Noncommutative Solitons and Integrable Systems

Masashi **H**AMANAKA

Nagoya University, Dept. of Math.

(visiting Math. Inst., Oxford)

Mathematics Seminar in Kent on Oct. 30th

Based on

- **MH**, ``Notes on exact multi-soliton solutions of NC integrable hierarchies ,'' **[hep-th/0610006]**
- **MH**, ``NC Ward's conjecture and integrable systems ,'' **NPB741 (2006) 368, [hep-th/0601209]**
- **MH**, ``Commuting flows and conservation laws for NC Lax hierarchies,'' **JMP46 (2005) 052701 [hep-th/0311206]**
- cf. **MH**, ``NC Solitons and D-branes,'' **Ph.D thesis (2003) [hep-th/0303256]**

1. Introduction

Successful points in NC theories

- Appearance of new physical objects
- Description of real physics
- Various successful applications to D-brane dynamics etc.

NC Solitons play important roles (Integrable!)

Final goal: NC extension of all soliton theories

Integrable equations in diverse dimensions

4	Anti-Self-Dual Yang-Mills eq. (instantons) $F_{\mu\nu} = -\tilde{F}_{\mu\nu}$	NC extension (Successful)
3	Bogomol'nyi eq. (monopoles)	NC extension (Successful)
2 (+1)	Kadomtsev-Petviashvili (KP) eq. Davey-Stewartson (DS) eq. …	NC extension (This talk)
1 (+1)	KdV eq. Boussinesq eq. NLS eq. Burgers eq. sine-Gordon eq. (affine) Toda field eq. …	NC extension (This talk)

Dim. of space

Ward's conjecture: Many (perhaps all?) integrable equations are reductions of the ASDYM eqs.

R.Ward, Phil.Trans.Roy.Soc.Lond.A315('85)451

ASDYM eq.

↓ Reductions

(KP eq.) (DS eq.) Ward's chiral model

KdV eq. Boussinesq eq.

NLS eq. Toda field eq.

sine-Gordon eq. Liouville eq.

Painleve eqs. Tops …

Almost confirmed by explicit examples !!!

NC Ward's conjecture: Many (perhaps all?) NC integrable equations are reductions of the NC ASDYM eqs.

MH&K.Toda, PLA316('03)77[hep-th/0211148]

NC ASDYM eq.

Successful

NC Reductions

Reductions

Many (perhaps all?)
NC integrable eqs.

Successful?

NC Sato's theory plays important roles in revealing integrable aspects of them

- Existence of physical pictures
- New physical objects
- Application to D-branes
- Classfication of NC integ. eqs.

Program of NC extension of soliton theories

- (i) Confirmation of NC Ward's conjecture
  - NC twistor theory → geometrical origin
  - D-brane interpretations → applications to physics
- (ii) Completion of NC Sato's theory
  - Existence of ``hierarchies'' → various soliton eqs.
  - Existence of infinite conserved quantities
    → infinite-dim. hidden symmetry
  - Construction of multi-soliton solutions
  - Theory of tau-functions → structure of the solution spaces and the symmetry

(i),(ii) → complete understanding of the NC soliton theories

Plan of this talk

1. Introduction
2. NC gauge theory in 4-dim. (ASDYM eq.)
3. NC Ward's conjecture
   --- Reduction of NC ASDYM to (1+1)-dim.
   (KdV, NLS, …)
4. Towards NC Sato's theory (KP, …)
   hierarchy, infinite conserved quantities,
   exact multi-soliton solutions,…
5. Conclusion and Discussion

2. NC Gauge Theory in 4-dimension

Here we discuss NC gauge theory of instantons.

(Ex.) 4-dim. (Euclidean) G=U(N) Yang-Mills theory

- Action

$$S=-\frac{1}{2}\int d^4x\, Tr\, F_{\mu\nu}F^{\mu\nu} \quad =-\frac{1}{4}\int d^4x\, Tr\left(F_{\mu\nu}F_{\mu\nu}+\tilde{F}_{\mu\nu}\tilde{F}_{\mu\nu}\right)$$

$$=-\frac{1}{4}\int d^4x\, Tr\Big[\underbrace{\left(F_{\mu\nu}\mp\tilde{F}_{\mu\nu}\right)^2}_{=0\ \Leftrightarrow\ \textbf{BPS}}\pm\underbrace{2F_{\mu\nu}\tilde{F}_{\mu\nu}}_{\leftrightarrow\ C_2}\Big]$$

$$(F_{\mu\nu}:=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu])$$

- Eq. Of Motion:

$$[D^\nu,[D_\nu,D_\mu]]=0$$

- BPS eq. (=(A)SDYM eq.)

$$F_{\mu\nu}=\pm\tilde{F}_{\mu\nu} \qquad \rightarrow \text{instantons}$$

$$(\Leftrightarrow F_{z_1\bar{z}_1}+F_{z_2\bar{z}_2}=0,\quad F_{z_1z_2}=0)$$

(Q) How we get NC version of the theories?

