Noncommutative Solitons and Integrable Systems

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Based on

- **MH**,``Conservation laws for NC Lax hierarchy,'' **JMP46(2005)052701[hep-th/0311206]**
- **MH**, ``NC Ward's conjecture and integrable systems,'' **NPB741 (2006) 368, [hep-th/0601209]**
- **MH**, ``Notes on exact multi-soliton solutions of NC integrable hierarchies ,'' **[hep-th/0610006]**

1. Introduction

Successful points in NC theories

- **Appearance of new physical objects**
- **Description of real physics**
- **Various successful applications to D-brane dynamics etc.**

NC Solitons play important roles
(Integrable!)

Final goal: NC extension of all soliton theories

Integrable equations in diverse dimensions

4	**Anti-Self-Dual Yang-Mills eq.** **(instantons)** $F_{\mu\nu} = -\tilde{F}_{\mu\nu}$	NC extension (Successful)
3	**Bogomol'nyi eq.** **(monopoles)**	NC extension (Successful)
2 (+1)	**Kadomtsev-Petviashvili (KP) eq.** **Davey-Stewartson (DS) eq. …**	NC extension (This talk)
1 (+1)	**KdV eq. Boussinesq eq.** **NLS eq. Burgers eq.** **sine-Gordon eq. (affine) Toda field eq. …**	NC extension (This talk)

Dim. of space

Ward's conjecture: Many (perhaps all?) integrable equations are reductions of the ASDYM eqs.

R.Ward, Phil.Trans.Roy.Soc.Lond.A315('85)451

ASDYM eq.

↓ Reductions

(KP eq.)	(DS eq.)	Ward's chiral model
KdV eq.	Boussinesq eq.	
NLS eq.	Toda field eq.	
sine-Gordon eq.	Liouville eq.	
Painleve eqs.	Tops …	

Almost confirmed by explicit examples !!!

NC Ward's conjecture: Many (perhaps all?) NC integrable equations are reductions of the NC ASDYM eqs.

MH&K.Toda, PLA316('03)77 [hep-th/0211148]

NC ASDYM eq.

Successful

↓ NC Reductions

Reductions

Many (perhaps all?) NC integrable eqs.

Successful?

NC Sato's theory plays important roles in revealing integrable aspects of them

- Existence of physical pictures
- New physical objects
- Application to D-branes
- Classfication of NC integ. eqs.

Program of NC extension of soliton theories

- **(i) Confirmation of NC Ward's conjecture**
  - **NC twistor theory → geometrical origin**
  - **D-brane interpretations → applications to physics**
- **(ii) Completion of NC Sato's theory**
  - **Existence of ``hierarchies'' → various soliton eqs.**
  - **Existence of infinite conserved quantities**

    → infinite-dim. hidden symmetry
  - **Construction of multi-soliton solutions**
  - **Theory of tau-functions → structure of the solution spaces and the symmetry**

(i),(ii) → complete understanding of the NC soliton theories

Plan of this talk

1. **Introduction**
2. **NC ASDYM equations (a master equation)**
3. **NC Ward's conjecture**
   --- Reduction of NC ASDYM to
   KdV, mKdV, Tzitzeica, …
4. **Towards NC Sato's theory (KP, …)**
   hierarchy, infinite conserved quantities,
   exact multi-soliton solutions,…
5. **Conclusion and Discussion**

2. NC ASDYM equations

Here we discuss G=GL(N) (NC) ASDYM eq. from the viewpoint of linear systems with a spectral parameter ζ .

- **Linear systems (commutative case):**

$$L\psi = (D_w - \zeta D_{\tilde z})\psi = 0,$$
$$M\psi = (D_z - \zeta D_{\tilde w})\psi = 0.$$

e.g. $\begin{pmatrix} \tilde z & w \\ \tilde w & z \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 + ix^1 & x^2 - ix^3 \\ x^2 + ix^3 & x^0 - ix^1 \end{pmatrix}$

- **Compatibility condition of the linear system:**

$$[L,M] = [D_w, D_z] + \zeta([D_z, D_{\tilde z}] - [D_w, D_{\tilde w}]) + \zeta^2 [D_{\tilde z}, D_{\tilde w}] = 0$$

$$\Leftrightarrow \begin{cases} F_{zw} = [D_z, D_w] = 0, \\ F_{\tilde z \tilde w} = [D_{\tilde z}, D_{\tilde w}] = 0, \\ F_{z\tilde z} - F_{w\tilde w} = [D_z, D_{\tilde z}] - [D_w, D_{\tilde w}] = 0 \end{cases}$$

:ASDYM equation

$$(F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])$$

Yang's form and Yang's equation

- **ASDYM eq. can be rewritten as follows**

$$\begin{cases} F_{zw} = [D_z, D_w] = 0, \quad \Rightarrow \quad \exists h,\, D_z h = 0,\ D_w h = 0 \\ F_{\tilde{z}\tilde{w}} = [D_{\tilde{z}}, D_{\tilde{w}}] = 0, \quad \Rightarrow \quad \exists \tilde{h},\, D_{\tilde{z}} \tilde{h} = 0,\ D_{\tilde{w}} \tilde{h} = 0 \\ F_{z\tilde{z}} - F_{w\tilde{w}} = [D_z, D_{\tilde{z}}] - [D_w, D_{\tilde{w}}] = 0 \end{cases}$$

If we define Yang's matrix: $J := \tilde{h}^{-1} h$

then we obtain from the third eq.:

$$\partial_z (J^{-1} \partial_{\tilde{z}} J) - \partial_w (J^{-1} \partial_{\tilde{w}} J) = 0$$

:Yang's eq.

