Noncommutative Ward's Conjecture and Integrable Systems

Masashi **H**AMANAKA

Nagoya University, Dept. of Math.

(visiting Oxford for one year)

13th NBMPS in Durham on Nov. 5th

Based on

- **MH**, ``On reductions of NC ASDYM eqs.,'' PLB625 (2005) 324, [hep-th/0507112]
- cf. **MH**, ``NC solitons and integrable systems,'' Proc. NCGP2004, [hep-th/0504001]

1. Introduction

Successful points in NC theories

- Appearance of new physical objects
- Description of real physics
- Various successful applications to D-brane dynamics etc.

NC Solitons play important roles (Integrable!)

Final goal: NC extension of all soliton theories

Integrable equations in diverse dimensions

4	Anti-Self-Dual Yang-Mills eq. (instantons) $F_{\mu\nu} = -\tilde{F}_{\mu\nu}$	NC extension (Successful)
3	Bogomol'nyi eq. (monopoles)	NC extension (Successful)
2 (+1)	Kadomtsev-Petviashvili (KP) eq. Davey-Stewartson (DS) eq. …	NC extension (This talk)
1 (+1)	KdV eq. Boussinesq eq. NLS eq. Burgers eq. sine-Gordon eq. (affine) Toda field eq. …	NC extension (This talk)

↑ Dim. of space

Ward's conjecture: Many (perhaps all?) integrable equations are reductions of the ASDYM eqs.

R.Ward, Phil.Trans.Roy.Soc.Lond.A315('85)451

ASDYM eq.

↓ Reductions

KP eq. DS eq. Ward's chiral model
KdV eq. Boussinesq eq.
NLS eq. Toda field eq.
sine-Gordon eq. Burgers eq.
Painleve eqs. Tops …

Almost confirmed by explicit examples !!!

NC Ward's conjecture: Many (perhaps all?) NC integrable equations are reductions of the NC ASDYM eqs.

MH&K.Toda, PLA316('03)77[hep-th/0211148]

NC ASDYM eq.

Successful

↓ NC Reductions

Reductions

Many (perhaps all?) NC integrable eqs.

Successful?

- Existence of physical pictures
- New physical objects
- Application to D-branes
- Classfication of NC integ. eqs.

Plan of this talk

1. Introduction
2. NC gauge theory in 4-dim. (ASDYM eq.)
3. Reduction to 3-dim. (Bogomol'nyi eq.)
4. Reduction to (1+1)-dim. (KdV, NLS, Burgers)
5. Reduction to (2+1)-dim. (KP, DS)
6. Conclusion and Discussion

2. NC Gauge Theory in 4-dimension

Here we discuss NC gauge theory of instantons.

(Ex.) 4-dim. (Euclidean) G=U(N) Yang-Mills theory

- Action

$$S=-\frac{1}{2}\int d^4x\,Tr\,F_{\mu\nu}F^{\mu\nu}=-\frac{1}{4}\int d^4x\,Tr\left(F_{\mu\nu}F_{\mu\nu}+\tilde{F}_{\mu\nu}\tilde{F}_{\mu\nu}\right)$$

$$=-\frac{1}{4}\int d^4x\,Tr\left[\underbrace{\left(F_{\mu\nu}\mp\tilde{F}_{\mu\nu}\right)^2}_{=0\Leftrightarrow \text{BPS}}\pm\underbrace{2F_{\mu\nu}\tilde{F}_{\mu\nu}}_{\leftrightarrow C_2}\right]$$

$$(F_{\mu\nu}:=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu])$$

- Eq. Of Motion:

$$[D^\nu,[D_\nu,D_\mu]]=0$$

- BPS eq. (=(A)SDYM eq.)

$$F_{\mu\nu}=\pm\tilde{F}_{\mu\nu}$$ → instantons

$$(\Leftrightarrow F_{z_1\bar{z}_1}+F_{z_2\bar{z}_2}=0,\quad F_{z_1z_2}=0)$$

(Q) How we get NC version of the theories?

