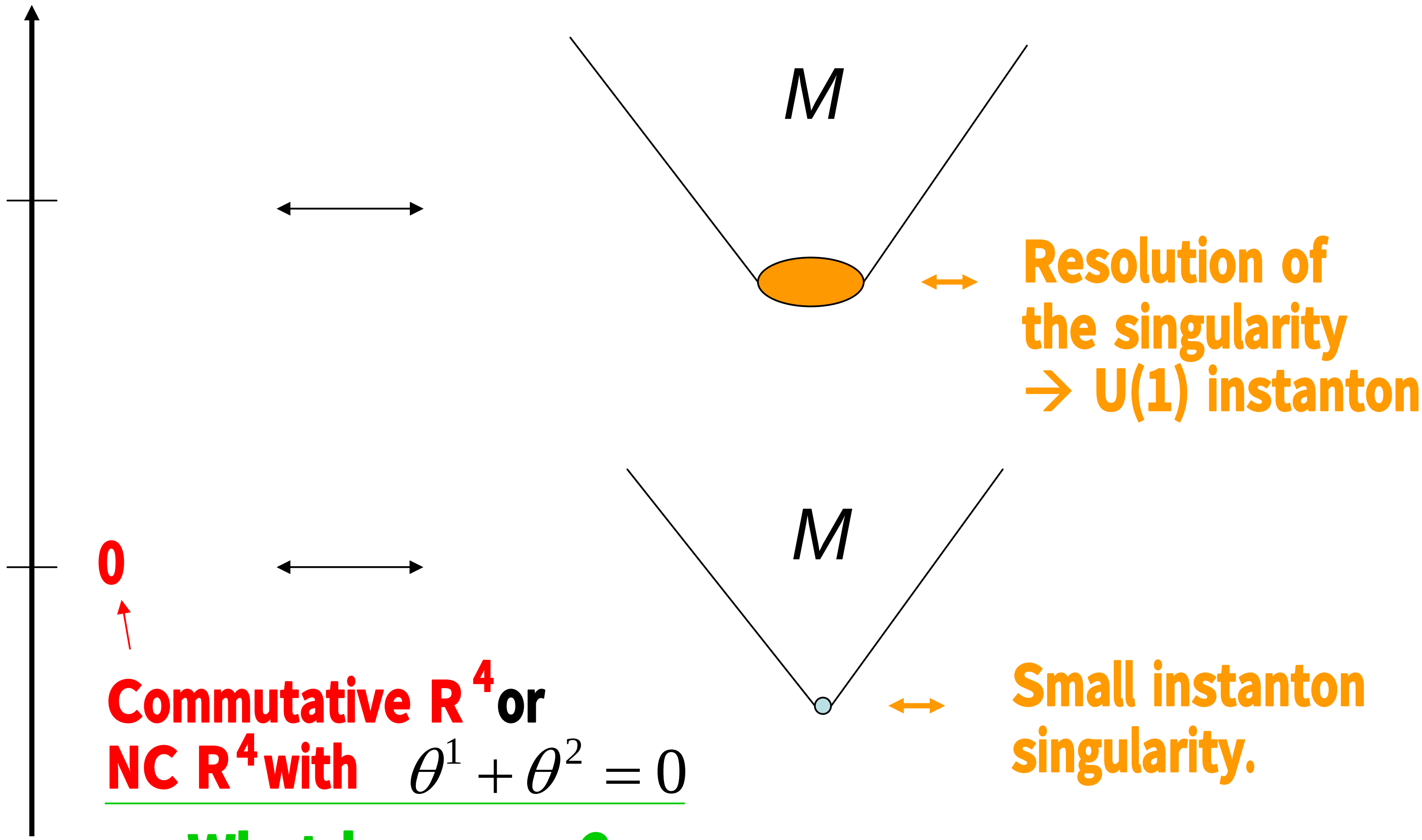


Resolution of the singularity

Comments on instanton moduli M_ζ

H. Nakajima

$$\zeta \propto \theta^1 + \theta^2$$



What happens ?