The solution J **reproduce the gauge fields as**

$$A_z = -h_z h^{-1},\ A_w = h_w h^{-1},\ A_{\tilde{z}} = -\tilde{h}_{\tilde{z}} \tilde{h}^{-1},\ A_{\tilde{w}} = \tilde{h}_{\tilde{w}} \tilde{h}^{-1}$$

(Q) How we get NC version of the theories?

(A) We have only to replace all products of fields in ordinary commutative gauge theories with star-products: $f(x)g(x) \rightarrow f(x) * g(x)$

- **The star product:**

$$f(x) * g(x) := f(x)\exp\left(\frac{i}{2}\theta^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right)g(x) = f(x)g(x) + i\frac{\theta^{ij}}{2}\partial_i f(x)\partial_j g(x) + O(\theta^2)$$

A deformed product

Presence of background magnetic fields

$$f * (g * h) = (f * g) * h$$ Associative

$$[x^i, x^j]_* := x^i * x^j - x^j * x^i = i\theta^{ij}$$ NC !

In this way, we get NC-deformed theories with infinite derivatives in NC directions. (integrable???)

Here we discuss G=GL(N) NC ASDYM eq. from the viewpoint of linear systems with a spectral parameter ζ.

(All products are star-products.)

- **Linear systems (NC case):**

$$L * \psi = (D_w - \zeta D_{\tilde{z}}) * \psi = 0,$$
$$M * \psi = (D_z - \zeta D_{\tilde{w}}) * \psi = 0.$$

e.g. $\begin{pmatrix} \tilde{z} & w \\ \tilde{w} & z \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 + ix^1 & x^2 - ix^3 \\ x^2 + ix^3 & x^0 - ix^1 \end{pmatrix}$

- **Compatibility condition of the linear system:**

$$[L, M]_* = [D_w, D_z]_* + \zeta([D_z, D_{\tilde{z}}]_* - [D_w, D_{\tilde{w}}]_*) + \zeta^2 [D_{\tilde{z}}, D_{\tilde{w}}]_* = 0$$

$$\Leftrightarrow \begin{cases} F_{zw} = [D_z, D_w]_* = 0, \\ F_{\tilde{z}\tilde{w}} = [D_{\tilde{z}}, D_{\tilde{w}}]_* = 0, \\ F_{z\tilde{z}} - F_{w\tilde{w}} = [D_z, D_{\tilde{z}}]_* - [D_w, D_{\tilde{w}}]_* = 0 \end{cases}$$

:NC ASDYM equation

$$(F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + \underline{[A_\mu, A_\nu]_*})$$

Don't omit even for G=U(1) $(\because U(1) \cong U(\infty))$

Yang's form and NC Yang's equation

- **NC ASDYM eq. can be rewritten as follows**

$$\begin{cases} F_{zw} = [D_z, D_w]_* = 0, \quad \Rightarrow \quad \exists h,\ D_z * h = 0,\ \ D_w * h = 0 \\ F_{\tilde{z}\tilde{w}} = [D_{\tilde{z}}, D_{\tilde{w}}]_* = 0, \quad \Rightarrow \quad \exists \widetilde{h},\ D_{\tilde{z}} * \widetilde{h} = 0,\ \ D_{\tilde{w}} * \widetilde{h} = 0 \\ F_{z\tilde{z}} - F_{w\tilde{w}} = [D_z, D_{\tilde{z}}]_* - [D_w, D_{\tilde{w}}]_* = 0 \end{cases}$$

If we define Yang's matrix: $J := \widetilde{h}^{-1} * h$

then we obtain from the third eq.:

$$\partial_z (J^{-1} * \partial_{\tilde{z}} J) - \partial_w (J^{-1} * \partial_{\tilde{w}} J) = 0$$ **:NC Yang's eq.**

The solution J reproduces the gauge fields as

$$A_z = -h_z * h^{-1},\ \ A_w = h_w * h^{-1},\ A_{\tilde{z}} = -\widetilde{h}_{\tilde{z}} * \widetilde{h}^{-1},\ \ A_{\tilde{w}} = \widetilde{h}_{\tilde{w}} * \widetilde{h}^{-1}$$

(All products are star-products.)

Backlund transformation for NC Yang's eq.

- **Yang's J matrix can be decomposed as follows**

$$J = \begin{pmatrix} A^{-1} - \tilde{B} * A * B & -\tilde{B} * \tilde{A} \\ \tilde{A} * B & \tilde{A} \end{pmatrix}$$

MH [hep-th/0601209, 0ymmnnn]
The book of Mason-Woodhouse

- **Then NC Yang's eq. becomes**

$$\partial_z (A * \tilde{B}_{\tilde{z}} * \tilde{A}) - \partial_w (A * \tilde{B}_{\tilde{w}} * \tilde{A}) = 0, \quad \partial_{\tilde{z}} (\tilde{A} * B_z * A) - \partial_{\tilde{w}} (\tilde{A} * B_w * A) = 0,$$

$$\partial_z (\tilde{A}^{-1} * \tilde{A}_{\tilde{z}}) * \tilde{A}^{-1} - \partial_w (\tilde{A}^{-1} * \tilde{A}_{\tilde{w}}) * \tilde{A}^{-1} + B_z * A * \tilde{B}_{\tilde{z}} - B_w * A * \tilde{B}_{\tilde{w}} = 0,$$

$$A^{-1} * \partial_z (A_{\tilde{z}} * A^{-1}) - A^{-1} * \partial_w (A_{\tilde{w}} * A^{-1}) + \tilde{B}_{\tilde{z}} * \tilde{A} * B_z - \tilde{B}_{\tilde{w}} * \tilde{A} * B_w = 0.$$