(A) They are obtained from ordinary commutative gauge theories by replacing products of fields with star-products: $f(x)g(x) \to f(x) * g(x)$

- The star product:

$$f(x) * g(x) := f(x) \exp\left(\frac{i}{2} \theta^{ij} \overleftarrow{\partial}_i \overrightarrow{\partial}_j \right) g(x) = f(x)g(x) + i\frac{\theta^{ij}}{2} \partial_i f(x) \partial_j g(x) + O(\theta^2)$$

A deformed product

$$f * (g * h) = (f * g) * h$$

Associative

$$[x^i, x^j]_* := x^i * x^j - x^j * x^i = i\theta^{ij}$$

NC !

Presence of background magnetic fields

In this way, we get NC-deformed theories with infinite derivatives in NC directions. (integrable???)

(Ex.) 4-dim. NC (Euclidean) G=U(N) Yang-Mills theory (All products are star products)

- Action

$$S=-\frac{1}{2}\int d^4x\,Tr\,F_{\mu\nu}*F^{\mu\nu}\quad =-\frac{1}{4}\int d^4x\,Tr\left(F_{\mu\nu}*F_{\mu\nu}+\tilde{F}_{\mu\nu}*\tilde{F}_{\mu\nu}\right)$$

$$=-\frac{1}{4}\int d^4x\,Tr\left[\underbrace{\left(F_{\mu\nu}\mp\tilde{F}_{\mu\nu}\right)_*^2}_{=0\Leftrightarrow\ \text{BPS}}\pm\underbrace{2F_{\mu\nu}*\tilde{F}_{\mu\nu}}_{\leftrightarrow C_2}\right]$$

$$(F_{\mu\nu}:=\partial_\mu A_\nu-\partial_\nu A_\mu+\underline{[A_\mu,A_\nu]_*})$$

Don't omit even for G=U(1)

$(\because U(1)\cong U(\infty))$

- Eq. Of Motion:

$$[D^\nu,[D_\nu,D_\mu]_*]_*=0$$

- BPS eq. (=NC (A)SDYM eq.)

$$F_{\mu\nu}=\pm\tilde{F}_{\mu\nu}$$ → NC instantons

$$(\Leftrightarrow F_{z_1\bar{z}_1}+F_{z_2\bar{z}_2}=0,\quad F_{z_1z_2}=0)$$

A deformed theory is obtained.

ADHM construction of (NC) instantons

Atiyah-Drinfeld-Hitchin-Manin, PLA65('78)185

ADHM eq. (G=``U(k)''): k × k matrix eq.

$$[B_1, B_1^*] + [B_2, B_2^*] + I I^* - J^* J = 0$$
$$[B_1, B_2] + IJ = 0$$

ADHM data $B_{1,2} : k \times k, \quad I : k \times N, \quad J : N \times k$

1:1

Instantons $A_\mu : N \times N$

ASD eq. (G=U(N), C_2=-k): N × N PDE

$$F_{z_1 \bar{z}_1} + F_{z_2 \bar{z}_2} = 0$$
$$F_{z_1 z_2} = 0$$

ADHM construction of BPST instanton (N=2,k=1)

ADHM eq. (G=``U(1)'')

$$[B_1, B_1^*]+[B_2, B_2^*]+I\,I^* - J^*J = 0$$
$$[B_1, B_2]+IJ=0$$

$$B_{1,2} = \alpha_{1,2}, \qquad I = (\rho, 0),\ J = \begin{pmatrix} 0 \\ \rho \end{pmatrix}$$

position **size**

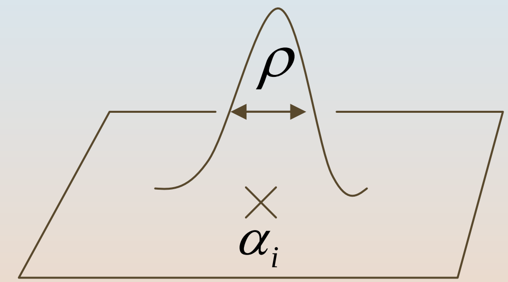
$$A_\mu = \frac{i(x-b)^\nu \eta^{(-)}_{\mu\nu}}{(x-b)^2+\rho^2},\ F_{\mu\nu} = \frac{2i\rho^2}{((x-b)^2+\rho^2)^2}\eta^{(-)}_{\mu\nu}$$