ADHM construction of NC U(1) instanton w/ ζ =0

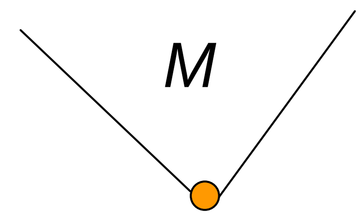
MH, PRD65('02)085022 [hep-th/0109070]

ADHM eq. (G=``U(1)'') 1 × 1 matrix eq.

$$[B_1, B_1^*] + [B_2, B_2^*] + I I^* - J^* J = 0$$
$$[B_1, B_2] + IJ = 0$$

$$B_{1,2} = \alpha_{1,2},\ I = 0,\ J = 0$$

position   size → singular?



Small instanton singularity

$$D_{z_i} = U_1^* \partial_{z_i} U_1 - \frac{\bar{\alpha}_i}{2\theta^i} |0\rangle\langle 0|, \quad F_{12} = -F_{34} = \frac{i}{\theta}|0\rangle\langle 0| \leftrightarrow \frac{2i}{\theta}\exp\left\{-\frac{r^2}{\theta}\right\}$$

Solution Generating Technique !

Regular!

Inconsistent?

ASD eq. (G=U(1), C_2=-1)

$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$$
$$F_{z_1 z_2} = 0$$

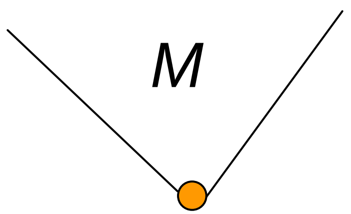
$$U_1 U_1^* = 1,$$
$$U_1^* U_1 = 1 - |0\rangle\langle 0|$$

ADHM construction of NC U(1) instanton w/ ζ =0

MH, PRD65('02)085022 [hep-th/0109070]

ADHM eq. (G=``U(1)'') 1 × 1 matrix eq.

$$[B_1, B_1^*]+[B_2, B_2^*]+I\,I^* - J^*J = 0$$
$$[B_1, B_2]+IJ = 0$$



$$B_{1,2} = \alpha_{1,2},\ I = 0,\ J = 0$$

position ↕ (for $B_{1,2}=\alpha_{1,2}$)   size → singular? ↕ (for $I=0,\ J=0$)

Small instanton singularity

consistent

$$D_{z_i} = U_1 \hat{\partial}_{z_i} U_1^* - \frac{\bar{\alpha}_i}{2\theta^i}|0\rangle\langle 0|, \quad F_{12} = -F_{34} = \frac{i}{\theta}|0\rangle\langle 0| \leftrightarrow \frac{2i}{\theta}\exp\left\{-\frac{r^2}{\theta}\right\}$$

<NC side>

Regular!

ASD eq. (G=U(1), C_2=-1)

$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$$
$$F_{z_1 z_2} = 0$$

<Com. side>

↓ **SW map**

$$J_{D0}(x) = \underline{\frac{2}{\theta^2}} + \underline{\delta^{(2)}(z_1-\alpha_1)\delta^{(2)}(z_2-\alpha_2)}$$
$$\| $$
$$F_{\mu\nu}\tilde{F}^{\mu\nu}$$

D4 (=infinite D0s)   **a D0 (singular)**

4. Applications to D-brane Dynamics

- Solution Generating Technique (SGT)

based on an auto-Backlund transformation:

$$D_{z_i} \to U_k^* D_{z_i} U_k - \sum_{m=1}^{k} \frac{\overline{\alpha}_i^{(m)}}{2\theta^i} |p_m\rangle\langle p_m|$$

: Almost gauge transformation

$$U_k U_k^* = 1, \qquad \langle p_m | p_n \rangle = \delta_{mn},$$

$$U_k^* U_k = 1 - P_k = 1 - \sum_{m=1}^{k} |p_m\rangle\langle p_m|$$

Almost unitary operator **[Harvey-Kraus-Larsen]**

which leaves EOM as it is: $\dfrac{\delta L}{\delta \phi} \to U_k^* \dfrac{\delta L}{\delta \phi} U_k$

$$e.g.\quad [D^\nu,[D_\nu,D_\mu]] \to [U_k^* D^\nu U_k,[U_k^* D_\nu U_k, U_k^* D_\mu U_k]] \qquad (U_k | p_m\rangle = 0)$$

$$= [U_k^* D^\nu,[D_\nu, D_\mu U_k]] = U_k^*[D^\nu,[D_\nu,D_\mu]]U_k$$

- SGT is independent of the details of actions
- SGT generates non-trivial soliton solutions from trivial solutions.
- (Ex.) Confirmation of Sen's conjecture on decay of D-branes via tachyon condensation