- **The following trf. leaves NC Yang's eq. as it is:**

$$\partial_z B^{new} = A * \tilde{B}_{\tilde{w}} * \tilde{A}, \quad \partial_w B^{new} = A * \tilde{B}_{\tilde{z}} * \tilde{A},$$

$$\partial_{\tilde{z}} \tilde{B}^{new} = \tilde{A} * B_w * A, \quad \partial_{\tilde{w}} \tilde{B}^{new} = \tilde{A} * B_z * A,$$

$$A^{new} = \tilde{A}^{-1}, \quad \tilde{A}^{new} = A^{-1}$$

We can generate new solutions from known (trivial) solutions

3. NC Ward's conjecture --- reduction to (1+1)-dim.

- **From now on, we discuss reductions of NC ASDYM on (2+2)-dimension, including NC KdV, mKdV, Tzitzeica...**
- **Reduction steps are as follows:**

  (1) take a simple dimensional reduction

  with a gauge fixing.

  (2) put further reduction condition on gauge field.
- **The reduced eqs. coincides with those obtained in the framework of NC KP and GD hierarchies, which possess infinite conserved quantities and exact multi-soliton solutions. (integrable-like)**

Reduction to NC KdV eq.

MH, PLB625, 324 [hep-th/0507112]

- **(1) Take a dimensional reduction and gauge fixing:**

$$(z,\tilde{z},w,\tilde{w}) \to (t,x) = (z, w+\tilde{w}), \qquad A_{\tilde{z}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The reduced NC ASDYM is:

$$(i) \quad [A_w, A_{\tilde{z}}]_* = 0$$

$$(ii) \quad A_w' - A_{\tilde{w}}' + [A_z, A_{\tilde{z}}]_* - [A_w, A_{\tilde{w}}]_* = 0$$

$$(iii) \quad A_z' - \dot{A}_w + [A_w, A_z]_* = 0$$

- **(2) Take a further reduction condition:**

NOT traceless !

$$A_w = \begin{pmatrix} q & -1 \\ q' + q*q & -q \end{pmatrix}, A_{\tilde{w}} = O, A_z = \begin{pmatrix} \frac{1}{2}q'' + \underline{q'*q} & -q' \\ f(q,q',q'',q''') & -\frac{1}{2}q'' - \underline{q*q'} \end{pmatrix}$$

We can get NC KdV eq. in such a miracle way !

$$(iii) \quad \Rightarrow \quad \dot{u} = \frac{1}{4}u''' + \frac{3}{4}(u'*u + u*u') \qquad u = 2q' \qquad [t,x] = i\theta$$

Note: $A, B, C \in gl(2) \xrightarrow{\theta \to 0} sl(2)$ U(1) part is necessary !

The NC KdV eq. has integrable-like properties:

- possesses infinite conserved densities:

Explicit!

$$\sigma_n = res_{-1}L^n + \frac{3}{4}\theta((res_{-1}L^n)\Diamond u'' - 2(res_{-2}L^n)\Diamond u')$$

MH, JMP46 (2005) [hep-th/0311206]

$res_r L^n$: coefficient of ∂_x^r in L^n

$\Diamond$: Strachan's product (commutative and non-associative)

$$f(x)\Diamond g(x) := f(x)\left(\sum_{s=0}^{\infty}\frac{(-1)^s}{(2s+1)!}\left(\frac{1}{2}\theta^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right)^{2s}\right)g(x)$$

$$[t,x] = i\theta$$

- has exact N-soliton solutions:

Etingof-Gelfand-Retakh, [q-alg/9701008]
MH, [hep-th/0610006]
cf. Paniak, [hep-th/0105185]

$$u = 2\partial_x \sum_{i=1}^{N}(\partial_x W_i) * W_i^{-1}$$

Explicit!

$$W_i := \left| W(f_1, \ldots, f_i) \right|_{i,i}$$:quasi-determinant of Wronski matrix

$$f_i = \exp\xi(x,\alpha_i) + a_i \exp(-\xi(x,\alpha_i))$$

$$\xi(x,\alpha) = x\alpha + t\alpha^3$$

Reduction to NC mKdV eq.

MH, NPB741, 368 [hep-th/0601209]

- **(1) Take a dimensional reduction and gauge fixing:**

$$(z,\tilde{z},w,\tilde{w}) \rightarrow (t,x) = (z, w+\tilde{w}), \qquad A_{\tilde{z}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The reduced NC ASDYM is:

$$(i) \quad [A_w, A_{\tilde{z}}]_* = 0$$

$$(ii) \quad A'_w - A'_{\tilde{w}} + [A_z, A_{\tilde{z}}]_* - [A_w, A_{\tilde{w}}]_* = 0$$

$$(iii) \quad A'_z - \dot{A}_w + [A_w, A_z]_* = 0$$

- **(2) Take a further reduction condition:**

$$A_w = \begin{pmatrix} p & -1 \\ 0 & -p \end{pmatrix}, A_{\tilde{w}} = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}, A_z = \begin{pmatrix} c & b \\ 0 & d \end{pmatrix}$$

