Soliton theories on noncommutative spaces

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MH, ``Commuting Flows and Conservation Laws for NC Lax Hierarchies,''[hep-th/0311206]

cf. MH,``NC Solitons and D-branes,''

Ph.D thesis (2003) [hep-th/0303256]

URL: http://www2.yukawa.kyoto-u.ac.jp/~hamanaka

1. Introduction

- Non-Commutative (NC) spaces are defined by noncommutativity of the coordinates:

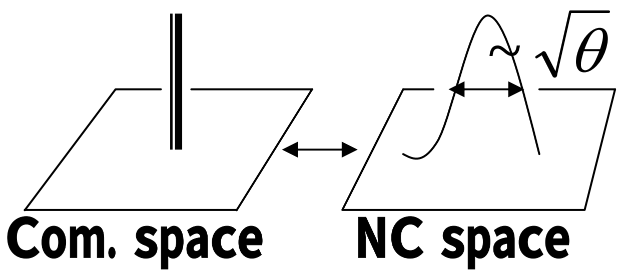
$$[x^i, x^j] = i\theta^{ij} \qquad \theta^{ij} \text{ : NC parameter}$$

This looks like CCR in QM: $[q, p] = i\hbar$

(→ ``space-space uncertainty relation'' →)

Resolution of singulality

(→ New physical objects)



e.g. resolution of small instanton singularity

(→ U(1) instantons) [Nekrasov-Schwarz]

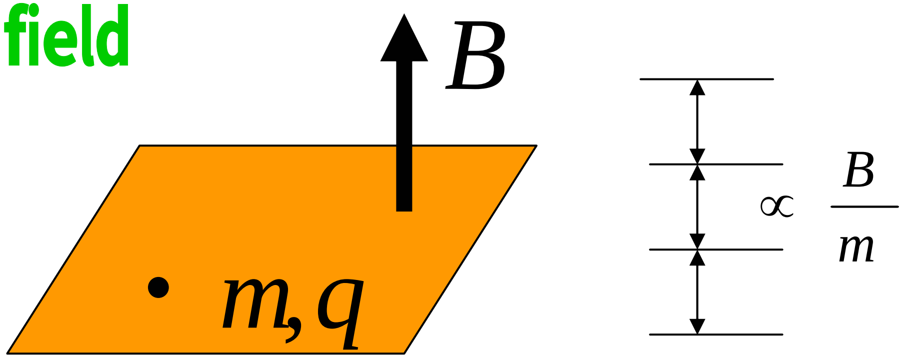
cf. Nakajima-Yoshioka, Eguchi-Kanno, Fujii-Minabe, Tachikawa, et. al.

NC gauge theories ⟷ Com. gauge theories in background of magnetic fields

↓

(real physics)

(Ex.) Motion of a charged particle in background magnetic field



$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + qBx\dot{y}$$

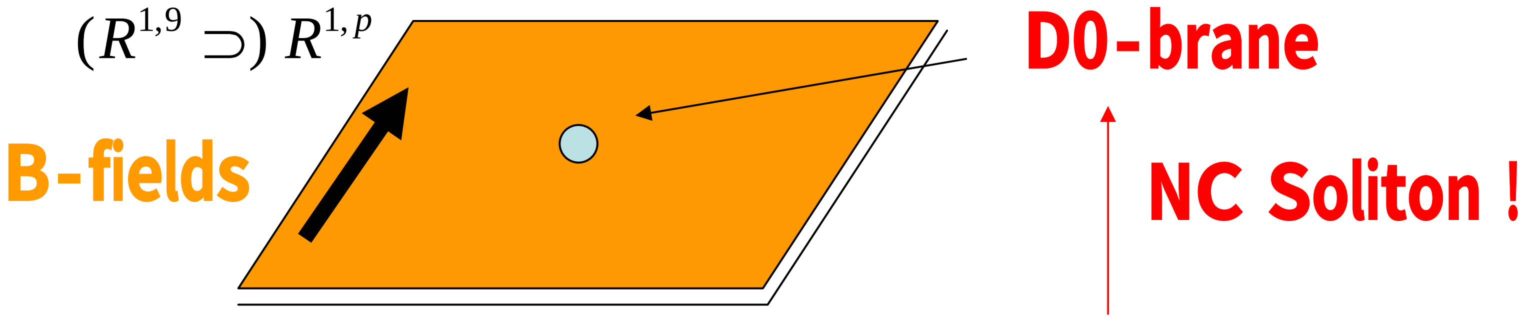
$$\downarrow \quad m \to 0$$

$$L = qBx\dot{y}$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = qB\underline{x} \xrightarrow{[y, \underline{p_y}] = i} [x, y] = -i\frac{1}{qB}$$