Effective action of D25-brane in SFT:

$$S = \frac{c}{G_s}\int d^{24}x (2\pi\theta Tr_H) L$$

$$L = -V(T-1)\sqrt{\det(G_{\mu\nu} + 2\pi\alpha'(F-B)_{\mu\nu})} + \cdots$$

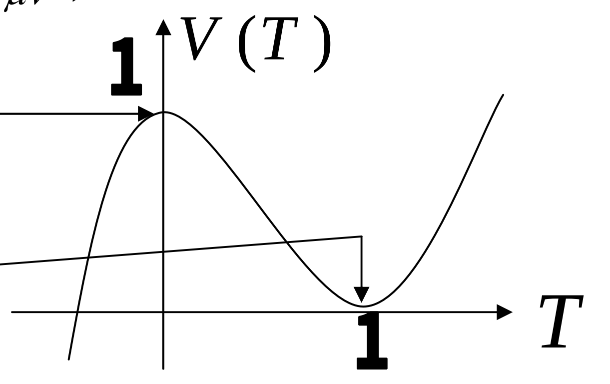
Unstable D25-brane

$$V(T=0)=1$$

$$c = T_{D25} g_s$$

Closed string vacuum

$$V(T=1)=0$$



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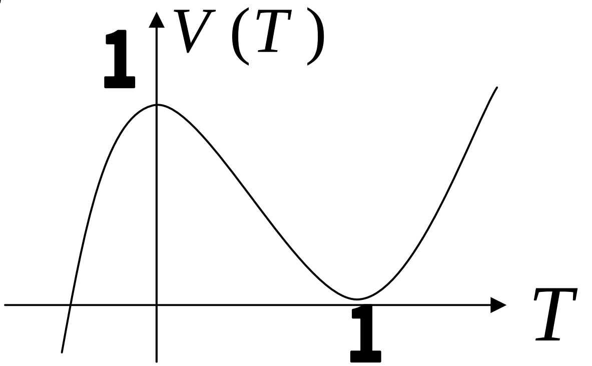
Sen's conjecture:

The EOM has lower-dim. D-brane solutions.

Solve it ! → impossible???



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- (Ex.) Confirmation of Sen's conjecture on decay of D-branes via tachyon condensation

Effective action of D25-brane in SFT:

$$S = \frac{c}{G_s} \int d^{24}x (2\pi\theta Tr_H) L$$

$L =$ **(complicated but gauge invariant !)**

$$T = 1, D_z = \hat{\partial}_z, A_t = 0$$ **: closed string vacuum**

$\downarrow$ **SGT**

$$T = 1 - P_k, D_z = U_k^* \hat{\partial}_z U_k, A_t = 0 \xrightarrow[\text{fluctuation}]{\text{tension}}$$ **k D23-branes !!! (exact result !)**

5. NC Extension of Soliton Theories

Soliton equations in diverse dimensions

4	Anti-Self-Dual Yang-Mills eq. (instantons) $F_{\mu\nu} = -\tilde{F}_{\mu\nu}$	NC extension (Successful)
3	Bogomol'nyi eq. (monopoles)	NC extension (Successful)
2 (+1)	KP eq. BCS eq. DS eq. …	NC extension (This section)
1 (+1)	KdV eq. Boussinesq eq. NLS eq. Burgers eq. sine-Gordon eq. Sawada-Kotera eq	NC extension (This section)

Dim. of space

Ward's observation:

Almost all integrable equations are reductions of the ASDYM eqs.

R.Ward, Phil.Trans.Roy.Soc.Lond.A315('85)451

ASDYM eq.

↓ Reductions

KP eq.   BCS eq.

KdV eq.   Boussinesq eq.

NLS eq. mKdV eq.

sine-Gordon eq.   Burgers eq. …

(Almost all !)

e.g. [Mason&Woodhouse]

NC Ward's observation: Almost all NC integrable equations are reductions of the NC ASDYM eqs.