(A) They are obtained from ordinary commutative gauge theories by replacing products of fields with star-products: $f(x)g(x) \to f(x) * g(x)$

- The star product:

$$f(x) * g(x) := f(x) \exp\left(\frac{i}{2} \theta^{ij} \overleftarrow{\partial}_i \overrightarrow{\partial}_j \right) g(x) = f(x)g(x) + i \frac{\theta^{ij}}{2} \partial_i f(x) \partial_j g(x) + O(\theta^2)$$

A deformed product

$$f * (g * h) = (f * g) * h$$

Associative

$$[x^i, x^j]_* := x^i * x^j - x^j * x^i = i\theta^{ij}$$

NC !

Presence of background magnetic fields

In this way, we get NC-deformed theories with infinite derivatives in NC directions. (integrable???)

(Ex.) 4-dim. NC (Euclidean) G=U(N) Yang-Mills theory (All products are star products)

- Action

$$S=-\frac{1}{2}\int d^4x\,Tr\,F_{\mu\nu}*F^{\mu\nu}\quad=-\frac{1}{4}\int d^4x\,Tr\left(F_{\mu\nu}*F_{\mu\nu}+\tilde{F}_{\mu\nu}*\tilde{F}_{\mu\nu}\right)$$

$$=-\frac{1}{4}\int d^4x\,Tr\left[\left(F_{\mu\nu}\mp\tilde{F}_{\mu\nu}\right)_*^2\pm 2F_{\mu\nu}*\tilde{F}_{\mu\nu}\right]$$

$=0\Leftrightarrow$ **BPS** $\quad\leftrightarrow C_2$

$$(F_{\mu\nu}:=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu]_*)$$

Don't omit even for G=U(1)

$$(\because U(1)\cong U(\infty))$$

- Eq. Of Motion:

$$[D^\nu,[D_\nu,D_\mu]_*]_*=0$$

- BPS eq. (=NC (A)SDYM eq.)

$$F_{\mu\nu}=\pm\tilde{F}_{\mu\nu}$$

→ NC instantons

$$(\Leftrightarrow F_{z_1\bar{z}_1}+F_{z_2\bar{z}_2}=0,\quad F_{z_1z_2}=0)$$

A deformed theory is obtained.

ADHM construction of (NC) instantons

Atiyah-Drinfeld-Hitchin-Manin, PLA65('78)185

ADHM eq. (G=``U(k)''): k × k matrix eq.

$$[B_1, B_1^*] + [B_2, B_2^*] + I I^* - J^* J = 0$$
$$[B_1, B_2] + IJ = 0$$

ADHM data $B_{1,2} : k \times k, \quad I : k \times N, \quad J : N \times k$

1:1

Instantons $A_\mu : N \times N$

ASD eq. (G=U(N), C_2=-k): N × N PDE

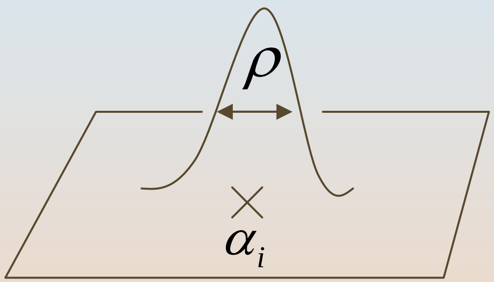
$$F_{z_1 \bar{z}_1} + F_{z_2 \bar{z}_2} = 0$$
$$F_{z_1 z_2} = 0$$

ADHM construction of BPST instanton (N=2,k=1)