$\rho \to 0$

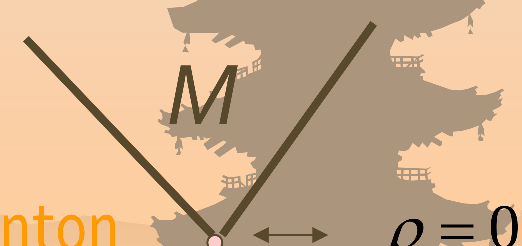
ASD eq. (G=U(2), C_2=-1)

$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$$
$$F_{z_1 z_2} = 0$$

Final remark: matrices B and coords. z always appear in pair: z-B



singular



Small instanton singularity

ADHM construction of NC BPST instanton (N=2,k=1)

Nekrasov&Schwarz, CMP198('98)689 [hep-th/9802068]

ADHM eq. (G=``U(1)'') 1×1 matrix eq.

$$[B_1, B_1^*] + [B_2, B_2^*] + I I^* - J^* J = \zeta$$
$$[B_1, B_2] + IJ = 0$$

$$B_{1,2} = \alpha_{1,2}, \quad I = (\sqrt{\rho^2 + \zeta}, 0),\ J = \begin{pmatrix} 0 \\ \rho \end{pmatrix}$$

position

size → slightly fat?

$A_\mu, F_{\mu\nu}$: something smooth $\xrightarrow{\rho \to 0}$ Regular! (U(1) instanton!)

ASD eq. (G=U(2), C_2=-1)

$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$$
$$F_{z_1 z_2} = 0$$

Resolution of the singularity

D-brane's interpretation of ADHM construction

Douglas, Witten, …
Hashimoto-Terashima,
Tong's excellent review

ADHM eq. (G=``U(k)''): k × k matrix eq.

$$[B_1, B_1^*]+[B_2, B_2^*]+I\,I^* - J^*J = 0$$
$$[B_1, B_2]+IJ = 0$$

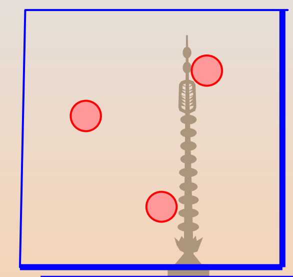
BPS

k D0-branes

ADHM data $B_{1,2} : k\times k,\quad I : k\times N,\quad J : N\times k$

1:1

Instantons $A_\mu : N\times N$



N D4-branes

ASD eq. (G=U(N), C_2=-k): N × N PDE

$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$$
$$F_{z_1 z_2} = 0$$

BPS

String theory is a treasure box of dualities

3. NC Ward's conjecture --- reduction to (1+1)-dim.

- From now on, we discuss reductions of NC ASDYM on (2+2)-dimension, including KdV, NLS, ...
- Reduction steps are as follows:

  (1) take a simple dimensional reduction

  with a gauge fixing.

  (2) put further reduction conditions on gauge fields.
- The reduced eqs. coincides with those obtained in the framework of NC KP and GD hierarchies,

  which possess infinite conserved quantities and

  exact multi-soliton solutions. (integrable-like)

Reduction to NC KdV eq.

MH, PLB625, 324 [hep-th/0507112]

- (1) Reduced ASDYM eq.: $x^\mu \to (t,x)$

$$(i) \quad B' = 0$$

$$(ii) \quad A' - \dot{B} + [C,B]_* = 0$$

$$(iii) \quad C' + \dot{A} + [A,C]_* = 0$$

A, B, C: 2 times 2 matrices (gauge fields) (D=0: gauge fixing)

(2) Further Reduction:

$$A = \begin{pmatrix} q & -1 \\ q' + q*q & -q \end{pmatrix},\ B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} \frac{1}{2}q'' + \underline{q'*q} & -q' \\ \frac{1}{4}q''' + \frac{1}{2}q'^2 + \frac{1}{2}\{q'',q\}_* + q*q'*q & -\frac{1}{2}q'' - \underline{q*q'} \end{pmatrix}$$

NOT Traceless !

$$(iii) \quad \Rightarrow \quad \dot{u} = \frac{1}{4}u''' + \frac{3}{4}(u'*u + u*u')$$

: NC KdV eq. !!!

$$[t,x] = i\theta$$

Note: $A,B,C \in gl(2) \xrightarrow{\theta \to 0} sl(2)$ U(1) part is necessary !