MH&K.Toda, PLA316('03)77[hep-th/0211148]

NC ASDYM eq.

Successful

↓ Reductions

NC KP eq. NC BCS eq.
NC KdV eq. NC Boussinesq eq.
NC NLS eq. NC mKdV eq.
NC sine-Gordon eq. NC Burgers eq. …
(Almost all !?)

Successful?

A general framework is needed

6. NC Sato's Theories

- Sato's Theory : one of the most beautiful theory of solitons
  - Based on the exsitence of hierarchies and tau-functions
- Sato's theory reveals essential aspects of solitons:
  - Construction of exact solutions
  - Structure of solution spaces
  - Infinite conserved quantities
  - Hidden infinite-dim. Symmetry

Let's discuss NC extension of Sato's theory

Derivation of soliton equations

- Prepare a Lax operator which is a pseudo-differential operator

$$L := \partial_x + u_2\partial_x^{-1} + u_3\partial_x^{-2} + u_4\partial_x^{-3} + \cdots \qquad u_k = u_k(\underline{x^1, x^2, x^3, \cdots})$$

Noncommutativity is introduced here:

$$[x^i, x^j] = i\theta^{ij}$$

(Arbitrarily)

- Introduce a differential operator

$$B_m := (\underbrace{L * \cdots * L}_{m\ times})_{\ge 0}$$

- Define NC (KP) hierarchy equation:

$$\boxed{\frac{\partial L}{\partial x^m} = [B_m, L]_*}$$

$$\partial_m u_2 \partial_x^{-1} + \qquad F_{m2}(u)\partial_x^{-1} +$$

$$\partial_m u_3 \partial_x^{-2} + \qquad F_{m3}(u)\partial_x^{-2} +$$

$$\partial_m u_4 \partial_x^{-3} + \cdots \qquad F_{m4}(u)\partial_x^{-3} + \cdots$$

Each coefficient yields a differential equation.

Negative powers of differential operators

$$\partial_x^n \circ f := \sum_{j=0}^{\infty} \underline{\binom{n}{j}} (\partial_x^j f)\partial_x^{n-j}$$

$$|||$$

$$\frac{n(n-1)(n-2)\cdots(n-(j-1))}{j(j-1)(j-2)\cdots 1}$$

: binomial coefficient which can be extended to negative n → negative power of differential operator (well-defined !)

Ex.)

$$\partial_x^3 \circ f = f\partial_x^3 + 3f'\partial_x^2 + 3f''\partial_x^1 + f'''$$

$$\partial_x^2 \circ f = f\partial_x^2 + 2f'\partial_x + f''$$

$$\partial_x^{-1} \circ f = f\partial_x^{-1} - f'\partial_x^{-2} + f''\partial_x^{-3} - \cdots$$

$$\partial_x^{-2} \circ f = f\partial_x^{-2} - 2f'\partial_x^{-3} + 3f''\partial_x^{-4} - \cdots$$

Closer look at NC (KP) hierarchy

For m=2

$$\partial_x^{-1})\quad \partial_2 u_2 = \underline{2u_3'} + u_2''$$

$$\partial_x^{-2})\quad \partial_2 u_3 = \underline{2u_4'} + u_3'' + 2u_2 * u_2' + 2[u_2, u_3]_*$$

$$\partial_x^{-3})\quad \partial_2 u_4 = \underline{2u_5'} + u_4'' + 4u_3 * u_2' - 2u_2 * u_2'' + 2[u_2, u_4]_*$$

$\vdots$

Infinite kind of fields are represented in terms of one kind of field $u_2 \equiv u$

MH&K.Toda, [hep-th/0309265]

$$u_x := \frac{\partial u}{\partial x}$$

$$\partial_x^{-1} := \int^x dx'$$

For m=3

$$\partial_x^{-1})\quad \partial_3 u_2 = u_2''' + 3u_3'' + 3u_4'' + 3u_2' * u_2 + 3u_2 * u_2'$$

etc.