ADHM eq. (G=``U(1)'')

$$[B_1, B_1^*]+[B_2, B_2^*]+I\,I^* - J^* J = 0$$
$$[B_1, B_2]+IJ = 0$$

Final remark: matrices B and coords. z always appear in pair: z-B



$$B_{1,2} = \alpha_{1,2}, \quad I = (\rho, 0),\ J = \begin{pmatrix} 0 \\ \rho \end{pmatrix}$$

position **size**

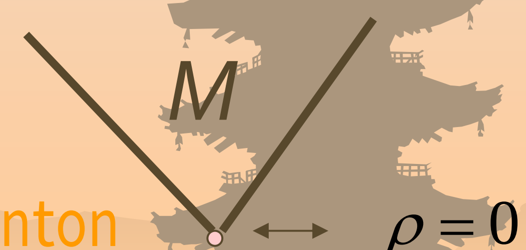
$$A_\mu = \frac{i(x-b)^\nu \eta^{(-)}_{\mu\nu}}{(x-b)^2+\rho^2},\quad F_{\mu\nu} = \frac{2i\rho^2}{((x-b)^2+\rho^2)^2}\eta^{(-)}_{\mu\nu}$$

$\rho \to 0$

singular

ASD eq. (G=U(2), C_2=-1)

$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$$
$$F_{z_1 z_2} = 0$$



Small instanton singularity

ADHM construction of NC BPST instanton (N=2,k=1)

Nekrasov&Schwarz,
CMP198('98)689
[hep-th/9802068]

ADHM eq. (G=``U(1)'') 1 × 1 matrix eq.

$$[B_1,B_1^*]+[B_2,B_2^*]+I\,I^*-J^*J=\zeta$$

$$[B_1,B_2]+IJ=0$$

$$B_{1,2}=\alpha_{1,2},\qquad I=(\sqrt{\rho^2+\zeta},0),\ J=\begin{pmatrix}0\\ \rho\end{pmatrix}$$

position — size → slightly fat?

$A_\mu, F_{\mu\nu}$: something smooth

$\rho\to 0$ → Regular!
(U(1) instanton!)

ASD eq. (G=U(2), C_2=-1)

$$F_{z_1\bar{z}_1}+F_{z_2\bar{z}_2}=0$$

$$F_{z_1 z_2}=0$$

Resolution of
the singularity

D-brane's interpretation of ADHM construction

Douglas, Witten

ADHM eq. (G=``U(k)''): k × k matrix eq.

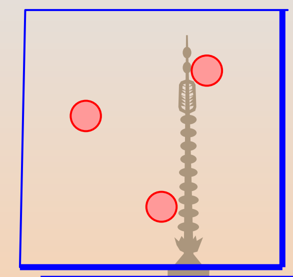
$$[B_1, B_1^*]+[B_2, B_2^*]+I\,I^* - J^*J = 0$$
$$[B_1, B_2]+IJ = 0$$

BPS

k D0-branes

ADHM data $B_{1,2} : k \times k,\quad I : k \times N,\quad J : N \times k$

1:1



Instantons $A_\mu : N \times N$

N D4-branes

ASD eq. (G=U(N), C_2=-k): N × N PDE

BPS

$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$$
$$F_{z_1 z_2} = 0$$

String theory is a treasure box of dualities

3. Reduction to 3-dimension

- Let's take a simple dimensional reduction of ASDYM to 3-dimension. $(\partial_4 = 0)$
- The reduced equation is known as Bogomol'yi eq.

$$B_i = \pm D_i \Phi$$ → monopoles

$(B_i = \varepsilon_{ijk} F_{jk}$:magnetic fields, Φ : Higgs field)

- Monopoles are also constructed from ADHM-like procedure (=Nahm constraction)

Nahm construction of (NC) monopoles

Nahm, in ``Monopoles in QFT'' (1982)