The NC KdV eq. has integrable-like properties:

- possesses infinite conserved densities:

Explicit!

$$\sigma_n = res_{-1}L^n - 3\theta((res_{-1}L^n)\lozenge u_3' + (res_{-2}L^n)\lozenge u_2')$$

MH, JMP46 (2005) [hep-th/0311206]

$res_r L^n$: coefficient of ∂_x^r in L^n

$\lozenge$: Strachan's product (commutative and non-associative)

$$f(x)\lozenge g(x) := f(x)\left(\sum_{s=0}^{\infty}\frac{(-1)^s}{(2s+1)!}\left(\frac{1}{2}\theta^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right)^{2s}\right)g(x)$$

$$[t,x]=i\theta$$

- has exact N-soliton solutions:

Etingof-Gelfand-Retakh, [q-alg/9701008]
MH, [hep-th/0610006]
cf. Paniak, [hep-th/0105185]

$$u = 2\partial_x \sum_{i=1}^{N}(\partial_x W_i)*W_i^{-1}$$

Explicit!

$$W_i := \left|W(f_1,\ldots,f_i)\right|_{i,i}$$:quasi-determinant of Wronski matrix

$$f_i = \exp\xi(x,\alpha_i) + a_i\exp(-\xi(x,\alpha_i))$$

$$\xi(x,\alpha) = x\alpha + t\alpha^3$$

Reduction to NC NLS eq.

- Reduced ASDYM eq.: $x^{\mu} \to (t, x)$

Legare, [hep-th/0012077]

$$(i) \quad B' = 0$$

$$(ii) \quad A' - \dot{B} + [C, B]_{*} = 0$$

$$(iii) \quad C' + \dot{A} + [A, C]_{*} = 0$$

A, B, C: 2 times 2 matrices (gauge fields) (D=0: gauge fixing)

Further Reduction: $A = \begin{pmatrix} 0 & \psi \\ -\overline{\psi} & 0 \end{pmatrix}, \; B = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; C = i\begin{pmatrix} \psi * \overline{\psi} & \overline{\psi}' \\ \psi' & -\overline{\psi} * \psi \end{pmatrix}$

NOT traceless

$$(iii) \Rightarrow \begin{pmatrix} 0 & i\dot{\psi} - \psi'' - 2\psi * \overline{\psi} * \psi \\ -i\dot{\overline{\psi}} + \overline{\psi}'' - 2\overline{\psi} * \psi * \overline{\psi} & 0 \end{pmatrix} = 0$$

$$i\dot{\psi} = \psi'' + 2\psi * \overline{\psi} * \psi$$ **: NC NLS eq. !!!**

[MH, PLB625,324]

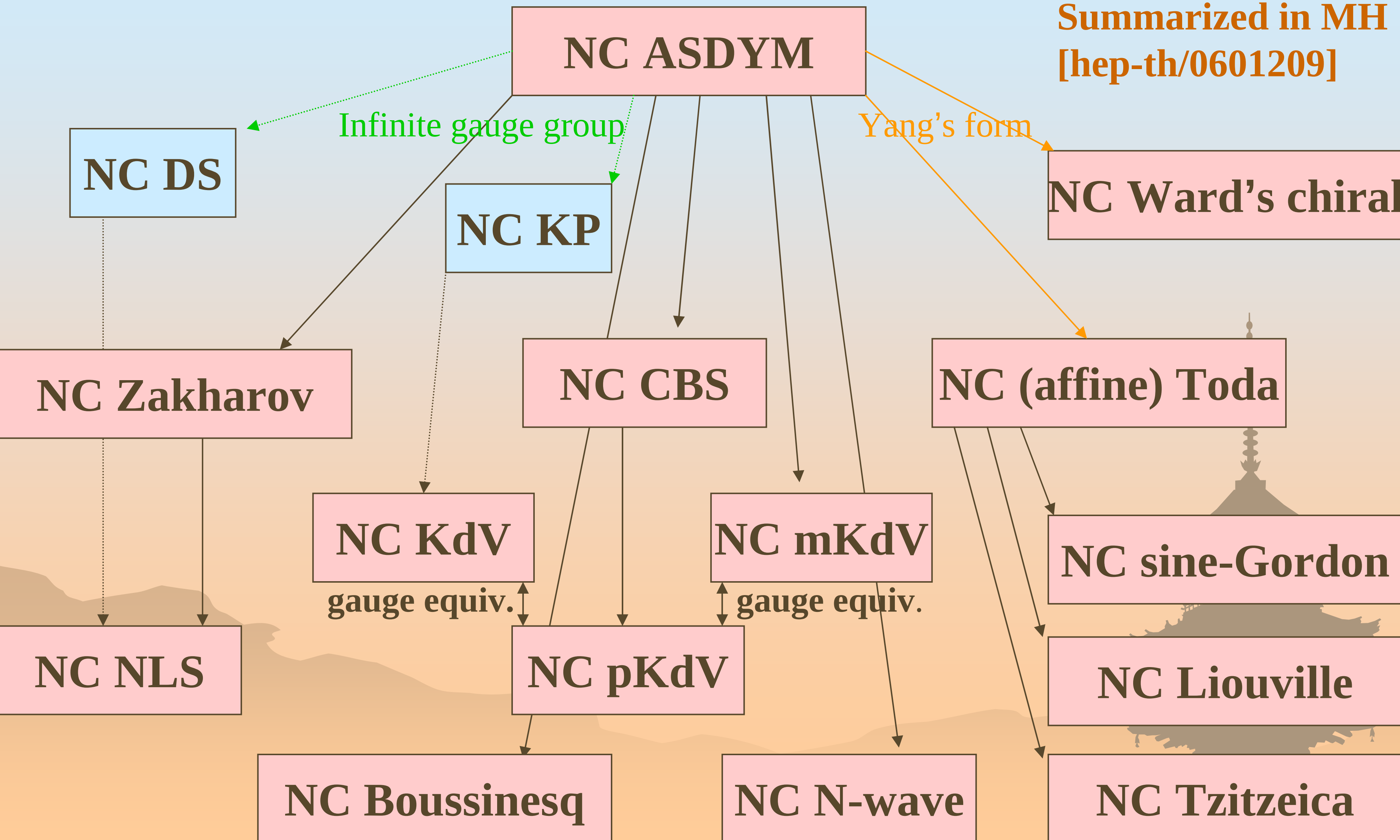
Note: $A, B, C \in u(2) \xrightarrow{\theta \to 0} su(2)$