$\vdots$

$\downarrow$

$$u_t = \frac{1}{4}u_{xxx} + \frac{3}{4}(u_x * u + u * u_x) + \frac{3}{4}\partial_x^{-1}u_{yy} + \frac{3}{4}[u, \partial_x^{-1}u_{yy}]_*$$

(2+1)-dim. NC KP equation

and other NC equations (NC hierarchy equations)

$$u = u(x^1, x^2, x^3, \cdots)$$

$$\updownarrow \quad \updownarrow \quad \updownarrow$$

$$x \quad y \quad t$$

(KP hierarchy) $\xrightarrow{\text{reductions}}$ (various hierarchies.)

- (Ex.) KdV hierarchy

Reduction condition

$$L^2 = B_2 (=: \partial_x^2 + u)$$: 2-reduction

gives rise to NC KdV hierarchy

which includes (1+1)-dim. NC KdV eq.:

$$u_t = \frac{1}{4} u_{xxx} + \frac{3}{4}(u_x * u + u * u_x)$$

Note $\frac{\partial u}{\partial x_{2N}} = 0$: dimensional reduction in x_{2N} directions

KP : $u(x^1, x^2, x^3, x^4, x^5, \ldots)$ — x, y, t **: (2+1)-dim.**

↓

KdV : $u(x^1, x^3, x^5, \ldots)$ — x, t **: (1+1)-dim.**

l-reduction of NC KP hierarchy yields wide class of other NC hierarchies

- No-reduction → NC KP $(x,y,t)=(x^1,x^2,x^3)$
- 2-reduction → NC KdV $(x,t)=(x^1,x^3)$
- 3-reduction → NC Boussinesq $(x,t)=(x^1,x^2)$
- 4-reduction → NC Coupled KdV …
- 5-reduction → …
- 3-reduction of BKP → NC Sawada-Kotera
- 2-reduction of mKP → NC mKdV
- Special 1-reduction of mKP → NC Burgers
- …

Noncommutativity should be introduced into space-time coords

7. Conservation Laws

- Conservation laws: $\partial_t \sigma = \partial_i J^i$ σ : Conserved density

time $\to \partial_t$, space $\to \partial_i$

Then $Q := \int_{space} dx\sigma$ is a conserved quantity.

$$\because \partial_t Q = \int_{space} dx \partial_t \sigma = \int_{\substack{spatial \\ \inf inity}} dS_i J^i = 0$$

Conservation laws for the hierarchies

Follwing G.Wilson's approach, we have:

$$\partial_m res_{-1} L^n = \partial_x J + \underbrace{[A, B]_*}_{\text{troublesome}} = \partial_x J + \underbrace{\theta^{ij} \partial_j \Xi_i}_{\text{I have found the explicit form !}}$$

$res_{-r} L^n$: coefficient of ∂_x^{-r} in L^n

∂_j should be space or time derivative

$\downarrow$

Noncommutativity should be introduced in space-time directions only.

Hot (old?) Results

Infinite conserved densities for NC hierarchy eqs. (n=1,2,..., ∞)

$$\sigma = res_{-1}L^n + \theta^{im}\sum_{k=0}^{m-1}\sum_{l=0}^{k}(-1)^{k-l}\binom{k}{l}(res_{-(l+1)}L^n)\Diamond(\partial_i\partial_x^{k-l}res_k L^m)$$

$t \equiv x^m$ $\qquad$ $res_r L^n$: coefficient of ∂_x^r in L^n

$\Diamond$: Strachan's product (commutative and non-associative)

$$f(x)\Diamond g(x) := f(x)\left(\sum_{s=0}^{\infty}\frac{(-1)^s}{(2s+1)!}\left(\frac{1}{2}\theta^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right)^{2s}\right)g(x)$$

MH, [hep-th/0311206]

We can calculate the explicit forms of conserved densities for the wide class of NC soliton equations.

- Space-Space noncommutativity:

  NC deformation is slight: $\sigma = res_{-1} L^n$

- Space-time noncommutativity

  NC deformation is drastical:

  – Example: NC KP and KdV equations $([t, x] = i\theta)$

$$\sigma = res_{-1} L^n - 3\theta((res_{-1} L^n) \Diamond u_3' + (res_{-2} L^n) \Diamond u_2')$$

meaningful ?

