TOWARDS Non-Commutative (=NC) integrable systems and soliton theories

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Based on

- **MH**, JMP46 (2005) 052701 [hep-th/0311206]
- **MH**, PLB625 (2005) 324 [hep-th/0507112]
- cf. **MH**, ``NC solitons and integrable systems'' Proc. of NCGP2004, [hep-th/0504001]

1. Introduction

Successful points in NC theories

- Appearance of new physical objects
- Description of real physics
- Various successful applications to D-brane dynamics etc.

NC Solitons play important roles (Integrable!)

Final goal: NC extension of all soliton theories

Integrable equations in diverse dimensions

4	Anti-Self-Dual Yang-Mills eq. (instantons) $F_{\mu\nu} = -\tilde{F}_{\mu\nu}$	NC extension (Successful)
3	Bogomol'nyi eq. (monopoles)	NC extension (Successful)
2 (+1)	KP eq. BCS eq. DS eq. …	NC extension (This talk)
1 (+1)	KdV eq. Boussinesq eq. NLS eq. Burgers eq. sine-Gordon eq. (affine) Toda field eq. …	NC extension (This talk)

Dim. of space

Ward's observation: Almost all integrable equations are reductions of the ASDYM eqs.

R.Ward, Phil.Trans.Roy.Soc.Lond.A315('85)451

ASDYM eq.

↓ Reductions

KP eq. BCS eq. Ward's chiral model
KdV eq. Boussinesq eq.
NLS eq. mKdV eq.
sine-Gordon eq. Burgers eq. …
(Almost all !?)

e.g. [The book of Mason&Woodhouse]

NC Ward's observation: Almost all NC integrable equations are reductions of the NC ASDYM eqs.

MH&K.Toda, PLA316('03)77[hep-th/0211148]

NC ASDYM eq.

Successful

↓ NC Reductions

Reductions ↓

NC KP eq. NC BCS eq. NC Ward's chiral model
NC KdV eq. NC Boussinesq eq.
NC NLS eq. NC mKdV eq.
NC sine-Gordon eq. NC Burgers eq. …
(Almost all !?)

Successful?

A general framework is needed

Program of NC extension of soliton theories

(i) Confirmation of NC Ward's conjecture

- NC twistor theory → geometrical origin
- D-brane interpretations → applications to physics

(ii) Completion of NC Sato's theory

- Existence of ``hierarchies'' → various soliton eqs.
- Existence of infinite conserved quantities
  → infinite-dim. hidden symmetry
- Construction of multi-soliton solutions
- Theory of tau-functions → structure of the solution spaces and the symmetry

(i),(ii) → complete understanding of the NC soliton theories

Brief notes on how to get NC equations

NC spaces: $[x^i, x^j] = i\theta^{ij}$ θ^{ij} **: NC parameter**

NC extension is realized by replacing products of fields with star-products: $f(x)g(x) \to f(x)*g(x)$

- Star-products: $f(x)*g(x) := f(x)\exp\left(\frac{i}{2}\theta^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right)g(x)$

  Star products realize "noncommutative" spaces:

  $$[x^i, x^j]_* := x^i * x^j - x^j * x^i = i\theta^{ij}$$

- Some examples of NC integrable eqs.

  NC KP: $u_t = \frac{1}{4}u_{xxx} + \frac{3}{4}\underline{(u_x * u + u * u_x)} + \frac{3}{4}\partial_x^{-1}u_{yy} + \underline{\frac{3}{4}[u, \partial_x^{-1}u_{yy}]_*}$ $\begin{cases}[x,y] = i\theta \\ [t,x] = i\theta\end{cases}$

  NC Burgers: $u_t = u_{xx} + \underline{2u * u_x}$ $[t,x] = i\theta$

  Ordering of non-linear terms and additional terms are determined to preserve integrable-like properties. (We discuss later.)

