Noncommutative Solitons and Integrable Systems

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Ref: MH,``NC solitons and integrable systems,''

(to appear soon) [hep-th/0504001].

cf. MH,``Commuting flows and conservation laws

for NC Lax hierarchies,'' [hep-th/0311206]

J.Math.Phys.46 (2005) to be published on April 1.

NC Ward's observation: Almost all NC integrable equations are reductions of the NC ASDYM eqs.

MH&K.Toda, PLA316('03)77[hep-th/0211148], Bad Honnef 03

NC ASDYM eq.

Successful

↓ NC Reductions

Reductions

NC KP eq. NC BCS eq. NC Ward's chiral model

NC KdV eq. NC Boussinesq eq.

NC NLS eq. NC mKdV eq.

NC sine-Gordon eq. NC Burgers eq. …

(Almost all !?)

Successful?

``Beyond the Standard integrable Models''

Program of NC extension of soliton theories

- (i) Confirmation of NC Ward's conjecture
  - NC twistor theory → geometrical origin
  - D-brane interpretations → applications to physics
- (ii) Completion of NC Sato's theory
  - Existence of ``hierarchies'' → various soliton eqs.
  - Existence of infinite conserved quantities
    → infinite-dim. hidden symmetry
  - Construction of multi-soliton solutions
  - Theory of tau-functions → structure of the solution spaces and the symmetry

(i),(ii) → complete understanding of the NC soliton theories

NC Lax hierarchy (m=1,2, …)

$$\frac{\partial L}{\partial x^m} = [B_m, L]_*$$

$$L := \partial_x + u_2 \partial_x^{-1} + u_3 \partial_x^{-2} + u_4 \partial_x^{-3} + \cdots, \quad B_m := (L * \cdots * L)_{\geq 0}$$

This yields wide class of NC soliton eqs.

- No-reduction → NC KP $\quad u_t = \frac{1}{4} u_{xxx} + \frac{3}{4}(u*u)_x + \frac{3}{4}\partial_x^{-1} u_{yy} + \frac{3}{4}[u, \partial_x^{-1} u_{yy}]_*$
- 2-reduction → NC KdV $\quad u_t = \frac{1}{4} u_{xxx} + \frac{3}{4}(u_x * u + u * u_x)$
- 3-reduction → NC Boussinesq
- 4-reduction → NC Coupled KdV
- 2-reduction of mKP → NC mKdV
- Special 1-reduction of mKP → NC Burgers
- Matrix generalization → NC AKNS, NLS, …

We've solved
"hierarchy problem"

Infinite conserved densities for the NC soliton eqs. (n=1,2,..., ∞)

$$\sigma_n = coef_{-1}L^n + \theta^{im}\sum_{k=0}^{m-1}\sum_{l=0}^{k}\binom{k}{l}(\partial_x^{k-l}coef_{-(l+1)}L^n)\lozenge(\partial_i coef_k L^m)$$

$t \equiv x^m$ $\qquad$ $coef_r L^n$: coefficient of ∂_x^r in L^n

$\lozenge$: Strachan's product (commutative and non-associative)

$$f(x)\lozenge g(x) := f(x)\left(\sum_{s=0}^{\infty}\frac{(-1)^s}{(2s+1)!}\left(\frac{1}{2}\theta^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right)^{2s}\right)g(x)$$

MH, JMP46 [hep-th/0311206]

This suggests infinite-dimensional symmetries would be hidden.

Exact N-soliton solutions of the NC KP hierarchy

$L = \Phi \partial_x \Phi^{-1}$ **solves the NC KP hierarchy !**

$\Phi f := \left| W(y_1, \ldots, y_N, f) \right|_{N+1, N+1}$ **quasi-determinant of Wronski matrix**

$y_i = \exp \xi(x, \alpha_i) + a_i \exp \xi(x, \beta_i)$

$\xi(x, \alpha) = x_1 \alpha + x_2 \alpha^2 + x_3 \alpha^3 + \cdots$

Etingof-Gelfand-Retakh, [q-alg/9701008]

The exact solutions are actually N-soliton solutions !