8. Exact Solutions and Ward's conjecture

- We have found exact N-soliton solutions for the wide class of NC hierarchies.
- 1-soliton solutions are all the same as commutative ones because of

$$f(x-vt)*g(x-vt)=f(x-vt)g(x-vt)$$

- Multi-soliton solutions behave in almost the same way as commutative ones except for phase shifts.
- Noncommutativity might affect the phase shifts

cf. MH, proc NCGP 2004 [hep-th/0501001] (to appear?)

NC Burgers hierarchy

MH&K.Toda,JPA36(‘03)11981[hep-th/0301213]

- NC (1+1)-dim. Burgers equation:

$$\dot{u} = u'' + 2u * u'$$: Non-linear &

Infinite order diff. eq. w.r.t. time ! (Integrable?)

NC Cole-Hopf transformation

$$u = \tau^{-1} * \tau' \quad \left(\xrightarrow{\theta\to 0} \partial_x \log \tau\right)$$

(NC) Diffusion equation:

$$\dot{\tau} = \tau''$$: Linear & first order diff. eq. w.r.t. time

(Integrable !)

A solution : $$\tau = 1 + \sum_{l=1}^{N} e^{k_l^2 t} * e^{\pm k_l x} = 1 + \sum_{l=1}^{N} \underline{e^{\frac{i}{2}k_l^3\theta}}\, e^{k_l^2 t \pm k_l x}$$

Deformed!

NC Ward's observation (NC NLS eq.)

- Reduced ASDYM eq.: $x^{\mu} \to (t, x)$

Legare, [hep-th/0012077]

$$(i) \quad B' = 0$$

$$(ii) \quad C' - \dot{A} + [A, C]_* = 0$$

$$(iii) \quad A' - \dot{B} + [C, B]_* = 0$$

A, B, C: 2 times 2 matrices (gauge fields)

Further Reduction: $A = \begin{pmatrix} 0 & q \\ -\overline{q} & 0 \end{pmatrix}, \; B = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; C = -i\begin{pmatrix} q * \overline{q} & q' \\ q' & -\overline{q} * q \end{pmatrix}$

NOT traceless

$$(ii) \Rightarrow \begin{pmatrix} 0 & i\dot{q} - q'' - 2q * \overline{q} * q \\ i\dot{\overline{q}} + \overline{q}'' + 2\overline{q} * q * \overline{q} & 0 \end{pmatrix} = 0$$

$$i\dot{q} = q'' + 2q * \overline{q} * q$$ **: NC NLS eq.**

Note: $A, B, C \in gl(2, C) \xrightarrow{\theta \to 0} sl(2, C)$

U(1) part is important

NC Ward's observation (NC Burgers eq.)

MH&K.Toda, JPA [hep-th/0301213]

- Reduced ASDYM eq.: $x^\mu \to (t, x)$

  G=U(1)

$$(i) \quad \dot{A} + \underline{[B, A]_*} = 0$$

$$(ii) \quad \dot{C} - B' + \underline{[B, C]_*} = 0$$

should remain

A, B, C: 1 times 1 matrices (gauge fields)

Further Reduction: $A = 0,\ B = u' - u^2,\ C = u$

$$(ii) \Rightarrow \quad \dot{u} = u'' + 2u' * u$$

: NC Burgers eq.

Note: Without the commutators [,], (ii) yields:

$$\dot{u} = u'' + \underline{u' * u + u * u'}$$

: neither linearizable nor Lax form

symmetric

9. Conclusion and Discussion

- In this talk, we discussed
  - NC instantons where we saw that resolution of singularities yields new physical objects
  - NC soliton theories motivated by Ward's conjecture where we proved existence of infinite conserved quantities and exact multi-soliton solutions which would suggest hidden infinite-dim. symmetry.

    → New study area of integrable systems and geometry

Further directions

- Completion of NC Sato's theory
  - Theory of tau-functions → hidden symmetry (deformed affine Lie algebras?)

    Cf. Dimakis&Mueller-Hoissen, Wadati group, …
  - Geometrical descriptions from NC extension of the theories of Krichever, Mulase and Segal-Wilson and so on.
- Confirmation of NC Ward's conjecture
  - NC twistor theory

    e.g. Kapustin&Kuznetsov&Orlov, Hannabuss, Hannover group,…
  - D-brane interpretations → physical meanings
- Foundation of Hamiltonian formalism with space-time noncommutativity
  - Initial value problems, Liouville's theorem, Noether's thm,…