Gauge theories are realized on D-branes
which are solitons in string theories

In this context, (NC) solitons are (lower-dim.) D-branes



N Dp-branes → G=U(N) (p+1)-dim. NC Gauge theory

Analysis of NC solitons ↔ Analysis of D-branes

(easy to treat)

↓

Various applications

e.g. confirmation of Sen's conjecture on decay of D-branes

Plan of this talk

1. Introduction
2. NC gauge theories
3. ADHM construction of (NC) instantons
4. Applications to D-brane dynamics (omitted)

5. NC extension of soliton theories
6. NC Sato's theories
7. Conservation Laws
8. Exact Solutions and Ward's conjecture
9. Conclusion and Discussion

2. NC Gauge Theories

Here we discuss NC gauge theory of instantons.

(Ex.) 4-dim. (Euclidean) G=U(N) Yang-Mills theory

- Action

$$S = -\frac{1}{2}\int d^4x\, Tr\, F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}\int d^4x\, Tr\left(F_{\mu\nu}F_{\mu\nu} + \tilde{F}_{\mu\nu}\tilde{F}_{\mu\nu}\right)$$

$$= -\frac{1}{4}\int d^4x\, Tr\left[\underbrace{\left(F_{\mu\nu} \mp \tilde{F}_{\mu\nu}\right)^2}_{=0 \Leftrightarrow \text{BPS}} \pm \underbrace{2F_{\mu\nu}\tilde{F}_{\mu\nu}}_{\leftrightarrow C_2}\right]$$

$$(F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])$$

- Eq. Of Motion:

$$[D^\nu, [D_\nu, D_\mu]] = 0$$

- BPS eq. (=(A)SDYM eq.)

$$F_{\mu\nu} = \pm\tilde{F}_{\mu\nu} \quad \rightarrow \text{instantons}$$

$$(\Leftrightarrow F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0, \quad F_{z_1 z_2} = 0)$$

(Q) How we get NC version of the theories?

(A) They are obtained from ordinary commutative gauge theories by replacing products of fields with star-products: $f(x)g(x) \rightarrow f(x)*g(x)$

- The star product:

$$f(x)*g(x) := f(x)\exp\left(\frac{i}{2}\theta^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right)g(x) = f(x)g(x) + i\frac{\theta^{ij}}{2}\partial_i f(x)\partial_j g(x) + O(\theta^2)$$

A deformed product

Presence of background magnetic fields

$$f*(g*h) = (f*g)*h$$ Associative

$$[x^i, x^j]_* := x^i * x^j - x^j * x^i = i\theta^{ij}$$ NC !

In this way, we get NC-deformed theories with infinite derivatives in NC directions. (integrable???)

(Ex.) 4-dim. NC (Euclidean) G=U(N) Yang-Mills theory (All products are star products)

- Action

$$S = -\frac{1}{2}\int d^4x\, Tr\, F_{\mu\nu} * F^{\mu\nu} \quad = -\frac{1}{4}\int d^4x\, Tr\left(F_{\mu\nu} * F_{\mu\nu} + \tilde{F}_{\mu\nu} * \tilde{F}_{\mu\nu}\right)$$

$$= -\frac{1}{4}\int d^4x\, Tr\left[\left(F_{\mu\nu} \mp \tilde{F}_{\mu\nu}\right)_*^2 \pm 2F_{\mu\nu} * \tilde{F}_{\mu\nu}\right]$$

$= 0 \Leftrightarrow$ **BPS** $\quad \leftrightarrow C_2$

$$(F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]_*)$$

Don't omit even for G=U(1)

$(\because U(1) \cong U(\infty))$

- Eq. Of Motion:

$$[D^\nu, [D_\nu, D_\mu]_*]_* = 0$$

- BPS eq. (=NC (A)SDYM eq.)

$$F_{\mu\nu} = \pm\tilde{F}_{\mu\nu}$$

→ NC instantons

$$(\Leftrightarrow F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0, \quad F_{z_1 z_2} = 0)$$

A deformed theory is obtained.

3. ADHM construction of (NC) instantons

Atiyah-Drinfeld-Hitchin-Manin, PLA65('78)185

D-brane's interpretation

Douglas, Witten

ADHM eq. (G=``U(k)''): k × k matrix eq.