Nahm eq. (G=``U(k)''): k × k ODE

$$\frac{dT_i}{d\xi} = i\varepsilon_{ijk}[T_j, T_k]$$

Nahm data $T_i : k \times k$

1:1

monopoles $\Phi,\ A_i : N \times N$

Bogomol'nyi eq. (G=U(N), k): N × N PDE

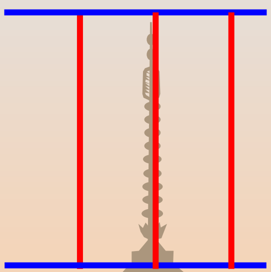
$$B_i = \partial_i \Phi$$

D-brane's interpretation

Diaconescu

BPS

k D1-branes



N D3-branes

BPS

A T-dualized configration of the D0-D4 brane systems

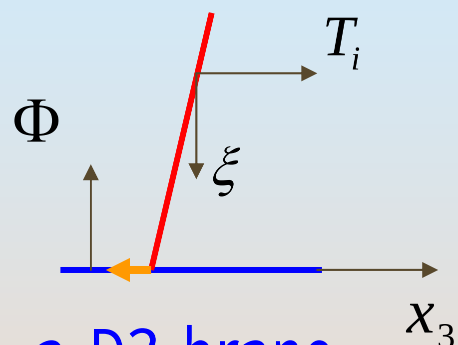
Nahm construction of NC Dirac monopoles

Gross&Nekrasov, JHEP[hep-th/0005204]

Nahm eq. (G=``U(1)”): 1 × 1 ODE

$$\frac{dT_i}{d\xi} = i\varepsilon_{ijk}[T_j, T_k] - \theta\delta_{i3}$$

a D1-brane



a D3-brane

1:1

Nahm data $T_{1,2} = 0,\ T_3 = \underline{-\theta\xi}$

Slope of D1-brane ⟷ Magnetic field pulls the end point of D1-brane

monopoles $\Phi = “\frac{1}{2r} + \frac{x_3}{\underline{\theta}}\exp\left(-\frac{x_1^2 + x_2^2}{\theta}\right)“,\ A_i =$ (smooth config.)