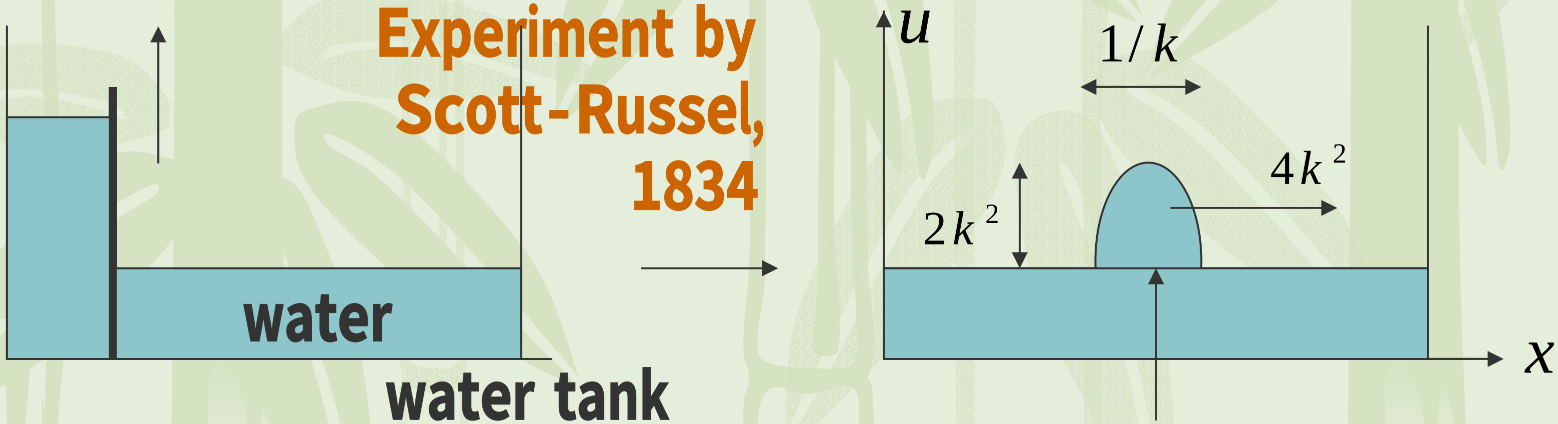
Plan of this talk

1. Introduction
2. Review of soliton theories
3. NC Sato’s theory
4. Conservation Laws
5. Exact Solutions and Ward’s conjecture
6. Conclusion and Discussion

2. Review of Soliton Theories

❖ KdV equation : describe shallow water waves

Experiment by Scott-Russel, 1834



solitary wave = soliton

This configuration satisfies

$$u = 2k^2 \cosh^{-2}(kx - 4k^3 t)$$

$$\dot{u} + u''' + 6u'u = 0$$ **: KdV eq.** **[Korteweg-de Vries, 1895]**

This is a typical integrable equation.

Let's solve it now !

- Hirota's method [PRL27(1971)1192]

$$\dot{u} + u''' + 6u'u = 0$$: naively hard to solve

$$\downarrow \quad u = 2\partial_x^2 \log \tau$$

$$\tau\dot{\tau}' - \tau'\dot{\tau} + 3\tau''\tau'' - 4\tau'\tau''' + \tau\tau'''' = 0$$

Hirota's bilinear relation : more complicated ?

A solution: $$\tau = 1 + e^{2(kx - \omega t)}, \quad \omega = 4k^3$$

$$\rightarrow \quad u = 2k^2 \cosh^{-2}(kx - 4k^3 t)$$ **: The solitary wave !**
(1-soliton solution)

2-soliton solution

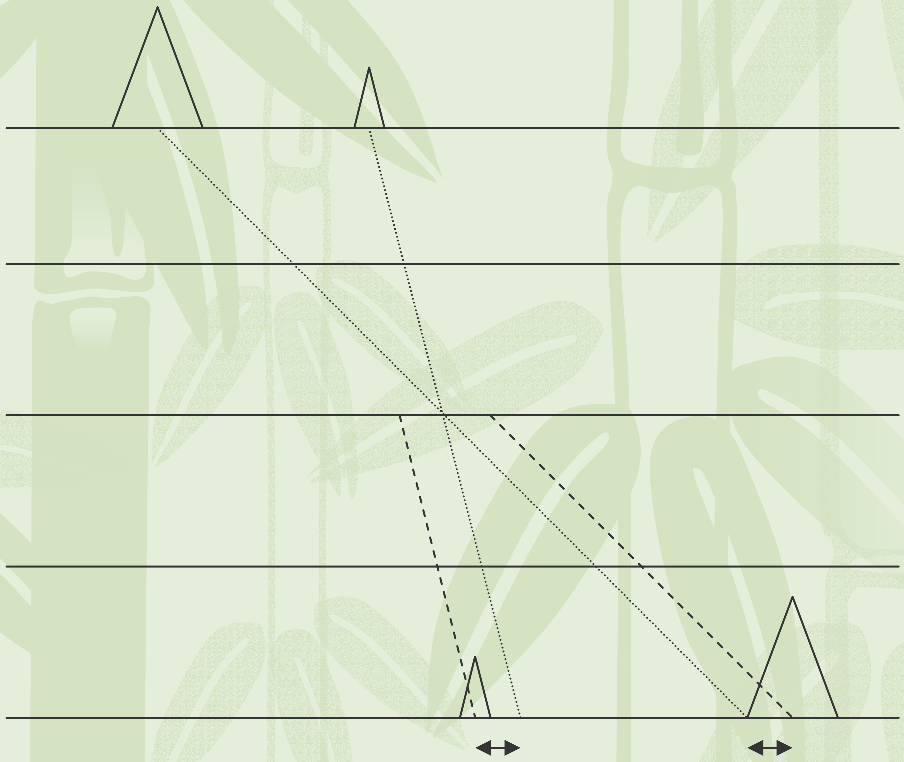
$$\tau = 1 + A_1 e^{2\theta_1} + A_2 e^{2\theta_2} + B A_1 A_2 e^{2(\theta_1+\theta_2)}$$