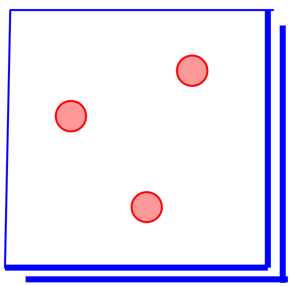
$$[B_1,B_1^*]+[B_2,B_2^*]+I\,I^*-J^*J=0$$
$$[B_1,B_2]+IJ=0$$

BPS

k D0-branes

ADHM data $B_{1,2}:k\times k,\quad I:k\times N,\quad J:N\times k$

1:1



Instantons $A_\mu:N\times N$

N D4-branes

ASD eq. (G=U(N), C_2=-k): N × N PDE

$$F_{z_1\bar{z}_1}+F_{z_2\bar{z}_2}=0$$
$$F_{z_1z_2}=0$$

BPS

[cf. MH, 素粒子論研究 106 (2002) 1]

String theory is a treasure house of dualities

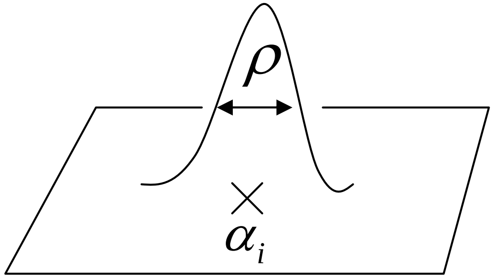
ADHM construction of BPST instanton (N=2,k=1)

ADHM eq. (G=``U(1)'')

$$[B_1, B_1^*]+[B_2, B_2^*]+I I^* - J^* J = 0$$

$$[B_1, B_2]+IJ = 0$$

Final remark: matrices B and coords. z always appear in pair: z-B



$$B_{1,2} = \alpha_{1,2}, \quad I = (\rho, 0),\ J = \begin{pmatrix} 0 \\ \rho \end{pmatrix}$$

position **size**

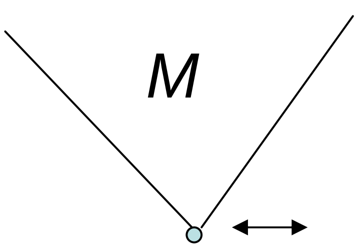
$$A_\mu = \frac{i(x-b)^\nu \eta_{\mu\nu}^{(-)}}{(x-b)^2 + \rho^2},\ F_{\mu\nu} = \frac{2i\rho^2}{((x-b)^2+\rho^2)^2}\eta_{\mu\nu}^{(-)}$$

$\rho \to 0$ singular

ASD eq. (G=U(2), C_2=-1)

$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$$

$$F_{z_1 z_2} = 0$$



$\rho = 0$

Small instanton singularity

ADHM construction of NC BPST instanton (N=2,k=1)

Nekrasov&Schwarz, CMP198('98)689 [hep-th/9802068]

ADHM eq. (G=``U(1)'') 1 × 1 matrix eq.

$$[B_1, B_1^*]+[B_2, B_2^*]+I\,I^* - J^*J = \zeta$$
$$[B_1, B_2]+IJ = 0$$

$$B_{1,2} = \alpha_{1,2}, \qquad I = (\sqrt{\rho^2 + \zeta}, 0),\ J = \begin{pmatrix} 0 \\ \rho \end{pmatrix}$$

$B_{1,2}$ ↔ position; I, J ↔ size → slightly fat?

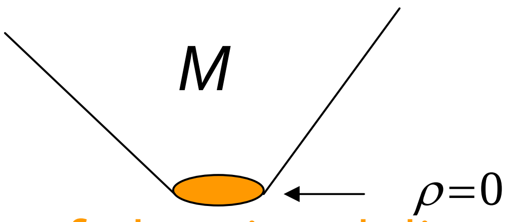
$A_\mu, F_{\mu\nu}$: something smooth

$\rho \to 0$ → Regular! (U(1) instanton!)

ASD eq. (G=U(2), C_2=-1)

$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$$
$$F_{z_1 z_2} = 0$$

[H.Nakajima]



Resolution of the singularity

5. NC Extension of Soliton Theories

Soliton equations in diverse dimensions

4	Anti-Self-Dual Yang-Mills eq. (instantons) $F_{\mu\nu} = -\tilde{F}_{\mu\nu}$	NC extension (Successful)
3	Bogomol'nyi eq. (monopoles)	NC extension (Successful)
2 (+1)	KP eq. BCS eq. DS eq. …	NC extension (This section)
1 (+1)	KdV eq. Boussinesq eq. NLS eq. Burgers eq. sine-Gordon eq. Sawada-Kotera eq	NC extension (This section)

↑ Dim. of space

Ward's observation:

Almost all integrable equations are reductions of the ASDYM eqs.