Bogomol’nyi eq. (G=U(1), k=1): 1× 1 PDE

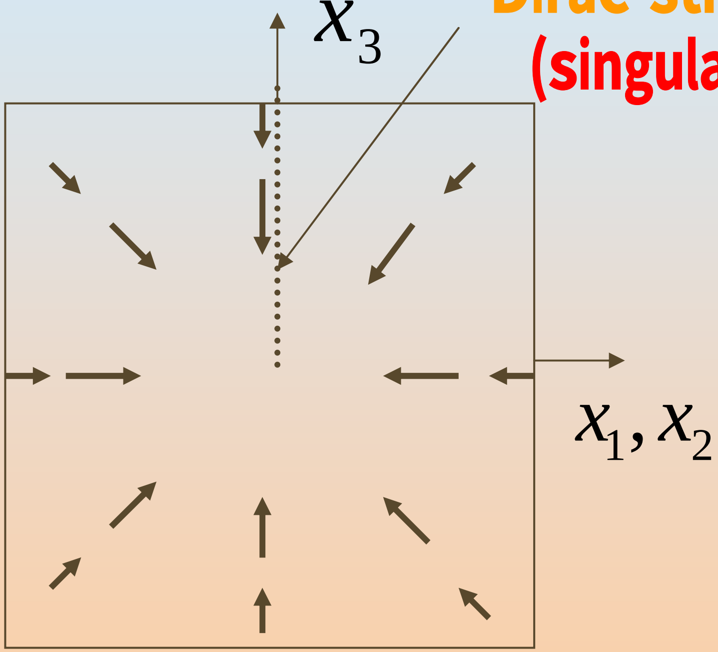
$$B_i = \partial_i \Phi$$

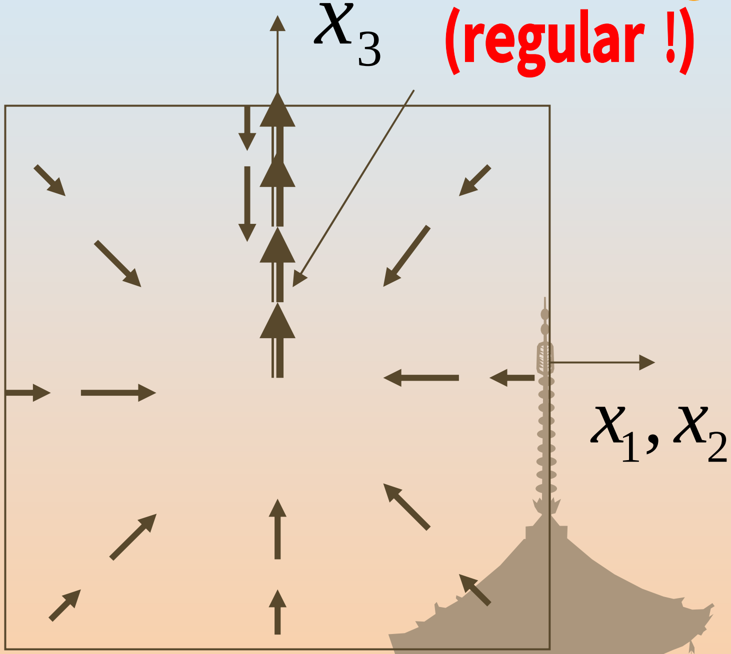
Φ :represents position of D3-brane

T_i :represents positions of D1-brane

Magnetic flux of Dirac monopoles

$$B_3 = \partial_3 \Phi = \text{``}\, \frac{-x_3}{2r^3} + \frac{2}{\theta} \exp\left(-\frac{x_1^2 + x_2^2}{\theta} \right) \text{''}$$





On commutative space

On NC space (roughly)

Solutions show interesting behaviors,
though the moduli space is the same as commutative one.

4. Reduction to (1+1)-dimension

- From now on, we discuss reductions of NC ASDYM on (2+2)-dimension, including KdV, NLS, Burgers...
- Reduction steps are as follows:

  (1) take a simple dimensional reduction

  with a gauge fixing.

  (2) put further reduction conditions on gauge fields.
- The reduced eqs. coincides with those obtained in the framework of NC KP or GD hierarchies,

which possess infinite conserved quantities and

exact multi-soliton solutions. (integrable-like)

Reduction to NC KdV eq.

MH, PLB625, 324 [hep-th/0507112]

- (1) Reduced ASDYM eq.: $x^\mu \to (t, x)$

$$(i) \quad B' = 0$$

$$(ii) \quad A' - \dot{B} + [C, B]_* = 0$$

$$(iii) \quad C' + \dot{A} + [A, C]_* = 0$$

A, B, C: 2 times 2 matrices (gauge fields)

(2) Further Reduction:

$$A = \begin{pmatrix} q & -1 \\ q' + q^2 & -q \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} \frac{1}{2}q'' + q' * q & -q' \\ \frac{1}{4}q''' + \frac{1}{2}q'^2 + \frac{1}{2}\{q'', q\}_* + q * q' * q & -\frac{1}{2}q'' - q * q' \end{pmatrix}$$

NOT Traceless !

$$(iii) \quad \Rightarrow \quad \dot{u} = \frac{1}{4}u''' + \frac{3}{4}(u' * u + u * u')$$

: NC KdV eq. !!!

$$[t, x] = i\theta$$

Note: $A, B, C \in gl(2) \xrightarrow{\theta \to 0} sl(2)$ U(1) part is necessary !