We get

$$a = -\frac{1}{2}p' - \frac{1}{2}p^2, b = -\frac{1}{2}p' + \frac{1}{2}p^2,$$

NOT traceless !

$$c = \frac{1}{4}p'' - \frac{1}{2}p^3 \underline{-\frac{1}{4}[p,p']_*}, d = -\frac{1}{4}p'' + \frac{1}{2}p^3 \underline{-\frac{1}{4}[p,p']_*}$$

and $(iii) \quad \Rightarrow \quad \dot{p} = \frac{1}{4}p''' - \frac{3}{4}(p' * p * p + p * p * p')$ **NC mKdV !**

$$[t,x] = i\theta$$

Relation between NC KdV and NC mKdV

- **(1) Take a dimensional reduction and gauge fixing:**

$$(z,\tilde{z},w,\tilde{w}) \to (t,x) = (z, w+\tilde{w}),$$

$$A_{\tilde{z}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Note: There is a residual gauge symmetry:

$$A_\mu \to g^{-1} * A_\mu * g + g^{-1} * \partial_\mu g, \quad g = \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}$$

- **(2) Compare the further reduction conditions:**

NCKdV: $A_w = \begin{pmatrix} q & -1 \\ q' + q * q & -q \end{pmatrix}, A_{\tilde{w}} = O, A_z = \begin{pmatrix} \frac{1}{2}q'' + q' * q & -q' \\ f(q,q',q'',q''') & -\frac{1}{2}q'' - q * q' \end{pmatrix}$

Gauge equivalent

The gauge trf. → $\beta = q - p, \quad \underline{2q' = p' - p^2}$

NC Miura map !

NCmKdV: $A_w = \begin{pmatrix} p & -1 \\ 0 & -p \end{pmatrix}, A_{\tilde{w}} = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}, A_z = \begin{pmatrix} c & b \\ 0 & d \end{pmatrix}$

MH, NPB741, 368 [hep-th/0601209]

Reduction to NC Tzitzeica eq.

MH, NPB741, 368 [hep-th/0601209]

- **Start with NC Yang's eq.**

$$\partial_z (J^{-1}\partial_{\tilde z} J) - \partial_w (J^{-1}\partial_{\tilde w} J) = 0$$

- **(1) Take a special reduction condition:**

$$J = \exp(-E_- \tilde w) * g(z,\tilde z) * \exp(E_+ w)$$

$$E_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

We get a reduced Yang's eq.

$$\partial_z (g^{-1} * \partial_{\tilde z} g) - [E_-, g^{-1} * E_+ g]_* = 0$$

$$E_- = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

- **(2) Take a further reduction condition:**

$$g = \exp(\rho) * diag\,(\exp(\omega), \exp(-\omega), 1)$$

We get (a set of) NC Tzitzeica eq.:

$$\partial_z(\exp(-\omega) * \partial_{\tilde z}\exp(\omega)) + \partial_z(\exp(-\omega) * V * \exp(\omega)) = \exp(\omega) - \exp(-2\omega),$$

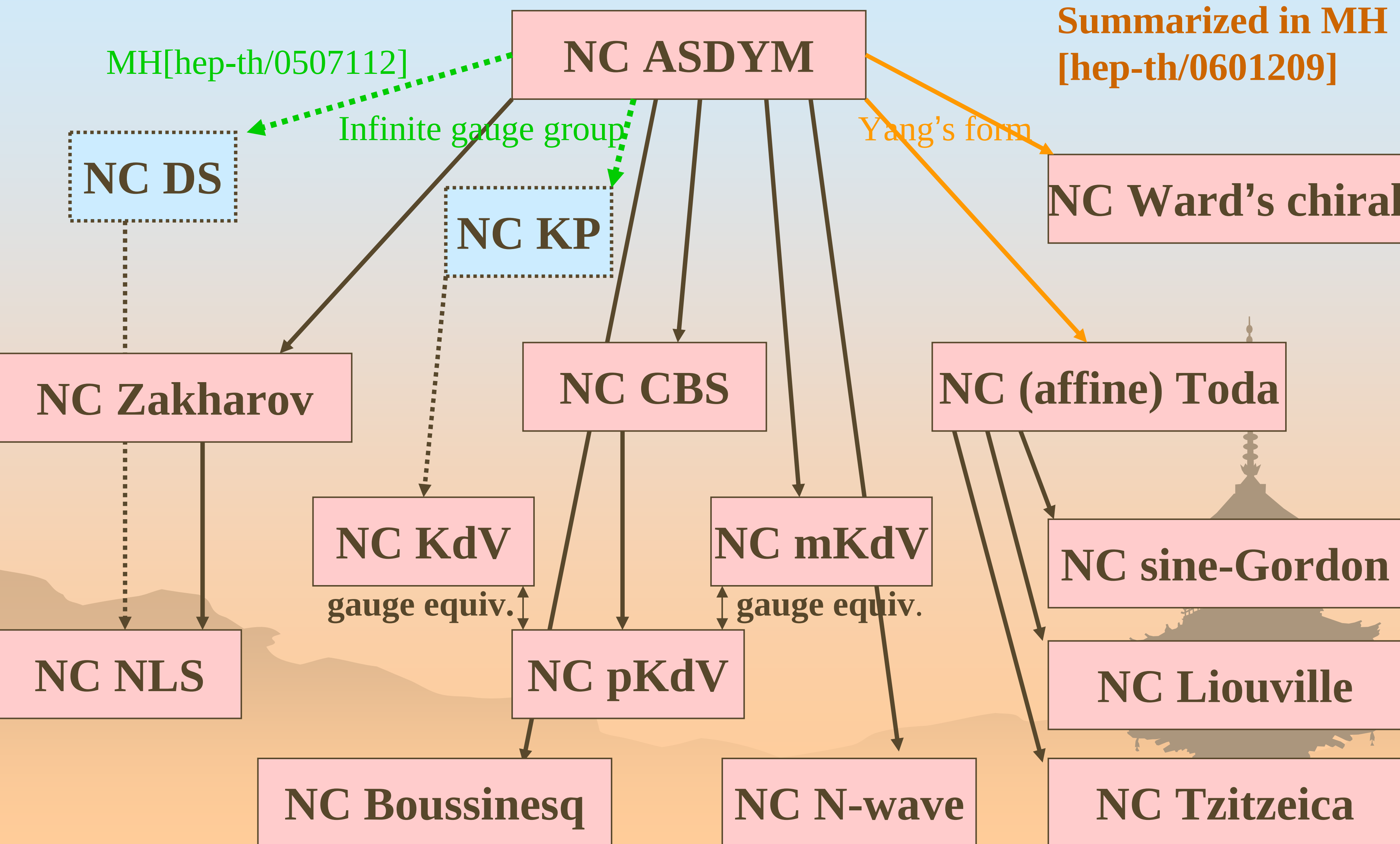
$$\partial_z(\exp(\omega) * \partial_{\tilde z}\exp(-\omega)) + \partial_z(\exp(\omega) * V * \exp(-\omega)) = \exp(-2\omega) - \exp(\omega),$$