U(1) part is necessary !

In this way, we can obtain various NC integrable equations from NC ASDYM !!!

Almost all ?

Summarized in MH [hep-th/0601209]



4. Towards NC Sato's Theory

- Sato's Theory : one of the most beautiful theory of solitons
  - Based on the exsitence of hierarchies and tau-functions
  - Various integrable equations in (1+1)-dim. can be derived elegantly from (2+1)-dim. KP equation.
- Sato's theory reveals essential aspects of solitons:
  - Construction of exact solutions
  - Structure of solution spaces
  - Infinite conserved quantities
  - Hidden infinite-dim. symmetry

Let's discuss NC extension of Sato's theory

Derivation of soliton equations

- Prepare a Lax operator which is a pseudo-differential operator

$$L := \partial_x + u_2\partial_x^{-1} + u_3\partial_x^{-2} + u_4\partial_x^{-3} + \cdots$$

$$u_k = u_k(x^1, x^2, x^3, \cdots)$$

Noncommutativity is introduced here:

$$[x^i, x^j] = i\theta^{ij}$$

- Introduce a differential operator

$$B_m := (\underbrace{L * \cdots * L}_{m\ times})_{\geq 0}$$

- Define NC (KP) hierarchy:

$$\boxed{\frac{\partial L}{\partial x^m} = [B_m, L]_*}$$

Here all products are star product:

$$\partial_m u_2 \partial_x^{-1} + \partial_m u_3 \partial_x^{-2} + \partial_m u_4 \partial_x^{-3} + \cdots$$

$$f_{m2}(u)\partial_x^{-1} + f_{m3}(u)\partial_x^{-2} + f_{m4}(u)\partial_x^{-3} + \cdots$$

$\longrightarrow$ **Each coefficient yields a differential equation.**

Negative powers of differential operators

$$\partial_x^n \circ f := \sum_{j=0}^{\infty} \binom{n}{j} (\partial_x^j f)\partial_x^{n-j}$$

$$|||$$

$$\frac{n(n-1)(n-2)\cdots(n-(j-1))}{j(j-1)(j-2)\cdots 1}$$

: binomial coefficient which can be extended to negative n → negative power of differential operator (well-defined !)

$$\partial_x^3 \circ f = f\partial_x^3 + 3f'\partial_x^2 + 3f''\partial_x^1 + f'''$$

$$\partial_x^2 \circ f = f\partial_x^2 + 2f'\partial_x + f''$$

$$\partial_x^{-1} \circ f = f\partial_x^{-1} - f'\partial_x^{-2} + f''\partial_x^{-3} - \cdots$$

$$\partial_x^{-2} \circ f = f\partial_x^{-2} - 2f'\partial_x^{-3} + 3f''\partial_x^{-4} - \cdots$$