Noncommutativity might affect the phase shift by $\theta^{ij} \omega_i k_j$

[MH, work in progress]

$\because) \exp(\omega_i t - k_i x) * \exp(\omega_j t - k_j x)$

$\cong \exp([t, x]_* \omega_{[i} k_{j]}) \exp((\omega_i + \omega_j) t - (k_i + k_j) x)$

Exactly solvable!

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  - Existence of ``hierarchies'' → Solved!
  - Existence of infinite conserved quantities Successful

    → infinite-dim. hidden symmetry
  - Construction of multi-soliton solutions Successful
  - Theory of tau-functions → description of the symmetry and the soliton solutions

NC Ward's conjecture (NC NLS eq.)

- Reduced ASDYM eq.: $x^{\mu} \to (t, x)$

Legare, [hep-th/0012077]

$$(i) \quad B' = 0$$

$$(ii) \quad C' - \dot{A} + [A, C]_* = 0$$

$$(iii) \quad A' - \dot{B} + [C, B]_* = 0$$

A, B, C: 2 times 2 matrices (gauge fields)

Further Reduction: $A = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}, B = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, C = i\begin{pmatrix} q * \bar{q} & \bar{q}' \\ q' & -\bar{q} * q \end{pmatrix}$

NOT traceless

$$(ii) \Rightarrow \begin{pmatrix} 0 & i\dot{q} - q'' - 2q * \bar{q} * q \\ i\dot{\bar{q}} + \bar{q}'' - 2\bar{q} * q * \bar{q} & 0 \end{pmatrix} = 0$$

$$i\dot{q} = q'' + 2q * \bar{q} * q$$ **: NC NLS eq. !!!**

Note: $A, B, C \in u(2) \xrightarrow{\theta \to 0} su(2)$ **U(1) part is necessary !**

NC Ward's conjecture (NC Burgers eq.)

- Reduced ASDYM eq.: $x^\mu \to (t, x)$

MH & K.Toda, JPA36 [hep-th/0301213 v2]

$$(i) \quad \dot{A} + \underline{[B, A]}_* = 0$$

$$(ii) \quad \dot{C} - B' + \underline{[B, C]}_* = 0, \qquad A, B, C \in \underline{u(1)} \simeq u(\infty)$$

should remain

Further Reduction: $A = 0,\ B = u' - u^2,\ C = u$

$$(ii) \Rightarrow \quad \dot{u} = u'' + 2u' * u \xrightarrow[u = \tau^{-1} * \tau']{} \dot{\tau} = \tau''$$

: NC Burgers eq. !!!

Linear eq. (Integrable !!!)

Note: Without the commutators [,], (ii) yields:

$$\dot{u} = u'' + \underline{u' * u + u * u'}$$: neither linearizable nor Lax form

Symmetric

NC Ward's conjecture (NC KdV eq.)

- Reduced ASDYM eq.: $x^{\mu} \to (t, x)$

MH, (HOT result) [hep-th/0504001]

$$(i) \quad [A, B]_* = 0$$

$$(ii) \quad A' - C' + [A, C]_* + [D, B]_* = 0$$

$$(iii) \quad \dot{C} - D' + [C, D]_* = 0$$

A, B, C: 2 times 2 matrices (gauge fields)

Further Reduction (different type of Mason-Sparling!):

$$A = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ -u & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 \\ -u & 0 \end{pmatrix}, D = \frac{1}{4}\begin{pmatrix} -u' & 2u \\ -u'' - 2u^2 & u' \end{pmatrix}$$

$$(iii) \Rightarrow \frac{1}{4}\begin{pmatrix} 0 & 0 \\ 4\dot{u} + u''' + 3(u' * u + u * u') & 0 \end{pmatrix} = 0$$

$$\dot{u} + \frac{1}{4}u''' + \frac{3}{4}(u' * u + u * u') = 0$$

: NC KdV eq. !!!

Note: $A, B, C \in sl(2, R)$

Bad news for D-brane picture?
Good news for twistor picture !

Program of NC extension of soliton theories

- Confirmation of NC Ward's conjecture Going well
  - NC twistor theory → geometrical origin
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  Work in progress I'll continue studying at Oxford from Oct.'05

- Completion of NC Sato's theory
  - Existence of ``hierarchies'' → Solved!
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