$$\partial_z V = \partial_z(\exp(-\rho) * \partial_{\tilde z}\exp(\rho)) = 0$$

$$\left(\xrightarrow{\theta \to 0} \omega_{z\tilde z} = \exp(\omega) - \exp(-2\omega)\right)$$

In this way, we can obtain various NC integrable equations from NC ASDYM !!!

Almost all ?



4. Towards NC Sato's Theory

- **Sato's Theory : one of the most beautiful theory of solitons**
  - **Based on the exsitence of hierarchies and tau-functions**
  - **Various integrable equations in (1+1)-dim. can be derived elegantly from (2+1)-dim. KP equation.**
- **Sato's theory reveals essential aspects of solitons:**
  - **Construction of exact solutions**
  - **Structure of solution spaces**
  - **Infinite conserved quantities**
  - **Hidden infinite-dim. symmetry**

Let's discuss NC extension of Sato's theory

Derivation of soliton equations

- **Prepare a Lax operator which is a pseudo-differential operator**

$$L := \partial_x + 2u\partial_x^{-1} + f(u)\partial_x^{-2} + g(u)\partial_x^{-3} + \cdots$$

$$u = u(\underline{x^1, x^2, x^3, \cdots})$$

Noncommutativity is introduced here:

$$[x^i, x^j] = i\theta^{ij}$$

- **Introduce a differential operator**

$$B_m := (\underbrace{L * \cdots * L}_{m \text{ times}})_{\geq 0}$$

- **Define NC KP hierarchy:**

$$\boxed{\frac{\partial L}{\partial x^m} = [B_m, L]_*}$$

$\longrightarrow$ yields NC KP equations and other NC hierarchy eqs.

$$\partial_m u_2 \partial_x^{-1} + \partial_m u_3 \partial_x^{-2} + \partial_m u_4 \partial_x^{-3} + \cdots$$

$$f_{m2}(u)\partial_x^{-1} + f_{m3}(u)\partial_x^{-2} + f_{m4}(u)\partial_x^{-3} + \cdots$$

Find a suitable L which satisfies NC KP hierarchy !
→ solutions of NC KP eq.

Negative powers of differential operators

$$\partial_x^n \circ f := \sum_{j=0}^{\infty} \binom{n}{j} (\partial_x^j f)\partial_x^{n-j}$$

$$\binom{n}{j} \equiv \frac{n(n-1)(n-2)\cdots(n-(j-1))}{j(j-1)(j-2)\cdots 1}$$

: binomial coefficient which can be extended to negative n → negative power of differential operator (well-defined !)

$$\partial_x^3 \circ f = f\partial_x^3 + 3f'\partial_x^2 + 3f''\partial_x^1 + f'''$$

$$\partial_x^2 \circ f = f\partial_x^2 + 2f'\partial_x + f''$$

$$\partial_x^{-1} \circ f = f\partial_x^{-1} - f'\partial_x^{-2} + f''\partial_x^{-3} - \cdots$$

$$\partial_x^{-2} \circ f = f\partial_x^{-2} - 2f'\partial_x^{-3} + 3f''\partial_x^{-4} - \cdots$$

Star product:

$$f(x) * g(x) := f(x)\exp\left(\frac{i}{2}\theta^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right)g(x)$$

which makes theories "noncommutative":

$$[x^i, x^j]_* := x^i * x^j - x^j * x^i = i\theta^{ij}$$

Closer look at NC KP hierarchy

For m=2

$\partial_x^{-1})\quad \partial_2 u_2 = \underline{2u_3'} + u_2''$

$\partial_x^{-2})\quad \partial_2 u_3 = \underline{2u_4'} + u_3'' + 2u_2 * u_2' + 2[u_2, u_3]_*$

$\partial_x^{-3})\quad \partial_2 u_4 = \underline{2u_5'} + u_4'' + 4u_3 * u_2' - 2u_2 * u_2'' + 2[u_2, u_4]_*$

$\vdots$

Infinite kind of fields are represented
in terms of one kind of field $u_2 \equiv u$

MH&K.Toda, [hep-th/0309265]

$u_x := \dfrac{\partial u}{\partial x}$

$\partial_x^{-1} := \int^x dx'$

etc.