The NC KdV eq. has integrable-like properties:

- possesses infinite conserved densities:

Explicit!

$$\sigma_n = res_{-1}L^n - 3\theta((res_{-1}L^n)\lozenge u_3' + (res_{-2}L^n)\lozenge u_2')$$

MH, JMP46 (2005) [hep-th/0311206]

$res_r L^n$: coefficient of ∂_x^r in L^n

$\lozenge$: Strachan's product (commutative and non-associative)

$$f(x)\lozenge g(x) := f(x)\left(\sum_{s=0}^{\infty}\frac{(-1)^s}{(2s+1)!}\left(\frac{1}{2}\theta^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right)^{2s}\right)g(x)$$

$$[t,x] = i\theta$$

- has exact N-soliton solutions:

Etingof-Gelfand-Retakh, [q-alg/9701008]
MH, work in progress
cf. Paniak, [hep-th/0105185]

$$u = 2\partial_x \sum_{i=1}^{N}(\partial_x W_i) * W_i^{-1}$$

Explicit!

$W_i := \left|W(y_1,\ldots,y_i)\right|_{i,i}$ **:quasi-determinant of Wronski matrix**

$$y_i = \exp\xi(x,\alpha_i) + a_i\exp(-\xi(x,\alpha_i)) \qquad \xi(x,\alpha) = x\alpha + t\alpha^3$$

Reduction to NC NLS eq.

- Reduced ASDYM eq.: $x^\mu \to (t,x)$

Legare, [hep-th/0012077]

$$(i)\quad B'=0$$

$$(ii)\quad A'-\dot{B}+[C,B]_*=0$$

$$(iii)\quad C'+\dot{A}+[A,C]_*=0$$

A, B, C: 2 times 2 matrices (gauge fields)

Further Reduction:

$$A=\begin{pmatrix}0 & \psi\\ -\overline{\psi} & 0\end{pmatrix},\ B=\frac{i}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\ C=i\begin{pmatrix}\psi*\overline{\psi} & \overline{\psi}'\\ \psi' & -\overline{\psi}*\psi\end{pmatrix}$$

NOT traceless

$$(iii)\Rightarrow\begin{pmatrix}0 & i\dot{\psi}-\psi''-2\psi*\overline{\psi}*\psi\\ -i\dot{\overline{\psi}}+\overline{\psi}''-2\overline{\psi}*\psi*\overline{\psi} & 0\end{pmatrix}=0$$

$$i\dot{\psi}=\psi''+2\psi*\overline{\psi}*\psi$$: **NC NLS eq. !!!**

[MH, PLB625,324]

Note: $A,B,C\in u(2)\xrightarrow{\theta\to 0} su(2)$

U(1) part is necessary !

Reduction to NC Burgers eq.

- Reduced ASDYM eq.: $x^\mu \to (t, x)$ MH & K.Toda, JPA36 [hep-th/0301213]

$$(i) \quad \dot{A} + \underline{[B, A]_*} = 0$$

$$(ii) \quad \dot{C} - B' + \underline{[B, C]_*} = 0, \qquad A, B, C \in \underline{u(1)}$$

should remain $\qquad \cong u(\infty)$

Further Reduction: $A = 0,\ B = u' - u^2,\ C = u$

$$(ii) \Rightarrow \quad \dot{u} = u'' + 2u * u' \longrightarrow \dot{\tau} = \tau''$$

$u = \tau^{-1} * \tau'$

: NC Burgers eq. !!!

Linear eq. (Integrable !!!)

Note: Without the commutators [,], (ii) yields:

$$\dot{u} = u'' + \underline{u' * u + u * u'}$$

: neither linearizable nor Lax form

Symmetric

5. Reduction to (2+1)-dimension

- From now on, we discuss reductions of NC ASDYM with infinite-dimensional gauge group,

  including KP, DS, N-wave ...

- This time, gauge fields value operators algebras s.t.:

$$g = \{a_0(x,y,t) + a_1(x,y,t)\partial_y + a_2(x,y,t)\partial_y^2\}$$

- Reduction steps are as follows:

  (1) take a simple dimensional reduction
  with a gauge fixing.

  (2) replace spectral parameters in linear systems
  with derivatives of y

  (3) put further reduction conditions on gauge fields.