Star product:

$$f(x) * g(x) := f(x)\exp\left(\frac{i}{2}\theta^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right)g(x)$$

which makes theories \`\`noncommutative'':

$$[x^i, x^j]_* := x^i * x^j - x^j * x^i = i\theta^{ij}$$

Closer look at NC KP hierarchy

For m=2

$$\partial_x^{-1})\quad \partial_2 u_2 = \underline{2u_3'} + u_2''$$

$$\partial_x^{-2})\quad \partial_2 u_3 = \underline{2u_4'} + u_3'' + 2u_2 * u_2' + 2[u_2, u_3]_*$$

$$\partial_x^{-3})\quad \partial_2 u_4 = \underline{2u_5'} + u_4'' + 4u_3 * u_2' - 2u_2 * u_2'' + 2[u_2, u_4]_*$$

$\vdots$

Infinite kind of fields are represented in terms of one kind of field $u_2 \equiv u$

MH&K.Toda, [hep-th/0309265]

$$u_x := \frac{\partial u}{\partial x}$$

$$\partial_x^{-1} := \int^x dx'$$

etc.

For m=3

$$\partial_x^{-1})\quad \partial_3 u_2 = u_2''' + 3u_3'' + 3u_4'' + 3u_2' * u_2 + 3u_2 * u_2'$$

$\vdots$

$$\downarrow$$

$$u_t = \frac{1}{4}u_{xxx} + \frac{3}{4}(u_x * u + u * u_x) + \frac{3}{4}\partial_x^{-1}u_{yy} + \frac{3}{4}[u, \partial_x^{-1}u_{yy}]_*$$

(2+1)-dim. NC KP equation

and other NC equations (NC KP hierarchy equations)

$$u = u(x^1, x^2, x^3, \cdots)$$

$$\updownarrow \quad \updownarrow \quad \updownarrow$$

$$x \quad y \quad t$$

(KP hierarchy) $\xrightarrow{\text{reductions}}$ (various hierarchies.)

- (Ex.) KdV hierarchy

Reduction condition

$$L^2 = B_2 (=: \partial_x^2 + u)$$: 2-reduction

gives rise to NC KdV hierarchy

which includes (1+1)-dim. NC KdV eq.:

$$u_t = \frac{1}{4} u_{xxx} + \frac{3}{4}(u_x * u + u * u_x)$$

Note $\dfrac{\partial u}{\partial x_{2N}} = 0$: dimensional reduction in x_{2N} directions

KP : $u(x^1, x^2, x^3, x^4, x^5, \ldots)$ — $x^1 = x$, $x^2 = y$, $x^3 = t$ **: (2+1)-dim.**

↓

KdV : $u(x^1, x^3, x^5, \ldots)$ — $x^1 = x$, $x^3 = t$ **: (1+1)-dim.**

l-reduction of NC KP hierarchy yields wide class of other NC hierarchies

- No-reduction → NC KP $\quad (x,y,t)=(x^1,x^2,x^3)$
- 2-reduction → NC KdV $\quad (x,t)=(x^1,x^3)$
- 3-reduction → NC Boussinesq $\quad (x,t)=(x^1,x^2)$
- 4-reduction → NC Coupled KdV $\quad \ldots$
- 5-reduction → …
- 3-reduction of BKP → NC Sawada-Kotera
- 2-reduction of mKP → NC mKdV
- Special 1-reduction of mKP → NC Burgers
- …

Noncommutativity should be introduced into space-time coords

Conservation Laws for the NC hierarchies

- Conservation laws: $\partial_t \sigma = \partial_i J^i$ σ : Conserved density

time space

Then $Q := \int_{space} dx \sigma$ is a conserved quantity.

$$\because \partial_t Q = \int_{space} dx \partial_t \sigma = \int_{\substack{spatial \\ \inf inity}} dS_i J^i = 0$$

Conservation laws for the hierarchies

$$\partial_m res_{-1} L^n = \partial_x J + \underline{\theta^{ij} \partial_j \Xi_i}$$

time space

I have succeeded in the evaluation explicitly !

$res_{-r} L^n$: coefficient of ∂_x^{-r} in L^n

Noncommutativity should be introduced in space-time directions only. →

$t \equiv x^m$

∂_j **should be space or time derivative**
→ ordinary conservation laws !

Infinite conserved densities for the NC KP hierarchy. (n=1,2,…, ∞)

$$\sigma_n = res_{-1}L^n + \theta^{im}\sum_{k=0}^{m-1}\sum_{l=0}^{k}\binom{k}{l}(\partial_x^{k-l} res_{-(l+1)}L^n)\lozenge(\partial_i res_k L^m)$$

$t \equiv x^m$ $\qquad res_r L^n$: coefficient of ∂_x^r in L^n

$\lozenge$: Strachan's product (commutative and non-associative)

$$f(x)\lozenge g(x) := f(x)\left(\sum_{s=0}^{\infty}\frac{(-1)^s}{(2s+1)!}\left(\frac{1}{2}\theta^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right)^{2s}\right)g(x)$$

MH, JMP46 (2005) [hep-th/0311206]

This suggests infinite-dimensional symmetries would be hidden.