For m=3

$\partial_x^{-1})\quad \partial_3 u_2 = u_2''' + 3u_3'' + 3u_4'' + 3u_2' * u_2 + 3u_2 * u_2'$

$\vdots$

$\downarrow$

$$u_t = \frac{1}{4}u_{xxx} + \frac{3}{4}(u_x * u + u * u_x) + \frac{3}{4}\partial_x^{-1}u_{yy} + \frac{3}{4}[u, \partial_x^{-1}u_{yy}]_*$$

(2+1)-dim.
NC KP equation

and other NC equations
(NC KP hierarchy equations)

$$u = u(x^1, x^2, x^3, \cdots)$$
$$\updownarrow\quad \updownarrow\quad \updownarrow$$
$$x\quad y\quad t$$

(KP hierarchy) $\xrightarrow{\text{reductions}}$ (various hierarchies.)

- (Ex.) KdV hierarchy

Reduction condition

$$L^2 = B_2 (=: \partial_x^2 + u)$$: 2-reduction

gives rise to NC KdV hierarchy

which includes (1+1)-dim. NC KdV eq.:

$$u_t = \frac{1}{4} u_{xxx} + \frac{3}{4} (u_x * u + u * u_x)$$

Note $\frac{\partial u}{\partial x_{2N}} = 0$: dimensional reduction in x_{2N} directions

KP : $u(x^1, x^2, x^3, x^4, x^5, \ldots)$: (2+1)-dim.
$\quad x \quad y \quad t$

$\downarrow$

KdV : $u(x^1, \quad x^3, \quad x^5, \ldots)$: (1+1)-dim.
$\quad x \quad t$

l-reduction of NC KP hierarchy yields wide class of other NC hierarchies

- No-reduction → NC KP $\qquad (x,y,t)=(x^1,x^2,x^3)$
- 2-reduction → NC KdV $\qquad (x,t)=(x^1,x^3)$
- 3-reduction → NC Boussinesq $\qquad (x,t)=(x^1,x^2)$
- 4-reduction → NC Coupled KdV $\qquad \ldots$
- 5-reduction → …
- 3-reduction of BKP → NC Sawada-Kotera
- 2-reduction of mKP → NC mKdV
- Special 1-reduction of mKP → NC Burgers
- …

Noncommutativity should be introduced into space-time coords

Exact N-soliton solutions of the NC KP hierarchy

$$L = \Phi * \partial_x \Phi^{-1}$$

solves the NC KP hierarchy !

$$= \partial_x + \frac{u}{2}\partial_x^{-1} + \cdots$$

$$\Phi f := \left| W(f_1, \ldots, f_N, f) \right|_{N+1, N+1}$$

quasi-determinant of Wronski matrix

Etingof-Gelfand-Retakh, [q-alg/9701008]

$$f_i = \exp \xi(x, \alpha_i) + a_i \exp \xi(x, \beta_i)$$

$$\xi(x, \alpha) = x_1 \alpha + x_2 \alpha^2 + x_3 \alpha^3 + \cdots$$

$$u = 2\partial_x \sum_{i=1}^{N} (\partial_x W_i) * W_i^{-1} \xrightarrow{\theta \to 0} 2\partial_x^2 \log\det W(f_1, \cdots, f_N)$$

$$W_i := \left| W(f_1, \ldots, f_i) \right|_{i,i}$$

Wronski matrix:

$$W(f_1, f_2, \cdots, f_m) = \begin{bmatrix} f_1 & f_2 & \cdots & f_m \\ \partial_x f_1 & \partial_x f_2 & \cdots & \partial_x f_m \\ \vdots & \vdots & \ddots & \vdots \\ \partial_x^{m-1} f_1 & \partial_x^{m-1} f_2 & \cdots & \partial_x^{m-1} f_m \end{bmatrix}$$

($\alpha_i^l = \beta_i^l$ **$\leftarrow\rightarrow$ l-reduction condition**)

Quasi-determinants

- **Quasi-determinants are not just a generalization of commutative determinants, but rather related to inverse matrices.**
- **For an n by n matrix** $X=(x_{ij})$ **and the inverse** $Y=(y_{ij})$ **of X, quasi-determinant of X is directly defined by**

$$\left|X\right|_{ij}=y_{ji}^{-1} \qquad \left(\xrightarrow{\theta\to 0} \frac{(-1)^{i+j}}{\underline{\det X^{ij}}}\det X\right)$$

some factor

- **Recall that**

$$X=\begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \Rightarrow$$

$$Y=X^{-1}=\begin{pmatrix} A^{-1}+A^{-1}B(D-CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1} \\ -(D-CA^{-1}B)^{-1}CA^{-1} & (D-CA^{-1}B)^{-1} \end{pmatrix}$$

→ **We can also define quasi-determinants recursively**

Quasi-determinants

- **Defined inductively as follows**

[For a review, see Gelfand et al., math.QA/0208146]

$$\left|X\right|_{ij} = x_{ij} - \sum_{i',j'} x_{ii'}((X^{ij})^{-1})_{i'j'} x_{j'j}$$
$$= x_{ij} - \sum_{i',j'} x_{ii'}(\left|X^{ij}\right|_{j'i'})^{-1} x_{j'j}$$