= A determinant of Wronski matrix (general property of soliton sols.)

``tau-functions''

$$\theta_i = k_i x - 4k_i^3 t, \quad B = \left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2$$

Scattering process



The shape and velocity is preserved ! (stable)

The positions are shifted ! (Phase shift)

There are many other soliton eqs. with (similar) interesting properties

- KP equation (2-dim. KdV equation)

  : describe 2-dim shallow water waves

$$u_t + u_{xxx} + 6u_x u + \partial_x^{-1} u_{yy} = 0 \quad \textbf{KP} \xrightarrow[\partial_y = 0]{} \textbf{KdV}$$

$$u_x := \frac{\partial u}{\partial x} \qquad \partial_x^{-1} := \int^x dx' \qquad \text{etc.}$$

Sato's theorem: [M.Sato & Y.Sato, 1981]

The solution space of KP eq. is an infinite-dim. Grassmann mfd. (determined by tau-fcns.)

Many other soliton eqs. are obtained from KP.

[See e.g. The book of Miwa-Jimbo-Date (Cambridge UP, 2000)]

3. NC Sato’s Theory

- Sato’s Theory : one of the most beautiful theory of solitons
  - Based on the exsitence of hierarchies and tau-functions
- Sato’s theory reveals essential aspects of solitons:
  - Construction of exact solutions
  - Structure of solution spaces
  - Infinite conserved quantities
  - Hidden infinite-dim. symmetry

Let’s discuss NC extension of Sato’s theory

Derivation of soliton equations

- Prepare a Lax operator which is a pseudo-differential operator

$$L := \partial_x + u_2 \partial_x^{-1} + u_3 \partial_x^{-2} + u_4 \partial_x^{-3} + \cdots$$

$$u_k = u_k(\underline{x^1, x^2, x^3, \cdots})$$

Noncommutativity is introduced here:

$$[x^i, x^j] = i\theta^{ij}$$

- Introduce a differential operator

$$B_m := (\underbrace{L * \cdots * L}_{m\ times})_{\geq 0}$$

- Define NC (KP) hierarchy:

$$\boxed{\frac{\partial L}{\partial x^m} = [B_m, L]_*}$$

Here all products are star product:

$$\partial_m u_2 \partial_x^{-1} + \partial_m u_3 \partial_x^{-2} + \partial_m u_4 \partial_x^{-3} + \cdots$$

$$f_{m2}(u)\partial_x^{-1} + f_{m3}(u)\partial_x^{-2} + f_{m4}(u)\partial_x^{-3} + \cdots$$

$\longrightarrow$ Each coefficient yields a differential equation.

Negative powers of differential operators

$$\partial_x^n \circ f := \sum_{j=0}^{\infty} \binom{n}{j} (\partial_x^j f) \partial_x^{n-j}$$

$$\binom{n}{j} = \frac{n(n-1)(n-2)\cdots(n-(j-1))}{j(j-1)(j-2)\cdots 1}$$

: binomial coefficient which can be extended to negative n
→ negative power of differential operator (well-defined !)

$$\partial_x^3 \circ f = f\partial_x^3 + 3f'\partial_x^2 + 3f''\partial_x^1 + f'''$$
$$\partial_x^2 \circ f = f\partial_x^2 + 2f'\partial_x + f''$$

$$\partial_x^{-1} \circ f = f\partial_x^{-1} - f'\partial_x^{-2} + f''\partial_x^{-3} - \cdots$$
$$\partial_x^{-2} \circ f = f\partial_x^{-2} - 2f'\partial_x^{-3} + 3f''\partial_x^{-4} - \cdots$$