We can calculate the explicit forms of conserved densities for the wide class of NC soliton equations.

- Space-Space noncommutativity:

  NC deformation is slight: $\sigma_n = res_{-1} L^n$

  involutive (integrable in Liouville's sense)

- Space-time noncommutativity

  NC deformation is drastical: $([t, x] = i\theta)$

  – Example: NC KP and KdV equations

$$\sigma_n = res_{-1} L^n - 3\theta((res_{-1} L^n) \Diamond u_3' + (res_{-2} L^n) \Diamond u_2')$$

meaningful ?

Exact N-soliton Solutions of NC KP hierarchy

- We have found exact N-soliton solutions for the wide class of NC hierarchies.
- Physical interpretations are non-trivial because when $f(x), g(x)$ are real, $f(x)*g(x)$ is not in general.
- However, the solutions could be real in some cases.
  - (i) 1-soliton solutions are all the same as commutative ones because of

$$f(x-vt)*g(x-vt)=f(x-vt)g(x-vt)$$

  - (ii) In asymptotic region, configurations of multi-soliton solutions could be real in soliton scatterings and the same as commutative ones.

Exact N-soliton solutions of the NC KP hierarchy

$$L = \Phi \partial_x \Phi^{-1}$$ **solves the NC KP hierarchy !**

$$\Phi f := \left| W(f_1, \ldots, f_N, f) \right|_{N+1,N+1}$$

quasi-determinant of Wronski matrix

$$f_i = \exp \xi(x, \alpha_i) + a_i \exp \xi(x, \beta_i)$$

$$\xi(x, \alpha) = x_1 \alpha + x_2 \alpha^2 + x_3 \alpha^3 + \cdots$$

Etingof-Gelfand-Retakh, [q-alg/9701008]

$$u_2 = \partial_x \sum_{i=1}^{N} (\partial_x W_i) * W_i^{-1} \xrightarrow{\theta \to 0} \partial_x^2 \log \det W(f_1, \cdots, f_N)$$

$$W_i := \left| W(f_1, \ldots, f_i) \right|_{i,i}$$

The exact solutions could be actually N-soliton solutions in asymptotic region ! **[MH, hep-th/0610006]**

Exactly solvable!

Quasi-determinants

- Quasi-determinant is not just a generalization of commutative determinant, but rather related to inverse matrices.
- For an n by n matrice $X=(x_{ij})$ and the inverse $Y=(y_{ij})$ of X, quasi-determinant of X is directly defined by

$$\left|X\right|_{ij}=y_{ji}^{-1}\qquad\left(\xrightarrow{\theta\to 0}\underbrace{\frac{(-1)^{i+j}}{\det X^{ij}}}_{\text{some factor}}\det X\right)$$

- Recall that

$$X=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\Rightarrow Y=X^{-1}=\begin{pmatrix}(A-BD^{-1}C)^{-1} & (C-DB^{-1}A)^{-1}\\ (B-AC^{-1}D)^{-1} & (D-CA^{-1}B)^{-1}\end{pmatrix}$$

→ **We can also define quasi-determinants recursively**

Quasi-determinants

- Defined inductively as follows

[For a review, see Gelfand et al., math.QA/0208146]

$$|X|_{ij} = x_{ij} - \sum_{i',j'} x_{ii'} (|X^{ij}|_{j'i'})^{-1} x_{j'j}$$

$$n=1: \quad |X|_{ij} = x_{ij}$$

$$n=2: \quad |X|_{11} = x_{11} - x_{12} \cdot x_{22}^{-1} \cdot x_{21}, \quad |X|_{12} = x_{12} - x_{11} \cdot x_{21}^{-1} \cdot x_{22},$$

$$|X|_{21} = x_{21} - x_{22} \cdot x_{12}^{-1} \cdot x_{11}, \quad |X|_{22} = x_{22} - x_{21} \cdot x_{11}^{-1} \cdot x_{12},$$

$$n=3: \quad |X|_{11} = x_{11} - x_{12} \cdot (x_{22} - x_{23} \cdot x_{33}^{-1} \cdot x_{32})^{-1} \cdot x_{21} - x_{13} \cdot (x_{32} - x_{33} \cdot x_{23}^{-1} \cdot x_{22})^{-1} \cdot x_{21}$$

$$- x_{12} \cdot (x_{23} - x_{22} \cdot x_{32}^{-1} \cdot x_{33})^{-1} \cdot x_{31} - x_{13} \cdot (x_{33} - x_{32} \cdot x_{22}^{-1} \cdot x_{23})^{-1} \cdot x_{31}$$

$$\cdots$$

Wronski matrix:

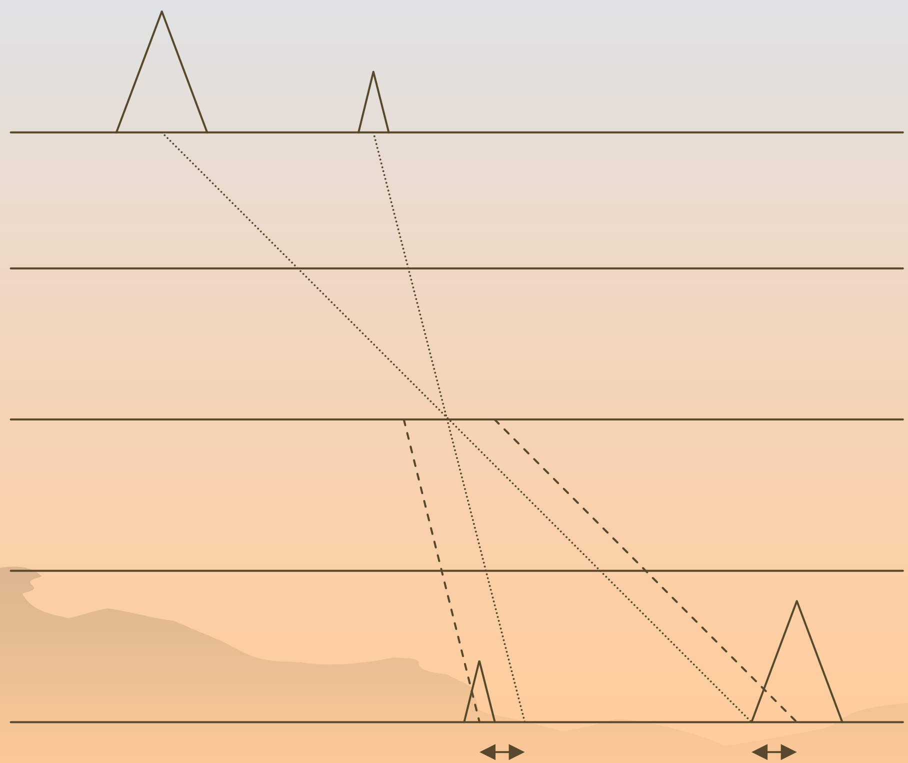
$$W(f_1, f_2, \cdots, f_m) = \begin{bmatrix} f_1 & f_2 & \cdots & f_m \\ \partial_x f_1 & \partial_x f_2 & \cdots & \partial_x f_m \\ \vdots & \vdots & \ddots & \vdots \\ \partial_x^{m-1} f_1 & \partial_x^{m-1} f_2 & \cdots & \partial_x^{m-1} f_m \end{bmatrix}$$

2-soliton solution of NC KdV

each packet has the configuration:

$$u^{(i)} = 2k_i^{\,2}\cosh^{-2}(k_i x - 4k_i^{\,3} t), \quad v_i = 4k_i^{\,2}, \quad h_i = 2k_i^{\,2}$$

Scattering process



The shape and velocity is preserved ! (stable)

The positions are shifted ! (Phase shift)

NC Burgers hierarchy

MH&K.Toda,JPA36('03)11981[hep-th/0301213]

- NC (1+1)-dim. Burgers equation:

$$\dot{u} = u'' + 2u * u'$$: Non-linear & Infinite order diff. eq. w.r.t. time ! (Integrable?)

NC Cole-Hopf transformation

$$u = \tau^{-1} * \tau' \quad \left(\xrightarrow{\theta \to 0} \partial_x \log \tau \right)$$

(NC) Diffusion equation:

$$\dot{\tau} = \tau''$$: Linear & first order diff. eq. w.r.t. time

(Completely Integrable !)

5. Conclusion and Discussion

- Confirmation of NC Ward's conjecture — Solved!
  - NC twistor theory → geometrical origin
  - D-brane interpretations → applications to physics

  Work in progress → [NC book of Mason&Woodhouse ?]

- Completion of NC Sato's theory
  - Existence of ``hierarchies'' → Solved!
  - Existence of infinite conserved quantities — Successful

    → infinite-dim. hidden symmetry?
  - Construction of multi-soliton solutions — Successful
  - Theory of tau-functions → description of the symmetry and the soliton solutions — Near at hand ?