X^{ij} **: the matrix obtained from X deleting i-th row and j-th column**

$$n=1: \quad \left|X\right|_{ij} = x_{ij}$$

$$n=2: \quad \left|X\right|_{11} = x_{11} - x_{12}\cdot x_{22}^{-1}\cdot x_{21}, \ \left|X\right|_{12} = x_{12} - x_{11}\cdot x_{21}^{-1}\cdot x_{22},$$
$$\left|X\right|_{21} = x_{21} - x_{22}\cdot x_{12}^{-1}\cdot x_{11}, \ \left|X\right|_{22} = x_{22} - x_{21}\cdot x_{11}^{-1}\cdot x_{12},$$

$$n=3: \quad \left|X\right|_{11} = x_{11} - x_{12}\cdot(x_{22} - x_{23}\cdot x_{33}^{-1}\cdot x_{32})^{-1}\cdot x_{21} - x_{13}\cdot(x_{32} - x_{33}\cdot x_{23}^{-1}\cdot x_{22})^{-1}\cdot x_{21}$$
$$- x_{12}\cdot(x_{23} - x_{22}\cdot x_{32}^{-1}\cdot x_{33})^{-1}\cdot x_{31} - x_{13}\cdot(x_{33} - x_{32}\cdot x_{22}^{-1}\cdot x_{23})^{-1}\cdot x_{31}$$
$$\cdots$$

Interpretation of the exact N-soliton solutions

- We have found exact N-soliton solutions for the wide class of NC hierarchies.
- Physical interpretations are non-trivial because when $f(x), g(x)$ are real, $f(x)*g(x)$ is not in general.
- However, the solutions could be **real** in some cases.
  - (i) **<u>1-soliton solutions</u> are all the same as commutative ones because of**

$$f(x-vt)*g(x-vt) = f(x-vt)g(x-vt)$$

  - (ii) **<u>In asymptotic region</u>, configurations of multi-soliton solutions could be real in soliton scatterings and the same as commutative ones.**

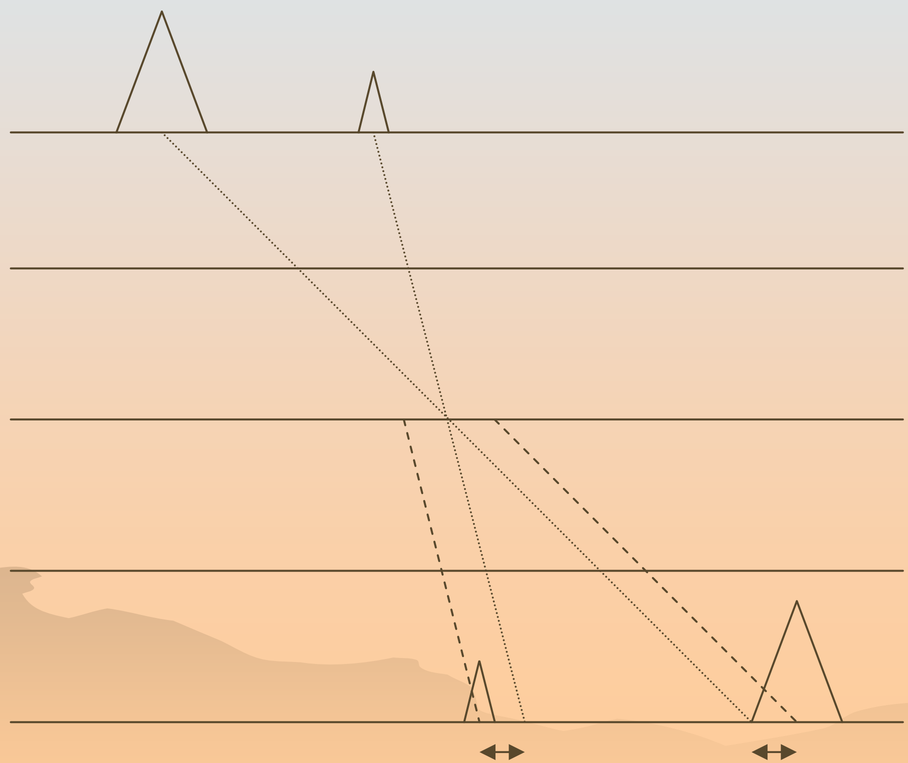
MH [hep-th/0610006]

2-soliton solution of KdV

each packet has the configuration:

$$u^{(i)} = 2k_i^{\,2} \cosh^{-2}(k_i x - 4k_i^{\,3} t), \quad \underbrace{v_i = 4k_i^{\,2}}_{\text{velocity}}, \quad \underbrace{h_i = 2k_i^{\,2}}_{\text{height}}$$

Scattering process (commutative case)



The shape and velocity is preserved ! (stable)