ASDYM from Linear systems

- ASDYM can be derived from compatibility condition of linear systems:

$$\left.\begin{array}{l} D_w * \Psi = \zeta\, D_{\tilde{z}} * \Psi \\ D_z * \Psi = \zeta\, D_{\tilde{w}} * \Psi \end{array}\right\} \rightarrow \quad F_{zw} = 0,\ F_{z\tilde{z}} - F_{w\tilde{w}} = 0,\ F_{\tilde{z}\tilde{w}} = 0$$

:ASDYM

(1) Dim. reduction into 2-dim.: $(t, x) = (z, w + \tilde{w})$

$$\partial_x \Psi = (A_w + \zeta\, A_{\tilde{z}}) * \Psi$$

$$\partial_t \Psi = (A_z + \zeta\, D_{\tilde{w}}) * \Psi = (A_z + \zeta (A_w + A_{\tilde{w}}) + \zeta^2 A_{\tilde{z}}) * \Psi$$

(2) Replace ζ with ∂_y as follows:

$$A_w = U + A\partial_y,\ A_z = V + \tilde{U}\partial_y + A\partial_y^2,\ A_{\tilde{z}} = A,\ A_{\tilde{w}} = \tilde{U} - U + A\partial_y,\ \Psi = e^{-\zeta y}\Phi$$

$$\left.\begin{array}{l} \partial_x \Phi = (U + A\partial_y) * \Phi \\ \partial_t \Phi = (V + \tilde{U}\partial_y + A\partial_y^2) * \Phi \end{array}\right\} \rightarrow \begin{array}{c} [U - \tilde{U}, A]_* = 0 \\ -\tilde{U}_x + A * \tilde{U}_y - 2A * U_y + [A, V]_* + [U, \tilde{U}]_* = 0 \\ U_t - V_x + [U, V]_* + A * (V - U_y)_y - \tilde{U} * U_y = 0 \end{array}$$

Reduction to DS eq.

- The reduced ASDYM

$$(i)\ [U-\tilde{U},A]_* = 0$$

$$(ii)\ -\tilde{U}_x + A*\tilde{U}_y - 2A*U_y + [A,V]_* + [U,\tilde{U}]_* = 0$$

$$(iii)\ U_t - V_x + [U,V]_* + A*(V-U_y)_y - \tilde{U}*U_y = 0$$

(3) Further Reduction:

$$A = \kappa\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\ U = \tilde{U} = \begin{pmatrix}0 & q\\ r & 0\end{pmatrix},$$

$$V = \frac{1}{2\kappa}\begin{pmatrix}R_1 & (\partial_x+\kappa\partial_y)q\\ -(\partial_x-\kappa\partial_y)r & R_2\end{pmatrix}$$

$$\begin{cases} 2\kappa\dot{q} = (\partial_x^2+\kappa^2\partial_y^2)q + R_1*q - q*R_2 \\ 2\kappa\dot{r} = -(\partial_x^2+\kappa^2\partial_y^2)r + R_2*r - r*R_1 \\ (\partial_x-\kappa\partial_y)R_1 = -(\partial_x+\kappa\partial_y)(q*r) \\ (\partial_x+\kappa\partial_y)R_2 = (\partial_x-\kappa\partial_y)(r*q) \end{cases}$$

NC Davey-Stewartson eq.

(New eq.)

(2-dim. generalization of NC NLS)

Reduction to KP eq.

- The reduced ASDYM

$$(i)\ [U-\tilde{U},A]_* = 0$$

$$(ii)\ -\tilde{U}_x + A*\tilde{U}_y - 2A*U_y + [A,V]_* + [U,\tilde{U}]_* = 0$$

$$(iii)\ U_t - V_x + [U,V]_* + A*(V-U_y)_y - \tilde{U}*U_y = 0$$

(3) Further Reduction:

$$A=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},\ U=\begin{pmatrix}0 & 1\\ u & 0\end{pmatrix},\ \tilde{U}=\begin{pmatrix}0 & 1\\ u/2 & 0\end{pmatrix},$$

$$V=\frac{1}{4}\begin{pmatrix}u_x - 3\partial_x^{-1}u_y & -2u\\ u_{xx}-2u*u+u_y & -u_x-3\partial_x^{-1}u_y\end{pmatrix}$$

$$u_t = \frac{1}{4}u_{xxx} - \frac{3}{4}(u_x*u+u*u_x) + \frac{3}{4}\partial_x^{-1}u_{yy} + \frac{3}{4}[u,\partial_x^{-1}u_{yy}]_*$$