Star product:

$$f(x) * g(x) := f(x) \exp\left(\frac{i}{2} \theta^{ij} \overleftarrow{\partial}_i \overrightarrow{\partial}_j \right) g(x)$$

which makes theories ``noncommutative'':

$$[x^i, x^j]_* := x^i * x^j - x^j * x^i = i\theta^{ij}$$

Closer look at NC KP hierarchy

For m=2

$$\partial_x^{-1})\quad \partial_2 u_2 = \underline{2u_3'} + u_2''$$

$$\partial_x^{-2})\quad \partial_2 u_3 = \underline{2u_4'} + u_3'' + 2u_2 * u_2' + 2[u_2, u_3]_*$$

$$\partial_x^{-3})\quad \partial_2 u_4 = \underline{2u_5'} + u_4'' + 4u_3 * u_2' - 2u_2 * u_2'' + 2[u_2, u_4]_*$$

$\vdots$

Infinite kind of fields are represented in terms of one kind of field $u_2 \equiv u$

$$u_x := \frac{\partial u}{\partial x}$$

MH&K.Toda, [hep-th/0309265]

$$\partial_x^{-1} := \int^x dx'$$

For m=3

$$\partial_x^{-1})\quad \partial_3 u_2 = u_2''' + 3u_3'' + 3u_4'' + 3u_2' * u_2 + 3u_2 * u_2'$$

etc.

$\vdots$

$$\downarrow$$

$$u_t = \frac{1}{4}u_{xxx} + \frac{3}{4}(u_x * u + u * u_x) + \frac{3}{4}\partial_x^{-1}u_{yy} + \frac{3}{4}[u, \partial_x^{-1}u_{yy}]_*$$

(2+1)-dim. NC KP equation

and other NC equations (NC KP hierarchy equations)

$$u = u(x^1, x^2, x^3, \cdots)$$

$$\updownarrow \quad \updownarrow \quad \updownarrow$$

$$x \quad y \quad t$$

(KP hierarchy) $\xrightarrow{\text{reductions}}$ (various hierarchies.)

- (Ex.) KdV hierarchy

Reduction condition

$$L^2 = B_2 (=: \partial_x^2 + u) \qquad \text{: 2-reduction}$$

gives rise to NC KdV hierarchy

which includes (1+1)-dim. NC KdV eq.:

$$u_t = \frac{1}{4} u_{xxx} + \frac{3}{4}(u_x * u + u * u_x)$$

Note $\dfrac{\partial u}{\partial x_{2N}} = 0$: dimensional reduction in x_{2N} directions

$$\begin{array}{lll}
\textbf{KP :} & u(x^1, x^2, x^3, x^4, x^5, \ldots) & \text{: (2+1)-dim.} \\
 & \quad\; x \;\;\; y \;\;\; t & \\
\big\downarrow & & \big\downarrow \\
\textbf{KdV :} & u(x^1, \quad x^3, \quad x^5, \ldots) & \text{: (1+1)-dim.} \\
 & \quad\; x \qquad\; t &
\end{array}$$

l-reduction of NC KP hierarchy yields wide class of other NC hierarchies

- No-reduction → NC KP $(x,y,t)=(x^1,x^2,x^3)$
- 2-reduction → NC KdV $(x,t)=(x^1,x^3)$
- 3-reduction → NC Boussinesq $(x,t)=(x^1,x^2)$
- 4-reduction → NC Coupled KdV $\cdots$
- 5-reduction → ...
- 3-reduction of BKP → NC Sawada-Kotera
- 2-reduction of mKP → NC mKdV
- Special 1-reduction of mKP → NC Burgers
- …

Noncommutativity should be introduced into space-time coords

4. Conservation Laws

- Conservation laws: $\partial_t \sigma = \partial_i J^i$ σ : Conserved density

time space

Then $Q := \int_{space} dx\,\sigma$ is a conserved quantity.

$$\because \partial_t Q = \int_{space} dx \partial_t \sigma = \int_{\substack{spatial \\ \inf inity}} dS_i J^i = 0$$

Conservation laws for the hierarchies

$$\partial_m res_{-1} L^n = \partial_x J + \underline{\theta^{ij} \partial_j \Xi_i}$$

time space

I have succeeded in the evaluation explicitly !

$res_{-r} L^n$: coefficient of ∂_x^{-r} in L^n

Noncommutativity should be introduced in space-time directions only. →

$t \equiv x^m$

∂_j **should be space or time derivative**
→ ordinary conservation laws !