R.Ward, Phil.Trans.Roy.Soc.Lond.A315('85)451

ASDYM eq.

↓ Reductions

KP eq.   BCS eq.
KdV eq.   Boussinesq eq.
NLS eq. mKdV eq.
sine-Gordon eq.   Burgers eq. …
(Almost all !)

e.g. [Mason&Woodhouse]

NC Ward's observation: Almost all NC integrable equations are reductions of the NC ASDYM eqs.

MH&K.Toda, PLA316('03)77[hep-th/0211148]

NC ASDYM eq.

↓ Reductions

NC KP eq. NC BCS eq.
NC KdV eq. NC Boussinesq eq.
NC NLS eq. NC mKdV eq.
NC sine-Gordon eq. NC Burgers eq. …
(Almost all !?)

Successful

Successful?

A general framework is needed

6. NC Sato’s Theories

- Sato’s Theory : one of the most beautiful theory of solitons
  - Based on the exsitence of hierarchies and tau-functions
- Sato’s theory reveals essential aspects of solitons:
  - Construction of exact solutions
  - Structure of solution spaces
  - Infinite conserved quantities
  - Hidden infinite-dim. Symmetry

Let’s discuss NC extension of Sato’s theory

Derivation of soliton equations

- Prepare a Lax operator which is a pseudo-differential operator

$$L := \partial_x + u_2 \partial_x^{-1} + u_3 \partial_x^{-2} + u_4 \partial_x^{-3} + \cdots \qquad u_k = u_k(x^1, x^2, x^3, \cdots)$$

Noncommutativity is introduced here:

$$[x^i, x^j] = i\theta^{ij}$$

(Arbitrarily)

- Introduce a differential operator

$$B_m := (\underbrace{L * \cdots * L}_{m \text{ times}})_{\geq 0}$$

- Define NC (KP) hierarchy equation:

$$\boxed{\frac{\partial L}{\partial x^m} = [B_m, L]_*}$$

$$\partial_m u_2 \partial_x^{-1} + \partial_m u_3 \partial_x^{-2} + \partial_m u_4 \partial_x^{-3} + \cdots$$

$$F_{m2}(u) \partial_x^{-1} + F_{m3}(u) \partial_x^{-2} + F_{m4}(u) \partial_x^{-3} + \cdots$$

→ **Each coefficient yields a differential equation.**

Negative powers of differential operators

$$\partial_x^n \circ f := \sum_{j=0}^{\infty} \underline{\binom{n}{j}} (\partial_x^j f)\partial_x^{n-j}$$

$$|||$$

$$\frac{n(n-1)(n-2)\cdots(n-(j-1))}{j(j-1)(j-2)\cdots 1}$$

: binomial coefficient which can be extended to negative n → negative power of differential operator (well-defined !)

Ex.)

$$\partial_x^3 \circ f = f\partial_x^3 + 3f'\partial_x^2 + 3f''\partial_x^1 + f'''$$

$$\partial_x^2 \circ f = f\partial_x^2 + 2f'\partial_x + f''$$

$$\partial_x^{-1} \circ f = f\partial_x^{-1} - f'\partial_x^{-2} + f''\partial_x^{-3} - \cdots$$

$$\partial_x^{-2} \circ f = f\partial_x^{-2} - 2f'\partial_x^{-3} + 3f''\partial_x^{-4} - \cdots$$

Closer look at NC (KP) hierarchy

For m=2

$\partial_x^{-1})\quad \partial_2 u_2 = \underline{2u_3'} + u_2''$

$\partial_x^{-2})\quad \partial_2 u_3 = \underline{2u_4'} + u_3'' + 2u_2 * u_2' + 2[u_2, u_3]_*$

$\partial_x^{-3})\quad \partial_2 u_4 = \underline{2u_5'} + u_4'' + 4u_3 * u_2' - 2u_2 * u_2'' + 2[u_2, u_4]_*$

$\vdots$

Infinite kind of fields are represented in terms of one kind of field $u_2 \equiv u$

MH&K.Toda, [hep-th/0309265]

$u_x := \frac{\partial u}{\partial x}$

$\partial_x^{-1} := \int^x dx'$

For m=3

$\partial_x^{-1})\quad \partial_3 u_2 = u_2''' + 3u_3'' + 3u_4'' + 3u_2' * u_2 + 3u_2 * u_2'$

etc.