:NC KP eq. !

$$\partial_y = 0$$

$$u_t = \frac{1}{4}u_{xxx} - \frac{3}{4}(u_x*u+u*u_x)$$

:NC KdV eq.

Note: This reduction to KdV is gauge equivalent to the previous result $g=\begin{pmatrix}1 & 0\\ q & 1\end{pmatrix}$

Infinite conserved densities for the NC KP hierarchy. (n=1,2,…, ∞)

$$\sigma_n = res_{-1}L^n + \theta^{im}\sum_{k=0}^{m-1}\sum_{l=0}^{k}\binom{k}{l}(\partial_x^{k-l} res_{-(l+1)}L^n)\Diamond(\partial_i res_k L^m)$$

$t \equiv x^m$ $\qquad$ $res_r L^n$: coefficient of ∂_x^r in L^n

$\Diamond$: Strachan's product (commutative and non-associative)

$$f(x)\Diamond g(x) := f(x)\left(\sum_{s=0}^{\infty}\frac{(-1)^s}{(2s+1)!}\left(\frac{1}{2}\theta^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right)^{2s}\right)g(x)$$

MH, JMP46 (2005) [hep-th/0311206]

This suggests infinite-dimensional symmetries would be hidden.

Exact N-soliton solutions of the NC KP hierarchy

$$L=\Phi\partial_x\Phi^{-1}$$ **solves the NC KP hierarchy !**

$$\Phi f := \left| W(y_1,\ldots,y_N,f) \right|_{N+1,N+1}$$

quasi-determinant of Wronski matrix

$$y_i = \exp\xi(x,\alpha_i) + a_i \exp\xi(x,\beta_i)$$

$$\xi(x,\alpha) = x_1\alpha + x_2\alpha^2 + x_3\alpha^3 + \cdots$$

Etingof-Gelfand-Retakh, [q-alg/9701008]

The exact solutions are actually N-soliton solutions !

Noncommutativity might affect the phase shift by $\theta^{ij}\omega_i k_j$

[MH, work in progress]

$$\because)\ \exp(\omega_i t - k_i x) * \exp(\omega_j t - k_j x)$$

$$\cong \exp([t,x]_* \omega_{[i} k_{j]}) \exp((\omega_i+\omega_j)t - (k_i+k_j)x)$$

Exactly solvable!

6. Conclusion and Discussion

- We derived many of NC integrable eqs. from NC ASDYM eqs. by reductions.
- This result could be a strong evidence for NC Ward's conjecture, which guarantees the existence of physical pictures.
- The derived eqs. coincide with those from the framework of NC KP or GD hierarchies,

  which possess infinite conserved quantities and exact multi-soliton solutions.
- Application to D-brane dynamics and description from NC twistor theory are expected.

NC Ward's conjecture: Many (perhaps all?) NC integrable equations are reductions of the NC ASDYM eqs.

MH&K.Toda, PLA316('03)77[hep-th/0211148]

NC ASDYM eq.

Successful

↓ NC Reductions

Reductions

NC KP eq. NC DS eq. NC Ward's chiral model
NC KdV eq. NC Boussinesq eq.
NC NLS eq. NC Toda field eq.
NC sine-Gordon eq. NC Burgers eq. …
(Many ! Perhaps all ?)

Successful!!!

In near future, almost confirmed by explicit examples !?