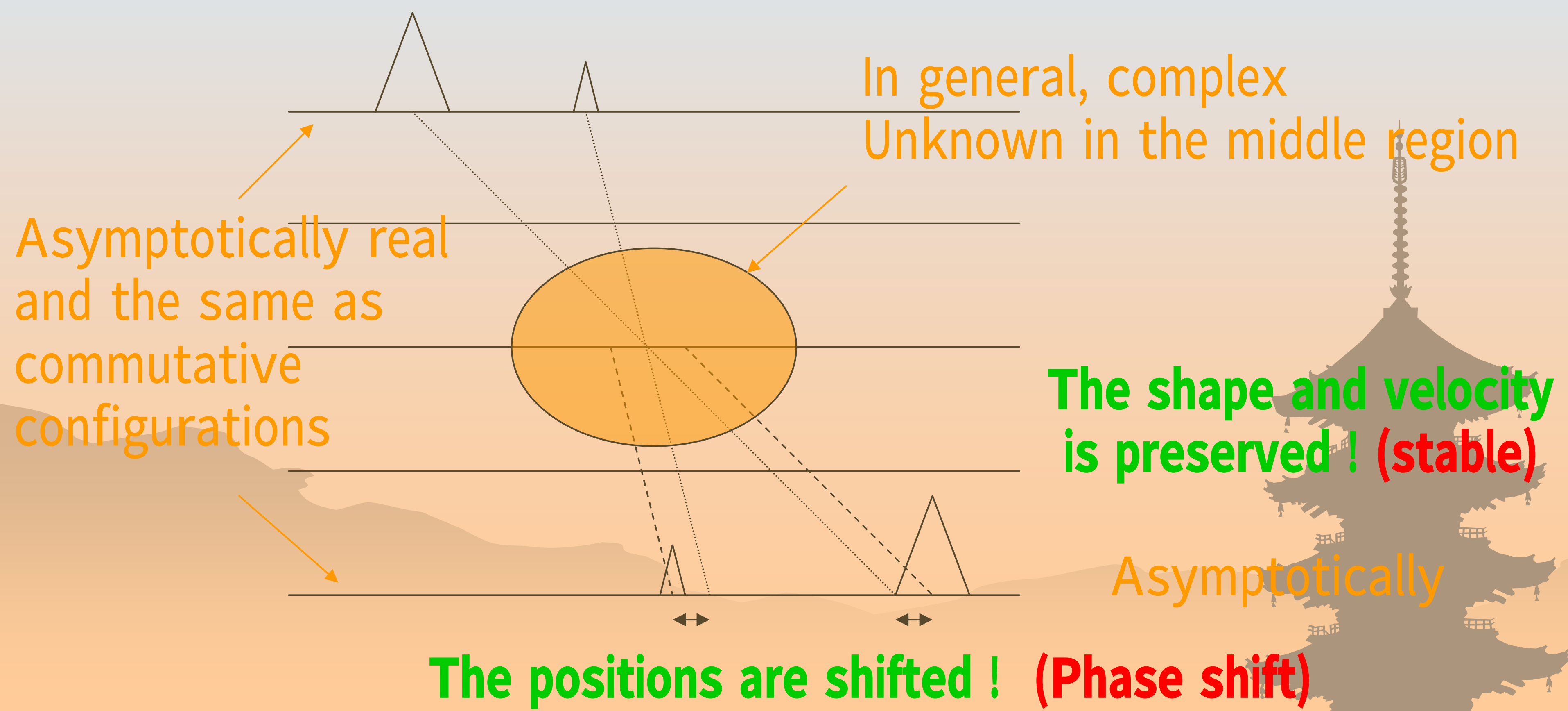
The positions are shifted ! (Phase shift)

2-soliton solution of NC KdV

each packet has the configuration:

$$u^{(i)} = 2k_i^{\ 2} \cosh^{-2}(k_i x - 4k_i^{\ 3} t), \quad \underbrace{v_i = 4k_i^{\ 2}}_{\text{velocity}}, \quad \underbrace{h_i = 2k_i^{\ 2}}_{\text{height}}$$

Scattering process (NC case)



Infinite conserved densities for the NC KP hierarchy. (n=1,2,…, ∞)

$$\sigma_n = res_{-1}L^n + \theta^{im}\sum_{k=0}^{m-1}\sum_{l=0}^{k}\begin{pmatrix}k\\ l\end{pmatrix}(\partial_x^{k-l} res_{-(l+1)}L^n)\Diamond(\partial_i res_k L^m)$$

$t \equiv x^m$ $\qquad$ $res_r L^n$: coefficient of ∂_x^r in L^n

$\Diamond$: Strachan's product (commutative and non-associative)

$$f(x)\Diamond g(x) := f(x)\left(\sum_{s=0}^{\infty}\frac{(-1)^s}{(2s+1)!}\left(\frac{1}{2}\theta^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right)^{2s}\right)g(x)$$

MH, JMP46 (2005) [hep-th/0311206]

This suggests infinite-dimensional symmetries would be hidden.

We can calculate the explicit forms of conserved densities for the wide class of NC soliton equations.

- Space-Space noncommutativity:

  NC deformation is slight: $\sigma_n = res_{-1} L^n$

  involutive (integrable in Liouville's sense)

- Space-time noncommutativity

  NC deformation is drastical: $([t, x] = i\theta)$

  - Example: NC KP and KdV equations

$$\sigma_n = res_{-1} L^n - 3\theta((res_{-1} L^n) \lozenge u_3' + (res_{-2} L^n) \lozenge u_2')$$

meaningful ?

5. Conclusion and Discussion

- Confirmation of NC Ward's conjecture — Solved!
  - NC twistor theory → geometrical origin
  - D-brane interpretations → applications to physics

  Work in progress → [NC book of Mason&Woodhouse ?]

- Completion of NC Sato's theory
  - Existence of ``hierarchies'' → Solved!
  - Existence of infinite conserved quantities — Successful

    → infinite-dim. hidden symmetry?
  - Construction of multi-soliton solutions — Successful
  - Theory of tau-functions → description of the symmetry in terms of infinite-dim. algebras. — Near at hand ?