$\vdots$

$\downarrow$

$$u_t = \frac{1}{4}u_{xxx} + \frac{3}{4}(u_x * u + u * u_x) + \frac{3}{4}\partial_x^{-1}u_{yy} + \frac{3}{4}[u, \partial_x^{-1}u_{yy}]_*$$

(2+1)-dim. NC KP equation

and other NC equations (NC hierarchy equations)

$u = u(x^1, x^2, x^3, \cdots)$

$x^1 \leftrightarrow x,\quad x^2 \leftrightarrow y,\quad x^3 \leftrightarrow t$

(KP hierarchy) $\xrightarrow{\text{reductions}}$ (various hierarchies.)

- (Ex.) KdV hierarchy

Reduction condition

$$L^2 = B_2 (=: \partial_x^2 + u) \quad \text{: 2-reduction}$$

gives rise to NC KdV hierarchy

which includes (1+1)-dim. NC KdV eq.:

$$u_t = \frac{1}{4} u_{xxx} + \frac{3}{4} (u_x * u + u * u_x)$$

Note $\dfrac{\partial u}{\partial x_{2N}} = 0$: dimensional reduction in x_{2N} directions

KP : $u(x^1, x^2, x^3, x^4, x^5, \ldots)$ with $x = x^1$, $y = x^2$, $t = x^3$ **: (2+1)-dim.**

↓

KdV : $u(x^1, x^3, x^5, \ldots)$ with $x = x^1$, $t = x^3$ **: (1+1)-dim.**

l-reduction of NC KP hierarchy yields wide class of other NC hierarchies

- No-reduction → NC KP $(x, y, t) = (x^1, x^2, x^3)$
- 2-reduction → NC KdV $(x, t) = (x^1, x^3)$
- 3-reduction → NC Boussinesq $(x, t) = (x^1, x^2)$
- 4-reduction → NC Coupled KdV …
- 5-reduction → …
- 3-reduction of BKP → NC Sawada-Kotera
- 2-reduction of mKP → NC mKdV
- Special 1-reduction of mKP → NC Burgers
- …

Noncommutativity should be introduced into space-time coords

7. Conservation Laws

- Conservation laws: $\partial_t \sigma = \partial_i J^i$ σ : Conserved density

time (t) space (i)

Then $Q := \int_{space} dx\sigma$ is a conserved quantity.

$$\because \partial_t Q = \int_{space} dx \partial_t \sigma = \int_{\substack{spatial \\ \inf inity}} dS_i J^i = 0$$

Conservation laws for the hierarchies

$res_{-r} L^n$: coefficient of ∂_x^{-r} in L^n

Follwing G.Wilson's approach, we have:

$$\partial_m res_{-1} L^n = \partial_x J + [A, B]_* = \partial_x J + \theta^{ij} \partial_j \Xi_i$$

$[A, B]_*$: troublesome

$\theta^{ij} \partial_j \Xi_i$: I have found the explicit form !

∂_j **should be space or time derivative**

↓

Noncommutativity should be introduced in space-time directions only.

Hot (old?) Results

Infinite conserved densities for NC hierarchy eqs. (n=1,2,…, ∞)

$$\sigma = res_{-1}L^n + \theta^{im}\sum_{k=0}^{m-1}\sum_{l=0}^{k}(-1)^{k-l}\binom{k}{l}(res_{-(l+1)}L^n)\Diamond(\partial_i\partial_x^{k-l}res_k L^m)$$

$t \equiv x^m$ $res_r L^n$: coefficient of ∂_x^r in L^n

$\Diamond$: Strachan's product (commutative and non-associative)

$$f(x)\Diamond g(x) := f(x)\left(\sum_{s=0}^{\infty}\frac{(-1)^s}{(2s+1)!}\left(\frac{1}{2}\theta^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right)^{2s}\right)g(x)$$

MH, [hep-th/0311206v2]

to appear soon

We can calculate the explicit forms of conserved densities for the wide class of NC soliton equations.

- Space-Space noncommutativity:

  NC deformation is slight: $\sigma = res_{-1} L^n$

- Space-time noncommutativity

  NC deformation is drastical:

  – Example: NC KP and KdV equations $([t, x] = i\theta)$

$$\sigma = res_{-1} L^n - 3\theta((res_{-1} L^n) \Diamond u_3' + (res_{-2} L^n) \Diamond u_2')$$

meaningful ?