Infinite conserved densities for the NC soliton eqs. (n=1,2,..., ∞)

$$\sigma_n = res_{-1}L^n + \theta^{im}\sum_{k=0}^{m-1}\sum_{l=0}^{k}\binom{k}{l}(\partial_x^{k-l} res_{-(l+1)}L^n)\Diamond(\partial_i res_k L^m)$$

$t \equiv x^m$ $\qquad$ $res_r L^n$: coefficient of ∂_x^r in L^n

$\Diamond$: Strachan's product (commutative and non-associative)

$$f(x)\Diamond g(x) := f(x)\left(\sum_{s=0}^{\infty}\frac{(-1)^s}{(2s+1)!}\left(\frac{1}{2}\theta^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right)^{2s}\right)g(x)$$

MH, JMP46 (2005) [hep-th/0311206]

This suggests infinite-dimensional symmetries would be hidden.

We can calculate the explicit forms of conserved densities for the wide class of NC soliton equations.

- Space-Space noncommutativity:

  NC deformation is slight: $\sigma_n = res_{-1} L^n$

  involutive (integrable in Liouville's sense)

- Space-time noncommutativity

  NC deformation is drastical:

  - Example: NC KP and KdV equations $([t,x]=i\theta)$

$$\sigma_n = res_{-1} L^n - 3\theta((res_{-1} L^n) \Diamond u_3' + (res_{-2} L^n) \Diamond u_2')$$

meaningful ?

5. Exact Solutions and Ward's conjecture

- We have found exact N-soliton solutions for the wide class of NC hierarchies.
- 1-soliton solutions are all the same as commutative ones because of

$$f(x-vt)*g(x-vt)=f(x-vt)g(x-vt)$$

- Multi-soliton solutions behave in almost the same way as commutative ones except for phase shifts.
- Noncommutativity affects the phase shifts

Exact multi-soliton solutions of the NC soliton eqs.

$$L = \Phi \partial_x \Phi^{-1}$$ **solves the NC Lax hierarchy !**

$$\Phi f := \left| W(y_1, \dots, y_N, f) \right|_{N+1, N+1}$$ **quasi-determinant of Wronski matrix**

$$y_i = \exp \xi(x, \alpha_i) + a_i \exp \xi(x, \beta_i)$$

$$\xi(x, \alpha) = x_1 \alpha + x_2 \alpha^2 + x_3 \alpha^3 + \cdots$$

Etingof-Gelfand-Retakh, [q-alg/9701008]

The exact solutions are actually N-soliton solutions !

Noncommutativity might affect the phase shift by $\theta^{ij} \omega_i k_j$

[MH, work in progress]

$$\because) \exp i(\omega_i t - k_i x) * \exp i(\omega_j t - k_j x)$$

$$\cong \exp(-i \theta^{ij} \omega_i k_j) \exp i((\omega_i + \omega_j) t - (k_i + k_j) x)$$

Exactly solvable!

NC Burgers hierarchy

MH&K.Toda,JPA36('03)11981[hep-th/0301213]

- NC (1+1)-dim. Burgers equation: $([t,x]=i\theta)$

$$\dot{u}=u''+2u*u'$$: Non-linear & Infinite order diff. eq. w.r.t. time ! (Integrable?)

NC Cole-Hopf transformation

$$u=\tau^{-1}*\tau' \quad (\xrightarrow{\theta\to 0}\partial_x\log\tau)$$

(NC) Diffusion equation:

$$\dot{\tau}=\tau''$$: Linear & first order diff. eq. w.r.t. time

(Integrable !)

A solution : $$\tau=1+\sum_{l=1}^{N}e^{k_l^2t}*e^{\pm k_lx}=1+\sum_{l=1}^{N}\underline{e^{\frac{i}{2}k_l^3\theta}}e^{k_l^2t\pm k_lx}$$

Deformed!

6. Conclusion and Discussion

- Confirmation of NC Ward's conjecture Going well
  - NC twistor theory → geometrical origin
  - D-brane interpretations → applications to physics

Talk at 13th NBMPS in Duham on Nov.5

Work in progress → [NC book of Mason&Woodhouse ?]

- Completion of NC Sato's theory
  - Existence of ``hierarchies'' → Solved!
  - Existence of infinite conserved quantities Successful
    → infinite-dim. hidden symmetry
  - Construction of multi-soliton solutions Successful
  - Theory of tau-functions → description of the symmetry and the soliton solutions Work in progress