8. Exact Solutions and Ward's conjecture

- We have found exact N-soliton solutions for the wide class of NC hierarchies.
- 1-soliton solutions are all the same as commutative ones because of

$$f(x-vt)*g(x-vt)=f(x-vt)g(x-vt)$$

- Multi-soliton solutions behave in almost the same way as commutative ones except for phase shifts.
- Noncommutativity might affect the phase shifts

cf. MH, proc NCGP 2004 [hep-th/0501001] (to appear?)

NC Burgers hierarchy

MH&K.Toda,JPA36('03)11981[hep-th/0301213]

- NC (1+1)-dim. Burgers equation:

$$\dot{u} = u'' + 2u * u'$$: Non-linear &

Infinite order diff. eq. w.r.t. time ! (Integrable?)

NC Cole-Hopf transformation

$$u = \tau^{-1} * \tau' \quad \left(\xrightarrow{\theta \to 0} \partial_x \log \tau \right)$$

(NC) Diffusion equation:

$$\dot{\tau} = \tau''$$: Linear & first order diff. eq. w.r.t. time

(Integrable !)

A solution : $$\tau = 1 + \sum_{l=1}^{N} e^{k_l^2 t} * e^{\pm k_l x} = 1 + \sum_{l=1}^{N} \underline{e^{\frac{i}{2} k_l^3 \theta}} e^{k_l^2 t \pm k_l x}$$

Deformed!

NC Ward's observation (NC NLS eq.)

- Reduced ASDYM eq.: $x^{\mu} \rightarrow (t, x)$

Legare, [hep-th/0012077]

$$(i) \quad B' = 0$$

$$(ii) \quad C' - \dot{A} + [A, C]_* = 0$$

$$(iii) \quad A' - \dot{B} + [C, B]_* = 0$$

A, B, C: 2 times 2 matrices (gauge fields)

Further Reduction: $A = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}, B = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, C = -i\begin{pmatrix} q * \bar{q} & q' \\ q' & -\bar{q} * q \end{pmatrix}$

NOT traceless

$$(ii) \Rightarrow \begin{pmatrix} 0 & i\dot{q} - q'' - 2q * \bar{q} * q \\ i\dot{\bar{q}} + \bar{q}'' + 2\bar{q} * q * \bar{q} & 0 \end{pmatrix} = 0$$

$$i\dot{q} = q'' + 2q * \bar{q} * q$$ **: NC NLS eq.**

Note: $A, B, C \in gl(2, C) \xrightarrow{\theta \to 0} sl(2, C)$

U(1) part is important

NC Ward's observation (NC Burgers eq.)

- Reduced ASDYM eq.: $x^{\mu} \to (t, x)$

MH&K.Toda, JPA [hep-th/0301213]

G=U(1)

$$(i) \quad \dot{A} + [B, A]_* = 0$$

$$(ii) \quad \dot{C} - B' + [B, C]_* = 0$$

should remain

A, B, C: 1 times 1 matrices (gauge fields)

Further Reduction: $A = 0,\ B = u' - u^2,\ C = u$

$$(ii) \Rightarrow \quad \dot{u} = u'' + 2u' * u$$

: NC Burgers eq.

Note: Without the commutators [,], (ii) yields:

$$\dot{u} = u'' + u' * u + u * u'$$

: neither linearizable nor Lax form

symmetric

9. Conclusion and Discussion

- In this talk, we discussed
  - NC instantons where we saw that resolution of singularities yields new physical objects
  - NC soliton theories motivated by Ward's conjecture where we proved existence of infinite conserved quantities and exact multi-soliton solutions which would suggest hidden infinite-dim. symmetry.

    → New study area of integrable systems and geometry

Further directions

- Completion of NC Sato's theory
  - Theory of tau-functions → hidden symmetry (deformed affine Lie algebras?)

    Cf. Dimakis&Mueller-Hoissen, Wadati group, …
  - Geometrical descriptions from NC extension of the theories of Krichever, Mulase and Segal-Wilson and so on.
- Confirmation of NC Ward's conjecture
  - NC twistor theory

    e.g. Kapustin&Kuznetsov&Orlov, Hannabuss, Hannover group,…
  - D-brane interpretations → physical meanings
- Foundation of Hamiltonian formalism with space-time noncommutativity
  - Initial value problems, Liouville's theorem, Noether's thm,…